Algebraic stacks #5

22/3 - 2017

Points of algebraic spaces (and stacks)

$X \in Sh((Sch/S)_{fppf})$

Def: $|X| := \{ \operatorname{Spec} k \xrightarrow{x} X : k \text{ field} \} / \sim$ (as a set)

$x \sim y$ if $\exists$

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[dr, "x"] & \\ \operatorname{Spec} k'' \arrow[ur] \arrow[dr] \arrow[rr, "z"] & & X \\ & \operatorname{Spec} k' \arrow[ur, "y"'] & \end{tikzcd}$$

Rmk: For $\mathfrak{X}$ stack, same definition of $|\mathfrak{X}|$ except that $\circ$ = 2-commutative.

Rmk: If X scheme, $|X|$ = underlying topological space and every point has a unique minimal representative $\operatorname{Spec} \kappa(x) \xrightarrow{x} X$ (and x monomorphism)

Rmk: $|-|$ is a functor: $\text{Spaces} = Sh((Sch/S)_{fppf}) \longrightarrow \text{Sets}$.

$$f : X \to Y \quad \rightsquigarrow \quad |f| : |X| \longrightarrow |Y|$$
$$[x] \longmapsto [fx]$$

Lemma: If $U \xrightarrow{\pi} X$ epimorphism of spaces (so $R = U \times_X U \rightrightarrows U \longrightarrow X$ coequalizer) then $|R| \rightrightarrows |U| \longrightarrow |X|$ coequalizer of sets.

More generally, $|\cdot|$ preserves colimits.

pf: If $[\operatorname{Spec} k \xrightarrow{x} X] \in |X|$ then $\exists\, V \xrightarrow{\rho} \operatorname{Spec} k$ fppf-covering and a lift to U: (*)

$$\begin{tikzcd} U \arrow[r] & X \\ V \arrow[u] \arrow[r, "\rho"] & \operatorname{Spec} k \arrow[u, "x"'] \\ \operatorname{Spec} k' \arrow[u] \arrow[ur] & \end{tikzcd}$$

Pick $\operatorname{Spec} k' \to V$ ~~above~~ any point $\rightsquigarrow [\operatorname{Spec} k' \to U] \in |U|$ lifting x.

Similarly, given two lifts $\operatorname{Spec} k_1 \xrightarrow{u_1} U$, $\operatorname{Spec} k_2 \xrightarrow{u_2} U$ s.th. $\pi u_1 \sim \pi u_2$, $\exists \operatorname{Spec} k_3 \xrightarrow{r} R$ s.th. $u_1 \sim \pi_1 r$, $u_2 \sim \pi_2 r$. (*)

(*) b/c X is sheafification of presheaf quotient U/R. $\square$

Topology of $|X|$

Def: $U \subset |X|$ open if $\exists$ functor $V \xrightarrow{j} X$ represented by open immersions s.th. $U = |j|(|V|)$.

Rmk: (i) j mono $\Rightarrow$ $|j|$ mono.

(ii) This defines a topology on $|X|$ (exercise!)

(iii) $|-|$: Spaces $\to$ Top is a functor (i.e. $|f|$ is continuous)

(iv) $|-|$: Spaces $\to$ Top preserves colimits, in particular, if $U \to X$ epimorphism then $|U| \to |X|$ topological quotient.

Thm 1 (Prop 6.3.4) Assume X quasi-separated algebraic space.

Then every point $\operatorname{Spec} k \xrightarrow{x} X$ has a unique minimal representative

$\operatorname{Spec} \kappa(x) \xhookrightarrow{x_0} X$ with x_0 a monomorphism.

Rmk: An algebraic space is **decent** if every point $\operatorname{Spec} k \xrightarrow{x} X$ is quasi-compact.

Thm 1 holds for decent spaces. Schemes and qs alg spaces are decent.

Ex: $X = \operatorname{Spec} \overline{\mathbb{Q}} / \operatorname{Gal}(\mathbb{Q})_{disc}$ is an alg space with $|X| = \{*\}$.

Does not admit a minimal representative.

Variant: $\operatorname{Spec} k(x) / \mathbb{Z}$ w/ $n.f(x) = f(x) + n$, char $k = 0$.

Ex: $X = \mathbb{A}^1_{\mathbb{C}} / \mathbb{C}$. Action $\underset{\mathbb{C} \times \mathbb{A}^1}{\underset{\cap}{(n, x)}} \longmapsto \underset{\mathbb{A}^1}{\underset{\cap}{x+n}}$. $|X| = \{x_0, x_1\}$

$\pi : \mathbb{A}^1_{\mathbb{C}} \longrightarrow X$, $\pi^{-1}(x_0) =$ closed points of $\mathbb{A}^1$
$\pi^{-1}(x_1) =$ generic point of $\mathbb{A}^1$

$\operatorname{Spec} \mathbb{C} \xrightarrow{0} \mathbb{A}^1 \xrightarrow{\pi} X$ is a minimal repr for x_0

but is not quasi-compact

(not decent)

$$\begin{tikzcd} \coprod_{\mathbb{C}} \operatorname{Spec} \mathbb{C} \arrow[r, hook] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r, hook, "x_0"] & X \end{tikzcd}$$

x_1 does not admit a monomorphic representative.

Thm 2 (Thm 6.4.1) Assume X quasi-separated algebraic space.

Then $\exists$ dense open $U \subseteq X$ s.th. U is a scheme.

Rmk: Thm 2 also holds for decent spaces.

Rmk: Not true in general: exists one-point alg spaces which are not schemes.

Stronger version for qcqs spaces:

Thm 3 (Raynaud-Gruson) Assume X quasi-compact and quasi-separated algebraic space. Then $\exists$ finite filtration $X = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n = \emptyset$ where U_k open and $\exists$ étale presentation $\pi_k : U_k' \longrightarrow U_k$ s.th. π_k isomorphism over $U_k \setminus U_{k+1}$. In particular $U_k \setminus U_{k+1}$ is a scheme.

Rmk: $\coprod_k U_k' \longrightarrow X$ is a Nisnevich presentation of X.

Rmk: Thm 3 $\Rightarrow$ Thm 1 + Thm 2.

Rmk: Natural construction gives U_k' quasi-affine but can arrange so that U_k' affine.

Method: Quotients of finite étale equivalence relations of quasi-affine schemes are quasi-affine schemes.

More generally: quotients of finite flat groupoids (needed later for coarse mod. spaces)

Quotients of finite flat groupoids

Thm 4 (FGA Exp 212, SGA3 Exp V, [R13, Thm 4.1])

Let $\underset{\operatorname{Spec} A_1}{\overset{\shortparallel}{X_1}} \underset{t}{\overset{s}{\rightrightarrows}} \underset{\operatorname{Spec} A_0}{\overset{\shortparallel}{X_0}}$ be a finite flat fin pres groupoid of affine schemes. (so A_1 is loc. free of finite rank as an A_0-module)

Let $A = \operatorname{eq}(A_0 \rightrightarrows A_1)$. Then $X_0 \xrightarrow{\pi} X = \operatorname{Spec} A$ is a geometric and categorical quotient.

This means:

geom quot:

- top quot:
  - (0) $\pi s = \pi t$
  - (i) $|X_1| \rightrightarrows |X_0| \to |X|$ is a coequalizer.
- (ii) $\mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0}$ is the subsheaf of invariants, i.e.,

$$0 \to \mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0} \xrightarrow{s^* - t^*} \pi_* s_* \mathcal{O}_{X_1}$$

is exact.

cat quot:

- (iii) $\forall X_0 \xrightarrow{p} T$, w/ T an <u>algebraic space</u> s.th. $ps = pt$ there $\exists ! X \xrightarrow{f} T$ s.th. $p = f\pi$, i.e., π is the colimit of $X_1 \rightrightarrows X_0$ in the category of <u>algebraic spaces</u>.

Moreover, π is <u>integral</u> (i.e. affine and $\mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0}$ integral extension)

Cor (FGA, SGA3, [R13, Thm 4.4], Olsson 6.2.2)

Let $X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0$ finite flat groupoid of schemes s.th.

(*) $\forall x \in X_0$, the orbit $o(x) := s(t^{-1}(x))$ is contained in an affine open subset of X_0.

Then $\exists$ geom and cat quot $\pi : X_0 \to X$ w/ X a <u>scheme</u>, and π is integral.

Rmk:
- (*) is necessary (easy).
- W/o (*), but assuming X_0 <u>separated</u>, $\exists$ geom + cat quot X w/ X an alg. space.

Rmk: In general, $\pi: X_0 \to X$ is not the quotient in the category of spaces ($Sh(Sch_{fppf})$).

But if $X_1 \rightrightarrows X_0$ finite étale (or flat) equivalence relation, then sheaf quotient $\tilde{X}$ is an algebraic space by definition (resp. by a theorem of Artin), thus $X = \tilde{X}$.

Consequence: If in Thm/Cor $X_1 \rightrightarrows X_0$ is an equiv relation (i.e. $X_1 \to X_0 \times X_0$ mono) then $\pi: X_0 \to X$ finite flat and $X_1 = X_0 \times_X X_0$.

Rmk. (*) holds when any finite set of points is contained in an affine subset. e.g. when X_0 is quasi-affine or more generally admits an ample line bundle (e.g. X_0 quasi-proj.)

Consequence: If X is an algebraic space w/ a finite étale (or flat) presentation $X_0 \xrightarrow{\pi} X$ w/ X_0 a quasi-affine scheme, then X (quasi-affine) scheme.

Rmk: Olsson only proves that quot in Thm 4/Cor is categorical in category of schemes. He therefore has to prove the two above consequences.

Rmk: As equalizers commute w/ flat base change, the quotients are uniform (i.e. commute w/ flat b.c.). In char 0 (or tame char p) they are universal (i.e. commute w/ any b.c.) but does not hold in general.

Proof of Thm 2:

Let $X^{sch} \subset X$ be the schematic locus: $x \in X^{sch} \iff \exists$ open nbhd $x \in U \subset X$ s.th. U a scheme.

Need to prove that X^{sch} dense.

WLOG X quasi-compact.

b/c X q-sep.

Pick $X' \xrightarrow{\pi} X$ étale presentation w/ X' affine. Note that π is étale, quasi-compact and separated

For any $n \in \mathbb{N}$, let $X^{(n)} \subset X$ be the locus of points $x \in X$ s.th. π is finite of rank n in a nbhd of x. This is open (possibly $\emptyset$) b/c local question on X.

Also qcpt b/c π qc.

$$\begin{tikzcd} X' \times_X X^{(n)} \arrow[r, phantom, "\subset"] \arrow[d, "\pi^{(n)}"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \\ X^{(n)} \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

$\pi^{(n)}$ is finite étale $\Rightarrow$ $X^{(n)}$ is a scheme.

$\Rightarrow \bigcup_n X^{(n)} \subseteq X^{sch}$

Easily seen that $\bigcup_n X^{(n)}$ dense.