

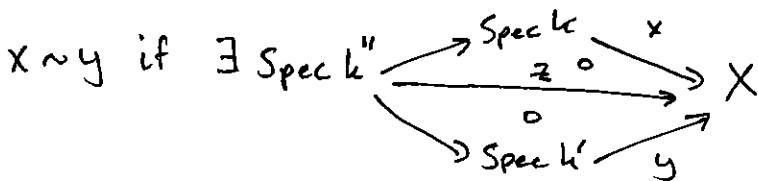
Algebraic stacks #5
22/3 - 2017

1/

Points of algebraic spaces (and stacks)

$$X \in \text{Sh}((\text{Sch}/S)_{\text{fppf}})$$

Def: $|X| := \{ \text{Spec } k \xrightarrow{x} X : k \text{ field} \} / \sim$ (as a set)



Rem: For \mathcal{X} stack, same definition of $|\mathcal{X}|$ except that $\circ = \mathbb{Z}$ -commutative.

Rem: If X scheme, $|X| =$ underlying topological space and every point has a unique minimal representative $\text{Spec } k(x) \xrightarrow{x} X$ (and x monomorphism)

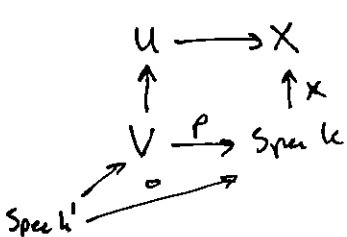
Rem: $|\cdot|$ is a functor: $\text{Spaces} = \text{Sh}((\text{Sch}/S)_{\text{fppf}}) \rightarrow \text{Sets}$.

$$\begin{aligned}
 f: X \rightarrow Y &\rightsquigarrow |f|: |X| \rightarrow |Y| \\
 [x] &\longmapsto [fx]
 \end{aligned}$$

Lemma: If $U \xrightarrow{\pi} X$ epimorphism of spaces (so $R = U \times_X U \rightrightarrows U \rightarrow X$ coequalizer) then $|R| \rightrightarrows |U| \rightarrow |X|$ coequalizer of sets.

More generally, $|\cdot|$ preserves colimits.

pf: If $[\text{Spec } k \xrightarrow{x} X] \in |X|$ then $\exists V \xrightarrow{p} \text{Spec } k$ fppf-covering and a lift to U : ^(*)



Pick $\text{Spec } k' \rightarrow V$ above any point $\rightsquigarrow [\text{Spec } k' \rightarrow U] \in |U|$ lifting x .

Similarly, given two lifts $\text{Spec } k_1 \xrightarrow{u_1} U, \text{Spec } k_2 \xrightarrow{u_2} U$ s.t. $\pi u_1 \sim \pi u_2, \exists \text{Spec } k_3 \xrightarrow{r} R$ s.t. $u_1 \sim \pi_1 r, u_2 \sim \pi_2 r$. ^(*)

(*) b/c X is sheafification of presheaf quotient U/R . □

Topology of $|X|$

Def: $U \subset |X|$ open if \exists functor $V \xrightarrow{j} X$ represented by open immersions
s.th. $U = |j|(V)$.

Rmk: (i) j mono $\Rightarrow |j|$ mono.

(ii) This defines a topology on $|X|$ (exercise!)

(iii) $|-|: \text{Spaces} \rightarrow \text{Top}$ is a functor (i.e. $|f|$ is continuous)

(iv) $|-|: \text{Spaces} \rightarrow \text{Top}$ preserves colimits, in particular, if $U \rightarrow X$ epimorphism
then $|U| \rightarrow |X|$ topological quotient.

Thm 1 (Prop 6.3.4) Assume X quasi-separated algebraic space.

Then every point $\text{Spec } k \xrightarrow{x} X$ has a unique minimal representative

$\text{Spec } k(x) \xrightarrow{x_0} X$ with x_0 a monomorphism.

Rmk: An algebraic space is decent if every point $\text{Spec } k \xrightarrow{x} X$ is quasi-compact.

Thm 1 holds for decent spaces. Schemes and qs alg spaces are decent.

Ex: $X = \text{Spec } \overline{\mathbb{Q}} / \text{Gal}(\overline{\mathbb{Q}})_{\text{disc}}$ is an alg space with $|X| = \{*\}$.

Does not admit a minimal representative.

Variant: $\text{Spec } k(x) / \mathbb{Z}$ w/ n.f. $f(x) = f(x) + n$, $\text{char } k = 0$.

Ex: $X = \mathbb{A}^1_{\mathbb{C}} / \mathbb{C}$. Action $(n, x) \mapsto x+n$. $|X| = \{x_0, x_1\}$

$\pi: \mathbb{A}^1_{\mathbb{C}} \rightarrow X$, $\pi^{-1}(x_0) = \text{closed points of } \mathbb{A}^1_{\mathbb{C}}$
 $\pi^{-1}(x_1) = \text{generic point of } \mathbb{A}^1_{\mathbb{C}}$

$\text{Spec } \mathbb{C} \xrightarrow{0} \mathbb{A}^1_{\mathbb{C}} \xrightarrow{\pi} X$ is a minimal repr for x_0
 but is not quasi-compact
 (not decent)

$$\begin{array}{ccc} \mathbb{C} & \text{Spec } \mathbb{C} & \hookrightarrow \mathbb{A}^1_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{C} & \text{Spec } \mathbb{C} & \xrightarrow{x_0} X \end{array}$$

x_1 does not admit a monomorphic representative.

Thm 2 (Thm 6.4.1) Assume X quasi-separated algebraic space.

Then \exists dense open $U \subseteq X$ s.t. U is a scheme.

Rmk: Thm 2 also holds for decent spaces.

Rmk: Not true in general: exists one-point alg spaces which are not schemes.

Stronger version for qcqs spaces:

Thm 3 (Raynaud-Crisson) Assume X quasi-compact and quasi-separated algebraic space. Then \exists finite filtration $X = U_1 \supseteq U_2 \supseteq \dots \supseteq U_n = \emptyset$ where U_k open and \exists étale presentation $\pi_k: U'_k \rightarrow U_k$ s.t.

π_k isomorphism over $U_k \setminus U_{k+1}$. In particular $U_k \setminus U_{k+1}$ is a scheme.

Rmk: $\coprod_k U'_k \rightarrow X$ is a Nisnevich presentation of X .

Rmk: Thm 3 \Rightarrow Thm 1 + Thm 2.

Rmk: Natural construction gives U'_k quasi-affine but can arrange so that U'_k affine.

Method: Quotients of finite étale equivalence relations of quasi-affine schemes are quasi-affine schemes.

More generally: ^{quotients of} finite flat groupoids (needed later for coarse mod. spaces)

Quotients of finite flat groupoids

Thm 4 (FGA Exp 212, SGAB Exp V, [R13, Thm 4.1])

Let $X_1 \xrightleftharpoons[t]{s} X_0$ be a finite flat fin pres groupoid of affine schemes.
 $\text{Spec } A_1 \quad \text{Spec } A_0$ (so A_1 is loc. free of finite rank as an A_0 -module)

Let $A = \text{eq}(A_0 \rightrightarrows A_1)$. Then $X_0 \xrightarrow{\pi} X = \text{Spec } A$ is a geometric and categorical quotient.

This means:

geom quot $\left\{ \begin{array}{l} \text{(0) } \pi s = \pi t \\ \text{(i) } |X_1| \rightrightarrows |X_0| \rightarrow |X| \text{ is a coequalizer.} \\ \text{(ii) } \mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0} \text{ is the subsheaf of invariants, i.e.,} \\ 0 \rightarrow \mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0} \xrightarrow{s^* - t^*} \pi_* s_* \mathcal{O}_{X_1} \end{array} \right.$
 is exact.

cat quot $\left\{ \begin{array}{l} \text{(iii) } \forall X_0 \xrightarrow{p} T, \text{ w/ } T \text{ an algebraic space s.th. } ps = pt \\ \text{there } \exists! X \xrightarrow{f} T \text{ s.th. } p = f\pi, \text{ i.e., } \pi \text{ is the colimit} \\ \text{of } X_1 \rightrightarrows X_0 \text{ in the category of algebraic spaces.} \end{array} \right.$

Moreover, π is integral (i.e. affine and $\mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_{X_0}$ integral extension)

Cor (FGA, SGAB, [R13, Thm 4.4], Olsson 6.2.2)

Let $X_1 \xrightleftharpoons[t]{s} X_0$ finite flat groupoid of schemes s.th.

(*) $\forall x \in X_0$, the orbit $\mathcal{O}(x) := s(t^{-1}(x))$ is contained in an affine open subset of X_0 .

Then \exists geom and cat quot $\pi: X_0 \rightarrow X$ w/ X a scheme, and π is integral.

Remk: • (*) is necessary (easy).

• W/o (*), but assuming X_0 separated, \exists geom+cat quot X w/ X an alg. space.

Remk: In general, $\pi: X_0 \rightarrow X_0$ is not the quotient in the category of spaces ($\text{Sh}(\text{Sch}_{\text{finit}}$).

But if $X_1 \rightrightarrows X_0$ finite étale (or flat) equivalence relation, then sheaf quotient \tilde{X} is an algebraic space by definition (resp. by a theorem of Artin), thus $X = \tilde{X}$.

Consequence: If in Thm/Cor $X_1 \rightrightarrows X_0$ is an equiv. relation (i.e. $X_1 \rightarrow X_0 \times X_0$ mono) then $\pi: X_0 \rightarrow X$ finite flat and $X_1 = X_0 \times_X X_0$.

Remk. (*) holds when any finite set of points is contained in an affine subset. e.g. when X_0 is quasi-affine or more generally admits an ample line bundle (i.e. X_0 quasi-proj.).

Consequence: If X is an algebraic space w/ a finite étale (or flat) presentation $X_0 \rightrightarrows X$ w/ X_0 a quasi-affine scheme, then X (quasi-affine) scheme.

Remk: Olsson only proves that quot in Thm 4/Cor is categorical in category of schemes. He therefore has to prove the two above consequences.

Remk: As equalizers commute w/ flat base change, the quotients are uniform (i.e. commute w/ flat b.c.). In char 0 (or some char p) they are universal (i.e. commute w/ any b.c.) but does not hold in general.

Proof of Thm 2:

Let $X^{\text{sch}} \subset X$ be the schematic locus: $x \in X^{\text{sch}} \Leftrightarrow \exists$ open nbhd $x \in U \subset X$ s.t. U a scheme.

Need to prove that X^{sch} dense.

WLOG X quasi-compact.

Pick $X' \xrightarrow{\pi} X$ étale presentation w/ X' affine. Note that π is étale, \downarrow quasi-compact and separated. b/c X q-compact.

For any $n \in \mathbb{N}$, let $X^{(n)} \subset X$ be the locus of points $x \in X$ s.t. π is finite of rank n in a nbhd of x . This is open (possibly \emptyset) b/c local question on X .

Also qcpt b/c π qcpt.

$$\begin{array}{ccc} X'_x X^{(n)} \subset X' & \pi^{(n)} \text{ is finite étale} \Rightarrow X^{(n)} \text{ is a } \underline{\text{scheme}}, & \\ \pi^{(n)} \downarrow \square \downarrow \pi & & \\ X^{(n)} \subset X & \Rightarrow \bigcup_n X^{(n)} \subseteq X^{\text{sch}} & \end{array}$$

Easily seen that $\bigcup_n X^{(n)}$ dense.