Algebraick Stacks

2017-02-24

Introduction

Schemes

- locally ringed spaces
- functors
- sheaves

Algebraic spaces

Stacks

- definition
- examples

Groupoids

- definition
- examples

Atlases

Schemes

<u>Atlas</u>: $X = \bigcup U_i$, $U_i = \operatorname{Spec} A_i \subset X$ Zariski-open

We could describe category of schemes purely in terms of atlases but this is awkward: different atlases give same scheme etc.

Two better approaches:

① Locally ringed spaces

A scheme <u>is</u> a loc. rng. space that <u>admits</u> an atlas

$$(\mathrm{Sch}) \underset{\substack{\text{fully}\\ \text{faithful}}}{\subset} (\mathrm{LocRS})$$

$\rightsquigarrow$ sheaves, stalks, cohomology, topological space, ...

② Functor of points

A scheme <u>is</u> a <u>functor</u> that admits an atlas.

$$(\mathrm{Sch}) \underset{\text{f.f.}}{\subset} (\mathrm{Functors}) \;/\!/\; \text{fun. } (\mathrm{Aff})^{op} \longrightarrow (\mathrm{Set}), \quad (\mathrm{Aff}) = (\mathrm{Rings})$$

$$X \longmapsto h_X$$

or fun.: $(\mathrm{Sch})^{op} \longrightarrow (\mathrm{Set}) \leftarrow (\text{Yoneda})$

$(\mathrm{Sch}/_k)^{op} \longrightarrow (\mathrm{Set})$

$\vdots$

$$h_X = \mathrm{Hom}(-, X)$$

<u>Base category</u>: Aff, Sch, $\mathrm{Sch}/_k$, ...

(more generally: Diff, Top, AnSp, AlgSp, ...)

Ex

Ex: $F : (\mathrm{Aff})^{op} = (\mathrm{Ring}) \longrightarrow (\mathrm{Set})$

(i) $F(A) = A^\times$ set of units (a comm. group) $\quad F = h_{\mathbb{G}_m} \quad \mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$

(ii) $F(A) = A$ set (a comm ring) $\quad F = h_{\mathbb{G}_a} \quad \mathbb{G}_a = \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$

(iii) $F(A) = \mathrm{Idem}(A)$ set $\quad F = h_{\operatorname{Spec}\mathbb{Z} \amalg \operatorname{Spec}\mathbb{Z}} = h_{\operatorname{Spec}(\mathbb{Z}\times\mathbb{Z})}$

$$\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}[e]/e^2 - e$$

Topological space of a functor

$$|F| = \{F(k) \ni \eta : \text{for some field } k\}/\sim \quad + \text{ topology (see later)}$$

Good setting for <u>moduli problems</u>:

$$F : (\mathrm{Sch})^{op} \longrightarrow (\mathrm{Set})$$
$$T \longmapsto \text{objects parametrized by } T$$

<u>Ex</u>: $\mathrm{Hilb}_{\mathbb{P}^n}(T) = \left\{ \begin{tikzcd} Z \arrow[r, hook, "\text{closed}"] \arrow[dr, "p"'] & \mathbb{P}^n \times T \arrow[d] \\ & T \end{tikzcd} : p \text{ proper and flat (and fin. pres)} \right\}$

$$M_g(T) = \left\{ \begin{tikzcd} C \arrow[d, "p"] \\ T \end{tikzcd} : p \text{ proper, smooth, fibers conn curves of genus } g \right\} \underbrace{/\sim}$$

<u>bad</u>, this is why we need <u>stacks</u>.

Functors are <u>presheaves</u> on the base category.

Equip base category w/ Grothendieck topology (Ét, fppf, fpqc)

$\rightsquigarrow$ <u>sheaves</u>.

$$(\mathrm{Sch}) \subset \underbrace{(\mathrm{Shv}_{fpqc}) \subset (\mathrm{Shv}_{fppf}) \subset \overset{\text{"Spaces"}}{\overset{\shortparallel}{(\mathrm{Shv}_{Ét})}}} \subset (\mathrm{PreShv}) = (\mathrm{Functors})$$

don't depend so much on precise choice of base category

Aff, Sch, AlgSp give same cat. of sheaves.

Interlude: Algebraic spaces

① $(\mathrm{AlgSp}) \underset{\text{f.f.}}{\subset} (\mathrm{LocRingTopoi})$ = "Spaces"

② $(\mathrm{AlgSp}) \underset{\text{f.f.}}{\subset} (\mathrm{Shv}_{ét})$ = "Spaces" (algebraic spaces are also fpqc-sheaves)

LHS's are those spaces admitting an <u>étale</u> atlas.
($\underset{\text{Artin}}{\Longleftrightarrow}$ —— " —— ~~flat~~ flat atlas)

Here atlas = <u>equivalence relation</u> (more on this later)

<u>Thm</u> (Deligne): G finite group $\circlearrowright X$ sep. scheme. Then X/G $\exists$ as an alg. space.

(X q-proj $\Rightarrow$ X/G q-proj)

<u>Rmk</u>: Unless G acts <u>freely</u>, X/G has poor properties.

$$\begin{array}{ccccc}
(\mathrm{AlgSp}) & \longrightarrow & (\mathrm{Shv}_{ét}) & \longrightarrow & (\mathrm{PreShv}) \\
X & \longmapsto & h_X & \longmapsto & h_X \\
 & & & & \\
X/G & & (h_X/G)_{ét} & & (h_X/G)_{\substack{pre\\shf}} \\
 & & \| & & \\
 & & (h_X/G)^{\sim}_{\substack{pre\\shf}} & &
\end{array}$$

Usually $h_{X/G} \overset{\not\cong}{\longleftarrow} (h_X/G)_{ét}$ unless G acts freely.

If G acts freely: $X \longrightarrow X/G$ G-torsor and in particular finite étale.

<u>Ex</u>: There exists <u>free</u> actions $G \circlearrowright X$ scheme s.th. X/G is an alg. space.

(a) $\mathbb{Z}/2 \circlearrowright$ Hironaka's proper smooth 3-fold

(b) $\mathbb{Z}/2 \circlearrowright$ 

quotient = "  "

(c) $\mathbb{Z}/2 \circlearrowright$  $= \mathbb{A}^1_{\mathbb{C}} \underset{\mathbb{A}^1 \setminus 0}{\amalg} \mathbb{A}^1_{\mathbb{C}}$

conj. action + swap

quotient = $= \mathbb{A}^1_{\mathbb{R}}$ except $\kappa(0) = \mathbb{C}$

$\mathbb{C}$-point

Groupoids

Def: A groupoid is a category such that every morphism has an inverse.

Ex: Fundamental groupoid of a topological space X:

$$\mathrm{ob} = X$$
$$\mathrm{mor}(x,y) = \{ [\gamma] \text{ homotopy class of paths } x \xrightarrow{\gamma} y \} \qquad (\gamma^{-1} = \gamma \text{ backwards})$$

Equivalently: a groupoid is two sets R, U and maps

R = arrows
U = objects

$$s,t : R \to U \quad \text{(source, target)}$$
$$e : U \to R \quad \text{(identity)}$$
$$i : R \to R \quad \text{(inverse)}$$
$$m : R \times_{s,U,t} R \to R \quad \text{(composition/multiplication)}$$

satisfying (long but natural) list of axioms. **Notation:** $R \rightrightarrows U$

Ex:
$$U = X \qquad R = \{ [\gamma] : \gamma : x \to y \}$$

$$s(\gamma : x \to y) = x$$
$$t(\gamma : x \to y) = y$$
$$i(\gamma) = \gamma^{-1}$$
$$m(\gamma, \delta) = \gamma \circ \delta = \xrightarrow{\delta}\xrightarrow{\gamma}$$

Groupoid in schemes: $R \overset{s}{\underset{t}{\rightrightarrows}} U$ where R, U schemes, s, t, e, i, m morphisms of schemes

$(\Leftrightarrow h_R(T) \rightrightarrows h_U(T)$ groupoid in sets functorially in $T)$

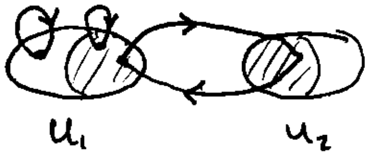
Examples of groupoids

(i) $X \rightrightarrows X$ s, t, e, i, m identities

(ii) $X = \bigcup U_i$ open covering $\coprod_{i,j} U_{ij} \rightrightarrows \coprod_i U_i$

$s : U_{ij} \hookrightarrow U_i$
$t : U_{ij} \hookrightarrow U_j$
$e : U_{ii} \xrightarrow{\sim} U_{ii}$
$i : U_{ij} \xrightarrow{\sim} U_{ji}$
$m : U_{ijk} \hookrightarrow U_{ik}$

Picture: $X =$ 

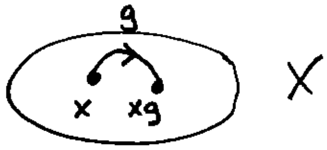


(iii) Any map of schemes $U \to X$ gives groupoid $U \times_X U \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} U$

(iv) Group scheme G acting on X gives action groupoid $X \times G \rightrightarrows X$

$s(x,g) = x$
$t(x,g) = xg$
$e(x) = (x,e)$
$i(x,g) = (xg, g^{-1})$
$m((x,g),(xg,h)) = (x, gh)$

$\left(\begin{array}{l} x : T \to X \\ g : T \to G \\ \text{generalized points} \end{array} \right)$



(iv') $G \circlearrowright X$ as above, $U \subset X$ subscheme $R = (X \times G)|_U \rightrightarrows U$

$$R = \{ (x,g) : x \in U,\ xg \in U \}$$

$R \rightrightarrows U$ is not coming from a group action $G \circlearrowright U$.

$\mathrm{Isom}(x,y) = s^{-1}(x) \cap t^{-1}(y)$

$\mathrm{Isom}(x,x)$ group scheme

$$\begin{tikzcd} \mathrm{Isom}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[d, "(s,t)"] \\ T \arrow[r, "(x,y)"'] & U \times U \end{tikzcd}$$

Def: $R \rightrightarrows U$ is an equiv relation if $\mathrm{Isom}(x,x)$ trivial $\forall x$.

(i), (ii), and free actions in (iv) are equivalence relations.

Stacks

Keep track of ~~auto~~ $\overset{\text{iso}}{\text{morphisms}}$ $\Rightarrow$ replace (Set) by (Grpd).

$$\begin{array}{ccccccc} & & (\mathrm{Shv}^{\mathrm{Grpd}}_{\text{ét}}) & & & & \\ & & \parallel & & & & \\ (\mathrm{AlgStacks}) & \subset & (\mathrm{Stacks}) & \subset & \mathrm{CFG} & = & \text{2-functors } (\mathrm{Aff})^{op} \to (\mathrm{Grpd}) \\ \cup & & \cup & & \cup & & \\ (\mathrm{AlgSp}) & \subset & (\mathrm{Spaces}) & \subset & \mathrm{CFS} & = & \text{Functors } (\mathrm{Aff})^{op} \to (\mathrm{Set}) \\ & & \parallel & & & & \\ & & (\mathrm{Shv}^{\mathrm{Set}}_{\text{ét}}) & & & & \end{array}$$

2-cat's

1-cat's

Algebraic $\Leftrightarrow$ have étale atlas. ($\underset{\text{Artin}}{\Leftrightarrow}$ have flat atlas) ($\Leftrightarrow$ satisfies Artin's axioms)

Stack/space $\Leftrightarrow$ sheaf condition

CFS/CFG = category fibered in sets/groupoids

$F : (\mathrm{Aff})^{op} \longrightarrow (\mathrm{Cat}) \quad \Longleftrightarrow \quad \mathcal{X} \xrightarrow{p} (\mathrm{Aff})^{*}$ fibered category

$F(T) \cong \mathcal{X}_T := p^{-1}(T)$

Examples of stacks

(i) $\mathcal{M}_g(T) =$ obj: $\left\{ \begin{array}{c} C \\ \downarrow p \\ T \end{array} \text{ proper, smooth, fibers curves of genus } g \right\}$

mor: $$\begin{tikzcd} C_1 \arrow[rr, "\sim"] \arrow[dr, "p_1"'] & & C_2 \arrow[dl, "p_2"] \\ & T & \end{tikzcd}$$

$\pi_0(\mathcal{M}_g(T)) = M_g(T)$

Fibered category: $\begin{array}{c} \mathcal{M}_g \\ \downarrow p \\ (Sch) \end{array}$ obj $= \{ (C,T,p) : p: C \to T \text{ as above} \}$

$p(C,T,p) = T$

mor $=$ $$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & C_2 \arrow[d, "p_2"] \\ T_1 \arrow[r, "g"'] & T_2 \end{tikzcd}$$

$p(f,g) = g$

NB! g does not determine C_1, f

e.g. $g = id_T$, $p_1 = p_2$

$$\begin{tikzcd} C \arrow[rr, "\sim", "f"'] \arrow[dr, "p"'] & & C \arrow[dl, "p"] \\ & T & \end{tikzcd}$$

could be several f.

Thm (Deligne-Mumford) $\mathcal{M}_g$ is a smooth connected algebraic stack

Variants: $\overline{\mathcal{M}}_g$, $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$, ...

(ii) G group (scheme)

$BG(T) =$ obj: $\begin{array}{c} E \\ \downarrow p \\ T \end{array}$ G-torsor, i.e., E has a free and transitive G-action

mor: $$\begin{tikzcd} E_1 \arrow[rr, "\sim", "f"'] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & T & \end{tikzcd}$$ f G-equivariant iso

Trivial torsor $\begin{array}{c} G \times T \\ \downarrow \\ T \end{array}$. Locally, every torsor is trivial.

$\Rightarrow$ locally, two elements $E_1, E_2 \in BG(T)$ are isom.

Ex: $G = \mathbb{Z}/2\mathbb{Z}$ $\begin{array}{c} \operatorname{Spec} \mathbb{C} \\ \downarrow \\ \operatorname{Spec} \mathbb{R} \end{array}$, $\begin{array}{c} \operatorname{Spec} \mathbb{R} \amalg \operatorname{Spec} \mathbb{R} \\ \downarrow \\ \operatorname{Spec} \mathbb{R} \end{array}$ $\in BG(\operatorname{Spec} \mathbb{R})$

becomes isomorphic over $\operatorname{Spec} \mathbb{C}$. $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ étale surj. i.e. covering.

$$\begin{array}{ccc} \mathbb{A}^1\setminus 0 & & z \\ \downarrow & & \downarrow \\ \mathbb{A}^1\setminus 0 & & z^2 \end{array} \qquad \mathbb{Z}/2\mathbb{Z}\text{-torsor}$$

<u>Picture:</u>

$= S^1$

$\downarrow$

$= S^1$

Étale-locally trivial

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

2-fiber product

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[loop above, "id"] \arrow[loop right, "\tau"] \end{tikzcd}$$

in 2-category of groupoids

An element $\begin{array}{c} E \\ \downarrow p \\ T \end{array} \in BG(T)$ gives a map $T \longrightarrow BG$ and

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ T \arrow[r, "{[p]}"'] & BG \end{tikzcd}$$

$\leftarrow$ corr to trivial torsor = universal family!

$$E = \mathrm{Isom}_{G\text{-tors}/T}(E, G\times T)$$

(iii) <u>Conics</u>

Up to isomorphism, there is over $\mathbb{C}$ 3 conics:

N = nonsingular $\cong \mathbb{P}^1$ $\qquad x^2+y^2+z^2=0$

L = two secant lines = $\times$ $\qquad xy = 0$

D = one double line = $//$ $\qquad x^2 = 0$

moduli of conics "=" $\{N, L, D\}$

stack: $\mathcal{M}(T) = \left\{ \begin{array}{c} C \\ \downarrow p \\ T \end{array} : p \text{ proper, flat, geom fibers are iso to conics} \right\}$

$$\cong \left\{ \begin{tikzcd} C \arrow[r, hook] \arrow[dr, "p"'] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & T \end{tikzcd} : p \text{ flat, } \mathcal{E} \text{ rk } 3 \text{ v.b., fibers are conics} \right\}$$

b/c canonical polarization $\omega^\vee_{C/T}$: $\mathcal{E} = p_* \omega^\vee_{C/T}$

Rigidify: $\left\{ \begin{tikzcd} C \arrow[r, hook] \arrow[dr, "p"'] & \mathbb{P}^2 \arrow[d] \\ & T \end{tikzcd} : p \text{ as above} \right\} \cong \mathbb{P}^5$

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \mapsto (a:b:c:d:e:f)$$

$\mathcal{M} = \left[\mathbb{P}^5 / PGL_3 \right]$

$\mathbb{P}^5$ has 3 PGL_3-orbits
corr to N (open, dim 5)
L (loc closed, dim 4)
D (closed, dim 2)

$\mathcal{M} = \{ B\mathrm{Aut}\, N, B\mathrm{Aut}\, L, B\mathrm{Aut}\, D \}$

$\dim \mathcal{M} = 5 - 8 = -3$ $\quad$ $\mathrm{stab}(N) = PGL_2$ has dim 3

$BPGL_2$ classifies families of smooth rational curves = smooth conics bundles

Atlases of stacks

Groupoid $R \rightrightarrows U$ $\rightsquigarrow$ stack quotient $[R \rightrightarrows U] = \left(T \longmapsto \left(R(T) \rightrightarrows U(T)\right)\right)_{\text{stackify}}$

Comes along with map $U = [U \rightrightarrows U] \xrightarrow{\pi} [R \rightrightarrows U] =: \mathcal{X}$

Recover $R \underset{\varphi}{\xrightarrow{\cong}} U \times_{\mathcal{X}} U$ (2-fiber product!) s.th. $s = \pi_1 \circ \varphi$, $t = \pi_2 \circ \varphi$, $\varphi \circ e = \Delta$ etc.

(generalization of example (iii))

If s (or equiv t) is étale/smooth/flat then so is $U \xrightarrow{\pi} \mathcal{X}$.

Rmk: If G smooth group acting ~~freely~~ on X, then $X \xrightarrow{\pi} X/G$ smooth and

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\pi} & X/G \end{array}$$

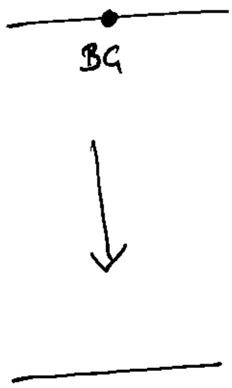
is cartesian. With $[X/G]$ instead of X/G this holds for non-free actions.

Rmk: If $R \rightrightarrows U$ equiv relation, then $[R \rightrightarrows U] \cong U/R = \left(T \to U(T)/R(T)\right)^{\sim}$ a space

eg if $G \circlearrowright X$ freely, then $[X/G] \cong X/G$.

Ex: $G = \mathbb{Z}/2 \circlearrowright \mathbb{A}^1$, $x \longmapsto -x$

$$\begin{array}{lll} & [\mathbb{A}^1/G] & \\ & \downarrow & \\ \pi_0[\mathbb{A}^1/G]^\sim = & (\mathbb{A}^1/G)_{Et} & \text{not algebraic} \\ & \downarrow & \\ & \mathbb{A}^1/G & \text{coarse quotient} \\ & \parallel & \\ & \operatorname{Spec}(k[x^2]) & \end{array}$$



Ex: $G \circlearrowright X$. Presheaf quotient: $T \longmapsto X(T)/G(T)$ $\quad (Sch)^{op} \longmapsto (Set)$

"Prestack" quotient: $T \longmapsto \left(X(T) \times G(T) \rightrightarrows X(T) \right)$

$\parallel$

CFG $\quad (Sch)^{op} \longmapsto (Grpd)$

Sheaf/stack quotient = presheaf/stack quotient + sheafify/stackify.

Ex: $G = \mathbb{Z}/2 = \langle id, \tau \rangle$ $\quad$ presheaf quot: $T \longmapsto \{*\}$ $\quad$ sheaf $= h_*$

$X = *$ $\quad$ prestack quot: $T \longmapsto \overset{id}{\circlearrowleft} * \circlearrowright^{\tau}$

stack quot: $T \longmapsto \left\{ \begin{array}{c} E \\ \downarrow \\ T \end{array} \ \mathbb{Z}/2\mathbb{Z}\text{-torsor} \right\}$