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CHOW VARIETIES

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Abstract. An important concept in Algebraic Geometry is *cycles*. The cycles of a variety X are formal sums of irreducible varieties in X. If all the varieties of the cycle have the same dimension r, it is an r-cycle. The degree of a cycle $\sum_i n_i[V_i]$ is $\sum_i n_i d_i$ where d_i is the degree of V_i. The cycles of a fixed dimension r and degree d of a projective variety X over a *perfect* field k, are parameterized by a projective variety $\mathrm{Chow}_{r,d}(X)$, the *Chow variety*.

We begin with an introduction to Algebraic Geometry and construct the Chow variety explicitly, giving defining equations. Some easy cases, such as 0-cycles, which are parameterized by $\mathrm{Chow}_{0,d}(X) = X^d/\mathfrak{S}_d = \mathrm{Sym}^d X$ when the base field has characteristic zero, are investigated. Finally, an overview on topics such as the independence of the embedding of $\mathrm{Chow}_{r,d}(X)$ and the existence of a Chow functor and a Chow scheme is given.

Contents

Introduction

The classical Chow theory dates back to the early decades of the twentieth century. In the modern view with Grothendieck's schemes the natural question is when the Chow variety can be extended into a Chow scheme, e.g. if the cycles of an algebraic scheme has a structure as a scheme. The classical construction is also problematic since the constructed variety $\mathrm{Chow}_r(X)$ a priori depends on the embedding of X into a projective space. Further it is not clear which functor it is that the Chow scheme should represent.

In [A], Angéniol shows that the cycles of codimension p of a scheme X are parameterized by an algebraic space $\mathscr{C}^p(X)$. This is under the conditions that X is a separated scheme of pure dimension n over an affine base scheme of characteristic zero, and that X is a closed subscheme of a smooth scheme. When X is a *variety* over $\mathbb{C}$, Angéniol also proves that $\mathscr{C}^p(X)$ is a scheme and that the reduced scheme is isomorphic with the variety given by the classical construction.

In the first chapter we define some basic concepts in Algebraic Geometry. In the second we introduce a more theoretical view using sheaves, which is not extensively used in the rest of the exposition but useful. In the third and fourth chapters, morphisms, projections and products are defined. The important notion of geometrically integral varieties is introduced in chapter five followed by some geometric properties such as the degree in chapter six. In the seventh chapter we define and investigate some properties of cycles, which in the eight chapter are shown to be parameterized by the Chow Variety. In the last chapter, we discuss some results on the Chow Functor and Chow Scheme.

The approach to Algebraic Geometry in this thesis is mostly classical using the language of A. Weil. The setting is as general as possible, allowing arbitrary fields and not only algebraically closed fields. Many authors define varieties to be irreducible sets, or even geometrically integral (absolutely irreducible). In this work however, varieties are not irreducible unless explicitly stated.

The reader is assumed to be familiar with algebraic notions such as localization, integral dependence and noetherian rings as well as elementary results on field extensions (such as in [AM] and [Mo]).

The notation closely follows Atiyah and MacDonald [AM]. When we write $A \subset B$ for sets, the set A is properly contained in B. Rings are always commutative rings with identity. Note that the zero ring in which $0 = 1$ is not excluded. We use the notation $\mathfrak{r}(\mathfrak{a})$ for the radical of the ideal $\mathfrak{a}$, which some authors denote $\sqrt{\mathfrak{a}}$. If $x_1, x_2, \ldots, x_n$ is a series of variables, it is abbreviated as x and we write $k[x]$ instead of $k[x_1, x_2, \ldots, x_n]$.

For those familiar with schemes, a k-variety is a *reduced* algebraic k-scheme, i.e. a reduced noetherian separated scheme over the base scheme $\mathrm{Spec}(k)$. We will also only consider affine varieties and projective varieties and not general varieties. Further all varieties are given with a closed embedding into $\mathbb{A}^n$ or $\mathbb{P}^n$. The product $X \times Y$ and base extension $X_{(k')}$ of varieties in the category of schemes is $(X \times_{\mathrm{Spec}(k)} Y)_{\mathrm{red}}$ and $\big(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')\big)_{\mathrm{red}}$ respectively.

Chapter 1

Classical Varieties

VARIETIES

We will consider polynomials in the polynomial ring $k[x_1, x_2, \ldots, x_n] = k[x]$ in the variables $x_1, x_2, \ldots, x_n$ over a field k, and their zeroes in the affine space K^n over an algebraically closed field extension K/k. We will often denote K^n by $\mathbb{A}^n(K)$ or $\mathbb{A}^n$.

Note that we will not require that K is *universal*, i.e. has an infinite transcendence degree over k and that every field is contained in K, as Samuel and Weil do [S, W]. The choice of K is not important, it is only an auxiliary field and the properties for varieties are independent of K, and we could choose $K = \bar{k}$. Sometimes, though, we need elements of K which are transcendent over k. If L/k is a field we can construct a new field K' which contains k-isomorphic copies of K and L. This is done taking the quotient of the tensor product $K \otimes_k L$ with any maximal ideal [Bourbaki, Algèbre, chap. V, §4, prop. 2.] and then its algebraic closure.

Definition 1.1 To each set of polynomials $\mathfrak{F} \subseteq k[x]$ we let $V_K(\mathfrak{F}) \subseteq \mathbb{A}^n(K)$ be the common zero locus of those polynomials, i.e. $V_K(\mathfrak{F}) = \{ P \in K^n \ : \ f(P) = 0 \ \ \forall f \in \mathfrak{F} \}$. A set $E \subseteq \mathbb{A}^n(K)$ is called *a k-variety* if $E = V_K(\mathfrak{F})$ for some set of polynomials $\mathfrak{F}$. Some authors denote the common zero locus $V_K(\mathfrak{F})$ with $Z(\mathfrak{F})$.

Remark 1.2 If $\mathfrak{F}$ is a set of polynomials, the common zero locus of $\mathfrak{F}$ is equal to the common zero locus of the ideal generated by $\mathfrak{F}$. We will therefore only use ideals and not sets of polynomials.

Definition 1.3 To every set $E \subseteq \mathbb{A}^n(K)$ we associate an ideal $\mathfrak{I}_k(E)$ consisting of all polynomials in $k[x]$ which vanish on E, i.e. $\mathfrak{I}_k(E) = \{ f \in k[x] \ : \ f(P) = 0 \ \ \forall P \in E \}$. It is clear that this is an ideal.

Based on these definitions we get a number of relations:

$$E \subseteq F \implies \mathfrak{I}_k(E) \supseteq \mathfrak{I}_k(F) \tag{1.1}$$
$$\mathfrak{a} \subseteq \mathfrak{b} \implies V_K(\mathfrak{a}) \supseteq V_K(\mathfrak{b}) \tag{1.2}$$

$$\mathfrak{a} \subseteq \mathfrak{I}_k\big(V_K(\mathfrak{a})\big) \tag{1.3}$$
$$E \subseteq V_K\big(\mathfrak{I}_k(E)\big) \tag{1.4}$$

$$V_K\left(\sum_{\alpha\in\mathscr{I}}\mathfrak{a}_\alpha\right) = \bigcap_{\alpha\in\mathscr{I}} V_K(\mathfrak{a}_\alpha) \tag{1.5}$$

$$V_K(\mathfrak{a}_1\mathfrak{a}_2\dots\mathfrak{a}_n) = \bigcup_{i=1}^{n} V_K(\mathfrak{a}_i) \tag{1.6}$$

$$\mathfrak{I}_k\left(\bigcup_{\alpha\in\mathscr{I}} E_\alpha\right) = \bigcap_{\alpha\in\mathscr{I}} \mathfrak{I}_k(E_\alpha) \tag{1.7}$$

Definition 1.4 We say that $\mathfrak{F}$ is a system of equations for a k-variety V if $\mathfrak{F}$ generate $\mathfrak{I}_k(V)$. By Hilbert's Basis theorem $k[x]$ is a noetherian ring and hence $\mathfrak{I}_k(V)$ is finitely generated, so there is always a finite system of equations.

Theorem 1.5 (Hilbert's Nullstellensatz) *Let* $\mathfrak{a}$ *be an ideal of* $k[x]$*. Then* $\mathfrak{I}_k(V_K(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

Proof. For a proof see e.g. Atiyah and MacDonald [AM, p. 85] or Mumford [Mu, Ch. I, Thm 2.1]. □

Corollary 1.6 *For any family of varieties* V_α *of* $\mathbb{A}^n$ *and a finite set of ideals* $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$*, we have:*

$$\mathfrak{I}_k\Big(\bigcap_{\alpha\in\mathscr{I}} V_\alpha\Big) = \mathfrak{r}\left(\sum_{\alpha\in\mathscr{I}}\mathfrak{I}_k(V_\alpha)\right) \tag{1.8}$$

$$V_K(\mathfrak{a}_1\cap\mathfrak{a}_2\cap\dots\cap\mathfrak{a}_n) = \bigcup_{i=1}^{n} V_K(\mathfrak{a}_i). \tag{1.9}$$

Proof. Follows from equations (1.5) and (1.6) and theorem 1.5. Note that $\mathfrak{r}(\mathfrak{a}\cap\mathfrak{b}) = \mathfrak{r}(\mathfrak{a}\mathfrak{b})$. □

Remark 1.7 Theorem 1.5 gives us a bijective correspondence between the k-varieties of $\mathbb{A}^n$ and the radical ideals in $k[x]$. Thus the k-varieties can be seen as *independent of the choice* of K, even though they are subsets of K^n.

We will now construct a topology based on the k-varieties.

Proposition 1.8 *The k-varieties as closed sets define a noetherian topology on* $\mathbb{A}^n$*, the k-*Zariski topology*.*

Proof. That this is a topology is easily verified: We have that $\emptyset = V_K\big((1)\big)$ and $\mathbb{A}^n = V_K\big((0)\big)$. Furthermore equations (1.5) and (1.6) assure us that arbitrary intersections and finite unions of closed subsets are closed.

It is also a noetherian topological space. In fact every descending chain of closed subsets corresponds to an ascending chain of ideals and these are stationary since $k[x_1, x_2, \dots, x_n]$ is a noetherian ring. □

Remark 1.9 The Zariski topology on $\mathbb{A}^n$ is a very unusual topology. The most striking property is that the open sets are very big. In fact, all non-empty open sets are *dense*, i.e. their closure is the whole space.

IRREDUCIBLE SETS

Definition 1.10 A topological space X is *irreducible* if it is non-empty and not a union of proper closed subsets. A subset Y of X is irreducible if the induced topological space Y is irreducible.

Remark 1.11 A topological space X is irreducible if and only if it is non-empty and every pair of non-empty open subsets of X intersect. Equivalently, all non-empty open subsets are dense.

Definition 1.12 The maximal irreducible subsets of X are called the *irreducible components* of X.

Proposition 1.13 *Let X be a topological set. Every irreducible subset Y of X is contained in an irreducible component of X and X is covered by its components, which are closed.*

Proof. Let Y be an irreducible subset of X. Consider all chains of irreducible subsets of X containing Y. For an ascending chain $\{Z_\alpha\}$, the union $Z = \bigcup_{\alpha \in \mathscr{I}} Z_\alpha$ is irreducible. In fact, let U and V be open subsets of Z. Then there is α and β such that $U \cap Z_\alpha$ and $V \cap Z_\beta$ are non-empty. We can assume that $Z_\alpha \subseteq Z_\beta$ and thus $U \cap Z_\beta$ and $V \cap Z_\beta$ are non-empty open subsets which intersect since Z_β is irreducible. Consequently $U \cap V \neq \emptyset$ and Z is irreducible. By Zorns lemma there is then a maximal irreducible set containing Y. Since $\overline{Z}$ is irreducible if Z is irreducible, the maximal irreducible set containing Y is closed.

Finally, since X is covered by the sets $\overline{\{x\}}$, $x \in X$, which are all contained in maximal irreducible subsets, the maximal irreducible subsets cover X. □

Proposition 1.14 *A topological space X is not covered by fewer than all its irreducible components.*

Proof. Let $X = \bigcup_{i=1}^{n} Y_i$ be a covering of irreducible components. Let Z be an irreducible subset of X not contained in any Y_i. Then $Z = \bigcup_{i=1}^{n} (Z \cap Y_i)$ and at least two of these sets are proper closed subsets of Z which is a contradiction since Z is irreducible. Thus the Y_i are all the irreducible components. □

Corollary 1.15 *A noetherian topological space X has a finite number of irreducible components.*

Proof. Assume that there is an infinite number of components $X_1, X_2, \ldots$. Then $X_1 \subset X_1 \cup X_2 \subset \cdots$ would be a non-stationary ascending chain of closed subsets. In fact the irreducible components of $X_1 \cup X_2 \cup \cdots \cup X_m$ are $X_1, X_2, \ldots, X_m$ and X_{m+1} is an irreducible component of $X_1 \cup X_2 \cup \cdots \cup X_{m+1}$ which is not covered by $X_1, X_2, \ldots, X_m$ by proposition 1.14. □

IRREDUCIBLE VARIETIES

Definition 1.16 *An irreducible k-variety* is an *irreducible* closed set in the k-Zariski topology, i.e. it is a non-empty k-variety and not a union of proper k-subvarieties.

Notation 1.17 Some authors call k-varieties and irreducible k-varieties for algebraic k-sets and k-varieties respectively.

The following propositions reduces many questions on k-varieties to irreducible k-varieties.

Proposition 1.18 *The irreducible k-varieties correspond to prime ideals in* $k[x]$.

Proof. The prime ideals are exactly those radical ideals which cannot be written as an intersection of two strictly bigger radical ideals. In fact, if $\mathfrak{p}$ is an ideal, the existence of $a, b \notin \mathfrak{p}$ such that $ab \in \mathfrak{p}$ is equivalent to the existence of two ideals $\mathfrak{a}, \mathfrak{b} \supset \mathfrak{p}$ such that $\mathfrak{ab} \subseteq \mathfrak{p}$. Further if $\mathfrak{p}$ is radical, it is equivalent to $\mathfrak{r}(\mathfrak{ab}) \subseteq \mathfrak{p} \subseteq \mathfrak{r}(\mathfrak{a} \cap \mathfrak{b})$ and thus equivalent to $\mathfrak{p} = \mathfrak{r}(\mathfrak{a}) \cap \mathfrak{r}(\mathfrak{b})$.

Finally, by equations (1.1) and (1.7), the non-irreducible k-varieties are those corresponding to a radical ideal which is the intersection of two strictly bigger radical ideals. □

Proposition 1.19 *There is a* unique representation *of every k-variety V as a finite union of irreducible k-varieties* $V = \bigcup_{i=1}^{n} V_i$ *which is minimal in the sense that* $V_i \nsubseteq V_j$. *The* V_i*:s are called the* components *of V and are the maximal irreducible k-subvarieties of V.*

Proof. Follows immediately from proposition 1.14 and corollary 1.15. □

Remark 1.20 Using the correspondence of k-varieties and radical ideals in remark 1.7, we reformulate proposition 1.19 algebraically as: There is a unique minimal representation of every radical ideal $\mathfrak{a} \subseteq k[x]$ as a finite intersection of prime ideals $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{p}_i$ such that $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$. This is a special case of the noetherian decomposition theorem.

Remark 1.21 The equivalent statement of proposition 1.13 in $k[x]$ is that if $\mathfrak{p} \supseteq \bigcap_{i=1}^{n} \mathfrak{p}_i$ then $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i.

Example 1.22 The affine space $\mathbb{A}^n$ is irreducible. In fact the minimal ideal $(0) \subset k[x]$ which corresponds to $\mathbb{A}^n$ is a prime ideal.

Example 1.23 The k-linear subspaces of $\mathbb{A}^n$ are irreducible k-varieties defined by a finite number of linear equations in k. They are bijective to $\mathbb{A}^m$ with $m \leq n$.

Example 1.24 The set $V_{\mathbb{C}}(x^2+1) \subset \mathbb{A}^1(\mathbb{C})$ is an irreducible $\mathbb{Q}$-variety because x^2+1 is irreducible in $\mathbb{Q}[x]$. It is not an irreducible $\mathbb{C}$-variety since it splits into two irreducible $\mathbb{C}$-varieties, $V_{\mathbb{C}}(x^2+1) = V_{\mathbb{C}}(x-i) \cup V_{\mathbb{C}}(x+i)$. Note that the space $\mathbb{A}^1(\mathbb{C})$ and the set $V_{\mathbb{C}}(x^2+1)$ are the same in these two cases but with different topologies.

Example 1.25 The line $V_K(x_1) \subset \mathbb{A}^2$ is an irreducible k-variety. It has irreducible k-subvarieties $V_K(x_1, f)$ for any irreducible polynomial $f \in k[x_2]$ but $V_K(x_1)$ is not a finite union of them.

Definition 1.26 Let V be a k-variety in $\mathbb{A}^n$. The *coordinate ring* of V in k is the ring $k[V] = k[x]/\mathfrak{I}_k(V)$. When V is irreducible $k[V]$ is an integral domain and we define the *function field* of V in k to be the quotient field $k(V)$ of the coordinate ring. The elements in the function field are called *rational functions* on V.

Remark 1.27 The coordinate ring of $\mathbb{A}^n$ are all polynomials, i.e. $k[\mathbb{A}^n] = k[x]$.

DIMENSION

Definition 1.28 The dimension of an irreducible k-variety V, denoted $\dim(V)$, is the transcendence degree of the function field $k(V)$ over k, denoted $\text{tr.deg}\big(k(V)/k\big)$. The dimension of a k-variety V is the supremum of the dimensions of its components. If all the components of V have the same dimension d, it is called *equidimensional* with *pure* dimension d. The empty set has dimension $-\infty$.

Remark 1.29 As we will prove later in 1.34 and 6.4, the dimension is equal to the *combinatorial* dimension $\dim_{comb}(V)$ of V which is defined as the supremum of the length n of all ascending chains

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

of irreducible subsets of V. By definition the empty set is not irreducible.

From proposition 1.18 it follows that $\dim_{comb}(V) = \dim(k[V])$ where the second dimension is the ring dimension (Krull dimension), i.e. the supremum of the length n of all descending chains

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

of prime ideals in $k[V]$. By definition the improper ideal R is not a prime ideal in R.

Remark 1.30 The dimension of $\mathbb{A}^n$ is n. In fact, it follows by induction on n since x_k is transcendent over $k(x_1, \ldots, x_{k-1})$.

Example 1.31 The k-variety $V(x,y) \cup V(z) \subset \mathbb{A}^3$ has dimension 2. A maximal chains of prime ideals in $k[x,y,z]/(xz,yz)$ is $(x,y,z) \supset (x,z) \supset (z)$.

Definition 1.32 The *codimension* of a k-variety V in $\mathbb{A}^n$ is $\dim(\mathbb{A}^n) - \dim(V) = n - \dim(V)$.

Theorem 1.33 *Let $W \subseteq V$ be two irreducible k-varieties. Then* $\dim(W) \leq \dim(V)$ *with equality if and only if $W = V$.*

Proof. The first assertion is trivial. In fact, we have $\mathfrak{I}_k(V) \subseteq \mathfrak{I}_k(W)$ and thus a surjection $k[V] \twoheadrightarrow k[W]$. A set of algebraic dependent elements in $k[V]$ maps onto a set of algebraic dependent elements in $k[W]$. Thus if $f_1, \ldots, f_d$ are algebraically independent elements of $k[W]$ any representatives in $k[V]$ are algebraically independent. Consequently $\text{tr.deg}\big(k(V)/k\big) \geq \text{tr.deg}\big(k(W)/k\big)$ since a transcendence basis for $k(W)$ can be extracted from the generators $w_1, w_2, \ldots, w_n$ of $k[W]$, which also are generators for $k(W)$.

For the second part, assume that $\dim(W) = \dim(V) = d$ and let $A = k[V]$ and $A/\mathfrak{p} = k[W]$. Then there are d elements $f_1, f_2, \ldots, f_d$ of A such that their images in $A/\mathfrak{p}$ are algebraically independent over k. Let $g \in \mathfrak{p}$. Then $g, f_1, \ldots, f_d$ are algebraically dependent over k in A and thus satisfies a nontrivial polynomial equation $Q(g, f_1, \ldots, f_d) = 0$ in A where Q is an irreducible polynomial with coefficients in k. If $g \neq 0$, the polynomial is not a multiple of g. But then $Q\left(0, \overline{f_1}, \ldots, \overline{f_d}\right) = 0$ is a nontrivial relation between the images of f_i in $A/\mathfrak{p}$ which thus are algebraically dependent. Consequently, $g = 0$ and thus $\mathfrak{p} = (0)$ and $W = V$. $\square$

Corollary 1.34 *For all k-varieties V there is an inequality* $\dim_{comb}(V) \leq \dim(V)$.

Proof. By theorem 1.33 every ascending chain of irreducible k-varieties gives an increasing sequence of dimensions, which proves the case when V is an irreducible k-variety. It then follows for arbitrary k-varieties since the (combinatorial) dimension of V is the maximum of the (combinatorial) dimensions of its components. □

ZERO-DIMENSIONAL VARIETIES AND HYPERSURFACES

Definition 1.35 Two points x and y in $\mathbb{A}^n(K)$ are *conjugate* over k if there is a k-automorphism $s \in \mathrm{Gal}(K/k)$ over K such that $s(x) = y$, i.e. $s(x_i) = y_i$ for all $i = 1, \ldots, n$. A point $x \in \mathbb{A}^n(K)$ is algebraic over k if all its components x_i are algebraic over k.

Remark 1.36 A 0-dimensional irreducible k-variety V corresponds to a maximal ideal in $k[x]$ and consists of an algebraic point over k and its conjugates over k. In fact, since $k(V)$ is an algebraic extension of k, the images v_i of x_i in $k[V] = k(V)$ are all algebraic over k and the points of V are $(v_1, v_2, \ldots, v_n)$ and its conjugates. Note that there is a finite number of conjugates and thus V has a finite number of points.

Example 1.37 The maximal ideals in $k[x]$ are *not* necessarily generated by n irreducible polynomials $f_i(x_i) \in k[x_i]$. As an example, the ideal $\mathfrak{a} = (x^2 - 2, y^2 - 2)$ in $\mathbb{Q}[x, y]$ is not maximal. In fact $(x - y)(x + y) \in \mathfrak{a}$. The maximal ideals containing $\mathfrak{a}$ are $\mathfrak{m}_1 = (x^2 - 2, x + y)$ and $\mathfrak{m}_2 = (y^2 - 2, x - y)$. It is however easy to see that $\mathfrak{m} \subseteq k[x]$ is a maximal ideal if and only if it is generated by n irreducible polynomials $f_i(x_i) \in k[x_1, x_2, \ldots, x_{i-1}]$.

Definition 1.38 A *k-hypersurface* is a k-variety corresponding to a principal ideal, i.e. a single equation in $k[x]$. A *k-hyperplane* is a linear k-variety corresponding to a single linear equation.

Proposition 1.39 *The k-hypersurfaces in* $\mathbb{A}^n$ *are the k-varieties with pure codimension* 1.

Proof. Let $V = V_K(\{f\})$ be a hypersurface and $f = \prod_i f_i^{n_i}$ a factorization of the defining equation f in irreducible polynomials. We have that $V = \bigcup_i V_K(\{f_i\})$ and thus the components of the hypersurface are the irreducible hypersurfaces corresponding to f_i. Further an irreducible hypersurface has codimension 1. In fact, there is a transcendence basis of $k(x) = k(\mathbb{A}^n)$ containing f_i and in the quotient field of the quotient ring $k[x]/(f_i)$ the other $n - 1$ elements form a transcendence basis.

Conversely, if V is an irreducible k-variety of codimension 1, choose an $f \in \mathfrak{I}_k(V)$ and let $W = V_K(\{f\})$. Then W contains V which is thus contained in an irreducible component W_i of W by proposition 1.13. But since $\dim(W_i) = n - 1 = \dim(V)$ we have that $V = W_i$ by theorem 1.33. Since a k-variety of pure codimension 1 is a finite union of irreducible k-varieties of codimension 1 this concludes the proof. □

Remark 1.40 (Complete intersections) Not every irreducible k-variety of codimension r is given by r equations. In fact the intersection of an irreducible variety and a hypersurface need not be irreducible. Those varieties that are the intersection of r hypersurfaces are called *set-theoretic complete intersections*. If the ideal of a variety of codimension r is generated by r elements, the variety is called a *strict complete intersection*. Trivially, every strict complete intersection is a set-theoretic complete intersection. But the converse is not true (cf. example 1.62). Note that even though $V_K\big(\mathfrak{I}_k(V)+\mathfrak{I}_K(W)\big) = V\cap W$ we only have that $\mathfrak{r}\big(\mathfrak{I}_k(V)+\mathfrak{I}_K(W)\big) = \mathfrak{I}_k(V\cap W)$.

PROJECTIVE VARIETIES

We will now extend our definitions to projective spaces. We will consider the projective space $\mathbb{P}^n(K)$ over K with points $a = (a_0 : a_1 : \cdots : a_n)$. The corresponding polynomial ring is $k[x] = k[x_0, x_1, \ldots, x_n]$.

Definition 1.41 To each set of polynomials $\mathfrak{F} \subseteq k[x]$ we define:

$$VP_K(\mathfrak{F}) = \{\ a \in \mathbb{P}^n(K)\ :\ f(ta) = 0\ \ \forall f \in \mathfrak{F},\ \forall t \in K\ \}$$

A set $E \subseteq \mathbb{P}^n(K)$ is called *a k-variety* if $E = VP_K(\mathfrak{F})$ for some set of polynomials $\mathfrak{F}$.

Remark 1.42 It is clear that a k-variety is the zero locus of all the *homogeneous* components f_i of every polynomial f in $\mathfrak{F}$, i.e.:

$$VP_K(\mathfrak{F}) = V_K\big(\{f_0, f_1, \ldots, f_s\ :\ \forall f = f_0 + f_1 + \cdots + f_s \in \mathfrak{F}\}\big)$$

Definition 1.43 To every set $E \subseteq \mathbb{P}^n(K)$ we associate an ideal, defined by

$$\mathfrak{I}\mathfrak{H}_k(E) = \{\ f \in k[x]\ :\ f(ta) = 0\ \ \forall a \in E,\ \forall t \in K\ \}.$$

This is clearly a *homogeneous* ideal, i.e. an ideal which contains all the homogeneous components of its elements.

The relations (1.1-1.7) holds if we replace V_K with VP_K and $\mathfrak{I}_k$ with $\mathfrak{I}\mathfrak{H}_k$. As in the affine case, the k-varieties as closed sets define a noetherian topology on $\mathbb{P}^n$ which we also call the k-Zariski topology.

Notation 1.44 We will call k-varieties in $\mathbb{A}^n$ and $\mathbb{P}^n$ for affine and projective varieties respectively.

Definition 1.45 If $V \subseteq \mathbb{P}^n$ we define the *representative cone* or *affine cone* $C(V) \subseteq \mathbb{A}^{n+1}$ as the union of the origin and the lines corresponding to points in V. Thus $(a_0 : a_1 : \cdots : a_n)$ is a point of V exactly when $(a_0, a_1, \ldots, a_n)$ is a point of $C(V) \setminus \big\{(0, 0, \ldots, 0)\big\}$.

Remark 1.46 The defining ideals $\mathfrak{I}\mathfrak{H}_k(V)$ of V and $\mathfrak{I}_k\big(C(V)\big)$ of $C(V)$ are equal and a projective set $V \subseteq \mathbb{P}^n$ is a k-variety if and only if $C(V) \subseteq \mathbb{A}^{n+1}$ is a k-variety.

Proposition 1.47 *The projective irreducible k-varieties correspond to homogeneous prime ideals in $k[x]$. Every projective k-variety has a unique representation as a minimal union of irreducible k-varieties which are called its components.*

Proof. The first part is proven exactly as proposition 1.18 and the second part follows from proposition 1.14 and corollary 1.15 as in proposition 1.19 for the affine case. □

Remark 1.48 A projective variety V is irreducible precisely when $C(V)$ is irreducible and the irreducible components $\{V_i\}$ of V corresponds to the irreducible components of $C(V)$, i.e. the irreducible components are $C(V_i)$.

Definition 1.49 Let $V \subseteq \mathbb{P}^n$ be a projective k-variety. The *homogeneous coordinate ring* is the graded ring $k[V] = k[x]/\mathfrak{I}\mathfrak{H}_k(V)$ which is an integral domain if V is irreducible. If that is the case we define the *function field* of V as the *zeroth graded part* of the quotient field, i.e. $k(V) = \{\ p/q : p, q \in k[V],\ p, q$ homogeneous of the same degree $\}$. As before we call the elements in $k(V)$ rational functions on V.

Remark 1.50 We let v_i be the images of x_i in $k[V]$. The function field is then generated by the quotients $\big(v_i/v_j\big)_{i=0}^n$ for any non-zero v_j.

Definition 1.51 The dimension of a projective irreducible k-variety is the transcendence degree of the function field $k(V)$ over k.

Remark 1.52 It is clear that the coordinate ring of a projective irreducible k-variety V is identical to the coordinate ring of its affine cone. Furthermore the function field of the affine cone is generated by $\big\{v_i/v_j\big\}_{i=0}^n$ and any non-zero v_i. Since every element of $k(V)$ has degree zero, v_i is transcendental over $k(V)$ and we have that $\dim\big(C(V)\big) = \dim(V) + 1$.

Theorem 1.53 *Let $W \subseteq V$ be two projective irreducible k-varieties. Then* $\dim(W) \leq \dim(V)$ *with equality if and only if* $W = V$.

Proof. Since $C(W) \subset C(V)$ if and only if $W \subset V$ and $\dim\big(C(V)\big) = \dim(V) + 1$ by remark 1.52, it follows immediate from theorem 1.33. □

Theorem 1.54 (Projective form of Hilbert's Nullstellensatz) *Let $\mathfrak{a}$ be a homogeneous ideal of $k[x]$, not equal to the "irrelevant ideal" $\mathfrak{a}^+ = (x_0, x_1, \ldots, x_n)$. Then $\mathfrak{I}\mathfrak{H}_k\big(VP_K(\mathfrak{a})\big) = \mathfrak{r}(\mathfrak{a})$.*

Proof. This follows immediate from the affine form, using the correspondence with the representative cones. □

Remark 1.55 A 0-dimensional irreducible k-variety projective variety V does not correspond to a maximal ideal. In fact, every non-empty projective k-variety corresponds to an ideal properly contained in $\mathfrak{a}^+$. The ideal of V is however a maximal ideal among those properly contained in $\mathfrak{a}^+$ by theorem 1.53. The elements of the generating set $\big\{v_i/v_j\big\}_{i=0}^n$ of $k(V)$ are algebraic over k. Thus V consists of an algebraic point over k and its conjugates. Note that a projective point a is algebraic over k if its quotients $\big\{a_i/a_j\big\}_{i=0}^n$ are algebraic over k.

Remark 1.56 (Affine cover) It is well known that by choosing a hyperplane at the infinity, given by a linear equation $f(x) = 0$, we can identify the subset $\{\ a : f(a) \neq 0\ \}$ of $\mathbb{P}^n$ with $\mathbb{A}^n$. In particular, we have the standard cover of affines using the hyperplanes given by $x_i = 0$ for $i = 0, 1, \ldots, n$.

Definition 1.57 Let $h : \mathbb{A}^n \to \mathbb{P}^n$ be the canonical affine embedding, which is defined by $h\big((a_1, a_2, \ldots, a_n)\big) = (1 : a_1 : a_2 : \cdots : a_n)$.

Definition 1.58 Let $V \subseteq \mathbb{A}^n$ be an affine k-variety. The *projective closure* $\overline{V}$ of V in $\mathbb{P}^n$ is the smallest projective k-variety containing $h(V)$, i.e. $\overline{h(V)}$. For any polynomial $f \in k[x_1, x_2, \ldots, x_n]$ we define the *homogenization*, $\overline{f} \in k[x_0, x_1, \ldots, x_n]$, as $\overline{f}\,(x_0, x_1, \ldots, x_n) = x_0^d f\,(x_1/x_0, x_2/x_0, \ldots, x_n/x_0)$ where d is the degree of f. Clearly

$\overline{f}$ is a homogeneous polynomial. For an ideal $\mathfrak{a} \in k[x_1, x_2, \ldots, x_n]$ we define $\overline{\mathfrak{a}}$ to be the homogeneous ideal generated by $\left(\overline{f}\right)_{f \in \mathfrak{a}}$.

Proposition 1.59 *If V is an affine k-variety, then the ideal of its projective closure $\mathfrak{I}\mathfrak{H}_k\left(\overline{V}\right)$ is $\overline{\mathfrak{I}_k(V)}$. Further the function fields $k(V)$ and $k(\overline{V})$ are equal.*

Proof. See [S, p. 13] □

Remark 1.60 If V is an affine k-variety of $\mathbb{A}^n$ then the coordinate ring $k[V]$ is equal to $k[v_1/v_0, v_2/v_0, \ldots, v_n/v_0] = k[\overline{V}]_{(v_0)}$, the zero degree part of the homogeneous localization of $k[\overline{V}]$ by $\{1, v_0, v_0^2, \ldots\}$.

Remark 1.61 The embedding h gives a canonical correspondence between k-varieties in $\mathbb{A}^n$ and k-varieties in $\mathbb{P}^n$ without any components contained in the hyperplane at infinity $x_0 = 0$. In fact, such a correspondence exists for any open $U \subset \mathbb{P}^n$ since the open sets are dense.

In particular, if V is a k-variety of $\mathbb{P}^n$ and we can choose a hyperplane $L = VP_K(f)$ not containing any components of V and restrict the variety to $\mathbb{P}^n \setminus L \simeq \mathbb{A}^n$ and thus get an affine variety V_{aff} with coordinate ring $k[V]_{(f)}$ and the same function field as $V = \overline{V_{\text{aff}}}$. As we will see later on in lemma 6.5, such a hyperplane always exists.

Example 1.62 (Twisted Cubic Curve) Let $V = V_K(x_2 - x_1^2, x_3 - x_1^3)$. This defines a curve in $\mathbb{A}^3$ which can be parameterized as $\left\{(t, t^2, t^3) \ : \ t \in K\right\}$. Its projective closure using the canonical embedding h is the set $\overline{V} = \left\{(1 : t : t^2 : t^3) : t \in K\right\} \cup \left\{(0 : 0 : 0 : 1)\right\}$. Its ideal $\mathfrak{I}\mathfrak{H}_k\left(\overline{V}\right)$ is equal to the homogenization $\overline{\mathfrak{I}_k(V)}$ which is *not* generated by $\{x_0x_2 - x_1^2, x_0^2x_3 - x_1^3\}$, the homogenization of the generators for the affine ideal. In fact, the homogenized ideal is not generated by fewer than three generators $\mathfrak{I}\mathfrak{H}_k\left(\overline{V}\right) = \left(x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2\right)$ and is thus not a strict complete intersection. On the other hand $\overline{V} = VP_K(x_1^3 - x_0^2x_3) \cap VP_K(x_2^3 - x_0x_3^2)$ and is therefore a set-theoretic complete intersection.

Chapter 2

Sheaves

SHEAVES

Definition 2.1 Let X be a topological space. A *presheaf* is a map $\mathscr{F}$ which for every open subset $U \subseteq X$ assigns a set $\mathscr{F}(U)$, together with *restriction* maps $\rho_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$ for all inclusions of open sets $U \subseteq V$, with the following two properties:

$$\begin{array}{lll} (P1) & \rho_U^U & = & \mathrm{id}_{\mathscr{F}(U)} \\ (P2) & \rho_U^W & = & \rho_U^V \rho_V^W \end{array}$$

The elements of $\mathscr{F}(U)$ are called the sections of $\mathscr{F}$ over U.

Definition 2.2 A morphism between presheaves $u : \mathscr{F} \to \mathscr{G}$ is a collection of maps $u_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for all inclusions of open sets $U \subseteq V$ the diagram

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "u_V"] \arrow[d, "(\rho_{\mathscr{F}})_U^V"'] & \mathscr{G}(V) \arrow[d, "(\rho_{\mathscr{G}})_U^V"] \\ \mathscr{F}(U) \arrow[r, "u_U"'] & \mathscr{G}(U) \end{tikzcd}$$

commutes.

Definition 2.3 A presheaf is a *sheaf* if for every cover $\{U_\alpha\}_{\alpha \in \mathscr{I}}$ of an open set U by open sets the following sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\prod_{\alpha \in \mathscr{I}} \rho_{U_\alpha}^U} \prod_{\alpha \in \mathscr{I}} \mathscr{F}(U_\alpha) \underset{\prod_{\alpha,\beta \in \mathscr{I}} \rho_{U_\alpha \cap U_\beta}^{U_\beta}}{\overset{\prod_{\alpha,\beta \in \mathscr{I}} \rho_{U_\alpha \cap U_\beta}^{U_\alpha}}{\rightrightarrows}} \prod_{\alpha,\beta \in \mathscr{I}} \mathscr{F}(U_\alpha \cap U_\beta)$$

is exact.

This is equivalent to the following two properties:

(S1) Given two sections $s, t \in \mathscr{F}(U)$ such that $\rho_{U_\alpha}^U(s) = \rho_{U_\alpha}^U(t)$ for all $\alpha \in \mathscr{I}$, then $s = t$.

(S2) Given a collection of sections, $s_\alpha \in \mathscr{F}(U_\alpha)$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for all $\alpha, \beta \in \mathscr{I}$, there exists a section, $s \in \mathscr{F}(U)$ such that $\rho^{U}_{U_\alpha}(s) = s_\alpha$.

Loosely speaking this says that sections are determined by their local values and any set of compatible local values comes from a section. Note that by (S1), the section in (S2) is unique.

Remark 2.4 From (S1) it follows that $\mathscr{F}(\emptyset)$ consist of exactly one element. In fact, using the empty covering $\{U_\alpha\}_{\alpha \in \mathscr{I}}$ with $\mathscr{I} = \emptyset$, of $\emptyset$, we have that $s = t$ for all $s, t \in \mathscr{F}(\emptyset)$.

Definition 2.5 A morphism of sheaves $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves where we consider the sheaves as presheaves.

Definition 2.6 Let $\mathscr{F}$ be a (pre)sheaf and x a point in X. The collection $\{\mathscr{F}(U)\}$, $U \ni x$ open, with the restriction maps, is an injective system. The direct limit of this system is termed the *stalk of $\mathscr{F}$ at x* and is denoted $\mathscr{F}_x$ and the corresponding maps are denoted ρ^U_x.

Remark 2.7 A morphism of (pre)sheaves $u : \mathscr{F} \to \mathscr{G}$ induces maps on the stalks $u_x : \mathscr{F}_x \to \mathscr{G}_x$.

Notation 2.8 Following common notation, we sometimes write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$. The sections over X are denoted $\Gamma(\mathscr{F})$ and are called *global* sections. Similarly $\Gamma(U, u) = u_U$ and $\Gamma(u) = u_X$ for a morphism u of (pre)sheaves.

Definition 2.9 When $\mathscr{F}(U)$ is a group (ring, module, etc) and ρ^V_U group homomorphisms (ring homomorphisms etc) for all U and $V \supseteq U$ we say that $\mathscr{F}$ is a *sheaf of groups* (*rings*, *modules*, *etc*). By definition $\mathscr{F}_x$ is then also a group (ring, module, etc) since we take the direct limit in the category of groups (rings, modules, etc). A morphism of sheaf of groups (rings, etc) u, is a morphism of sheaves such that the morphisms u_U are group (ring, etc) homomorphisms. Then by definition the stalk maps u_x are also group (ring, etc) homomorphisms. Note that $\mathscr{F}(\emptyset) = \{0\}$, i.e. the zero group (ring, module, etc).

Definition 2.10 The *generic* stalk of $\mathscr{F}$ is the direct limit of the injective system consisting of *all* non-empty open sets with the restriction maps. We will denote the generic stalk by $\mathscr{F}_\xi$ and the corresponding maps by ρ^U_ξ.

Remark 2.11 Every element of $\mathscr{F}_x$ can be represented by an element of $\mathscr{F}_U$ for some open $U \ni x$. In fact, if $s_1 \in \mathscr{F}_{U_1}$ and $s_2 \in \mathscr{F}_{U_2}$ are two sections, then $\rho^{U_1}_x(s_1) + \rho^{U_2}_x(s_2)$ and $\rho^{U_1}_x(s_1)\rho^{U_2}_x(s_2)$ are restrictions of the elements $\rho^{U_1}_{U_1 \cap U_2}(s_1) + \rho^{U_2}_{U_1 \cap U_2}(s_2)$ and $\rho^{U_1}_{U_1 \cap U_2}(s_1)\rho^{U_2}_{U_1 \cap U_2}(s_2)$ in $\mathscr{F}_{U_1 \cap U_2}$.

The corresponding fact for the generic stalk $\mathscr{F}_\xi$ is only true if X is irreducible. In fact, if X is not irreducible, two non-empty open subsets may have an empty intersection.

Ringed spaces

Definition 2.12 A *ringed space* is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X, its structure sheaf.

Definition 2.13 Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two ringed spaces. A *morphism of ringed spaces* $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\psi : X \to Y$ together with a morphism of sheaves $\theta : \mathscr{O}_Y \to \psi_* \mathscr{O}_X$, i.e. a collection of ring homomorphisms $\theta_U : \mathscr{O}_Y(U) \to \mathscr{O}_X\left(\psi^{-1}(U)\right)$ such that for all inclusions of open sets $U \subseteq V$ in Y the diagram

$$\begin{tikzcd} \mathscr{O}_Y(V) \arrow[r, "\theta_V"] \arrow[d, "(\rho_{\mathscr{O}_Y})^V_U"'] & \mathscr{O}_X\left(\psi^{-1}(V)\right) \arrow[d, "(\rho_{\mathscr{O}_X})^{\psi^{-1}(V)}_{\psi^{-1}(U)}"] \\ \mathscr{O}_Y(U) \arrow[r, "\theta_U"] & \mathscr{O}_X\left(\psi^{-1}(U)\right) \end{tikzcd}$$

commutes.

Proposition 2.14 *A morphism of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ induces a ring homomorphism $\theta^\sharp_x : \mathscr{O}_{Y,\psi(x)} \to \mathscr{O}_{X,x}$ between the stalks. Further, if ψ is* dominant, *i.e. the image of ψ is dense in Y, we also have a ring homomorphism between the generic stalks $\theta^\sharp_\xi : \mathscr{O}_{Y,\xi} \to \mathscr{O}_{X,\xi}$*

Proof. The ring homomorphism $\theta^\sharp_x$ is given by taking direct limits of the injective systems consisting of every open U containing $\psi(x)$ in Y and the open sets $\psi^{-1}(U)$ in X which all contain x. Explicitly the homomorphism is defined as follows: Let f be an element in $\mathscr{O}_{Y,\psi(x)}$. Then $f = \rho^U_{\psi(x)}(g)$ for some $U \ni x$ and $g \in \mathscr{O}_Y(U)$. The image of f is then $\rho_x^{\psi^{-1}(U)}(\theta_U(g))$ and is well-defined because the commuting diagram of 2.13. The generic stalk homomorphism is defined in the same way, but we need the condition that ψ is dominant to ensure that $\psi^{-1}(U)$ is non-empty for every non-empty open $U \subseteq Y$. $\square$

Regular functions

Notation 2.15 In this chapter X is an affine or projective k-variety. Its *ambient space* is the space $\mathbb{A}^n$ or $\mathbb{P}^n$ in which X is embedded. Note that in general the coordinate ring $A = k[X]$ is not a polynomial ring. When X is a projective variety we will see A as a graded ring using the natural grading.

Remark 2.16 The topology of X is the induced topology of the Zariski topology of its ambient space, $\mathbb{A}^n$ or $\mathbb{P}^n$. The k-varieties of X, i.e. the closed subsets of X in the k-Zariski topology of X corresponds to radical ideals in $A = k[X]$. Note that the irrelevant ideal $\mathfrak{a}^+$, consisting of all elements of positive degree in A, is excluded in the projective case.

Remark 2.17 If X is an affine k-variety, the elements of the coordinate ring $A = k[X]$ can be seen as functions from X to K. In fact, the elements of the polynomial ring $k[\mathbb{A}^n]$ defines functions from the ambient space $\mathbb{A}^n$ to K. If we for an element $f \in A$, take any representative in $k[\mathbb{A}^n]$ and restrict the corresponding function to X, we get a well-defined map from X to K.

Further, the quotient f/g of two elements $f, g \in A$, $g \neq 0$ defines a map from a non-empty open subset $U = \{x : g(x) \neq 0\}$ of X to K. In fact, g is not identically zero on X and thus vanishes on a closed proper subset V of X.

If X is a projective k-variety the homogeneous elements of $A = k[X]$ do not define functions from X to K. To get a function, we need to take a quotient f/g of homogeneous elements $f, g \in A$ of the same degree. This defines a function from the open subset $g(x) \neq 0$ of X to K.

Remark 2.18 The field K is isomorphic to $\mathbb{A}^1(K)$. When we speak of K as a topological space, it is the k-Zariski topology of $\mathbb{A}^1(K)$ that is used.

Proposition 2.19 *A quotient $f = g/h$ of polynomials $g, h \in A$, $h \neq 0$, homogeneous of the same degree in the projective case, defines a* continuous function *from a non-empty open set $U \subset X$ to K in the k-Zariski topology.*

Proof. Since h is not identically zero on X, it vanishes on a closed proper subset $V = V_K\big((h)\big)$ of X. As we have seen in remark 2.17 the quotient g/h defines a function from $U = X \setminus V$ to K.

In $k[\mathbb{A}^1] = k[t]$ every prime ideal is maximal and thus the irreducible k-varieties of $K = \mathbb{A}^1$ correspond to maximal ideals in $k[t]$. Since all closed sets are finite unions of irreducible sets, it is enough to show that the inverse image $f^{-1}(V)$ of an irreducible set V of K is closed to prove that f is continuous. Let $p(t) \in k[t]$ be the irreducible polynomial corresponding to V and d its degree. The points in $f^{-1}(V)$ then fulfill the equation $p\big(f(a)\big) = p\big(\frac{g}{h}(a)\big) = 0$ or equivalently $h^d(a)p\big(\frac{g}{h}(a)\big) = 0$ since $h(a) \neq 0$ for all $a \in U$. Thus the inverse image $f^{-1}(V)$ is the closed set $V_K\big(h^d p(g/h)\big)$ in the affine case and $VP_K\big(h^d p(g/h)\big)$ in the projective case. □

Definition 2.20 Let U be an open subset of X. A function $f : U \to K$ is regular at a point $x \in U$ if there is an open $V \ni x$ and polynomials $g, h \in A$, homogeneous of the same degree in the projective case, such that $f(x) = g(x)/h(x)$ for every $x \in V$. If f is regular at every point, we say that f is regular.

Proposition 2.21 *A regular function $f : U \to K$ is continuous in the k-Zariski topology.*

Proof. Let $\{U_x\}_{x \in X}$ be open neighborhoods such that $f|_{U_x}$ is equal to quotient of polynomials in A and let V be a closed set of K. By proposition 2.19 the restriction of the inverse image $f^{-1}(V)|_{U_x}$ is closed in U_x. Since $\{U_x\}$ is a covering of U it follows that $f^{-1}(V)$ is closed and hence f continuous. □

Remark 2.22 The regular functions on U is a k-algebra. The ring structure is given by addition and multiplication of the local representations as polynomials. On the empty set, the regular functions are the zero ring.

Proposition 2.23 *If f and f' are two regular functions on an open set $U \subseteq X$ which are equal on an open subset $W \subseteq U$ which is dense in U, i.e. $\overline{W} = U$ taking the closure in U, then $f = f'$ on U.*

Proof. Let $s = f - f'$. Let U_x be an open neighborhood of $x \in U$ and $g, h \in A$ be such that $s = g/h$ on U_x. Then s is zero on a closed subset of U_x. Since $\{U_x\}_{x \in U}$ is an open covering of U, the difference s is zero on a closed subset $Z \subseteq U$. But $Z \supseteq W$ which is dense and thus $Z = U$ which proves that $f = f'$ everywhere on U. □

Proposition 2.24 *If $U \subseteq V$ are open subsets of X, the restriction of a function on V to U induces a k-algebra homomorphism from the regular functions on V to the regular functions on U. Further the homomorphism is* injective *if V is irreducible.*

Proof. Let f be a regular function on V. Then $f|_U$ is a regular function on U. Since the k-algebra structure is given by the local representations it is clear that the map $f \mapsto f|_U$ is a k-algebra homomorphism. If V is irreducible then U is dense in V. Thus by proposition 2.23, two regular functions f and f' on V are equal exactly when they are equal on U which proves that the map $f \mapsto f|_U$ is injective. □

Sheaf of affine algebraic sets

Definition 2.25 Let X be an affine k-variety. For any point $x \in X$ we let $\mathfrak{j}_x$ be the prime ideal $\mathfrak{I}_k(\{x\})$.

Remark 2.26 The ideal $\mathfrak{j}_x$ is maximal if x is $\bar{k}$-rational, i.e. the coordinates are elements of $\bar{k}$. In fact, there is a finite number of k-conjugate points to x and thus $V_K(\mathfrak{j}_x) = \overline{\{x\}}$ is the irreducible zero-dimensional variety which consists of x and its conjugates.

Definition 2.27 Let X be an affine variety. For any $f \in A$, we define $D(f) = X \setminus V_K(\{f\}) = \{ x \in X : f(x) \neq 0 \}$.

Proposition 2.28 *The open sets $\{D(f)\}_{f \in X}$ form a* basis *for X.*

Proof. Let U be an arbitrary open set of X. Then there is an ideal $\mathfrak{a}$ of A such that $U = X \setminus V_K(\mathfrak{a})$. Since $U = \bigcup_{f \in \mathfrak{a}} D(f)$, the sets $\{D(f)\}_{f \in X}$ form a *basis* for X. □

Definition 2.29 Let X be an affine k-variety. For any empty open set U we let $\Gamma(U, \mathscr{O}_X) = \mathscr{O}_X(U)$ be the set of regular functions on U. Then $\mathscr{O}_X$ with the restriction maps is a presheaf of k-algebras. By the local nature of the definition of regular functions, this is also clearly a sheaf and is called the *structure sheaf* of X.

Proposition 2.30 *The sections of $\mathscr{O}_X$ on the open sets $D(f)$ are $\mathscr{O}_X(D(f)) = A_f$, the localization of A in $\{1, f, f^2, \ldots\}$.*

Proof. There is a natural k-algebra homomorphism $\psi : A_f \to \mathscr{O}_X(D(f))$ which maps $\frac{g}{f^m}$ on the regular function which is defined by $\frac{g}{f^m}$ everywhere on $D(f)$. The map ψ is injective. In fact, if $\frac{g}{f^m}$ is the zero function on $\mathscr{O}_X(D(f))$ we have that $g(x) = 0$ on $x \in D(f)$ and thus $(fg)(x) = 0$ on $x \in X$. But then $fg = 0$ in A and $\frac{g}{f^m} = 0$ in A_f.

We will proceed to show that ψ is a surjection and thus an isomorphism. Let $s \in \mathscr{O}_X(D(f))$ be a regular function. By definition there is an open covering $\bigcup_\alpha U_\alpha$ of $D(f)$ such that $s = g_\alpha / h_\alpha$ on U_α. The basis $\{D(r)\}$ of X induces a basis $\{D(fr)\}$ on $D(f)$. We can thus assume that $U_\alpha = D(r_\alpha)$. Since $h_\alpha(x) \neq 0$ for all $x \in D(r_\alpha)$ we have that $D(h_\alpha r_\alpha) = D(r_\alpha)$. If we let $g'_\alpha = g_\alpha r_\alpha$ and $h'_\alpha = h_\alpha r_\alpha$ we have that $s = g'_\alpha / h'_\alpha$ on $U_\alpha = D(h'_\alpha)$.

The open set $D(f)$ can be covered by a finite number of $D(h'_\alpha)$. In fact, $D(f) \subseteq \bigcup_\alpha D(h'_\alpha)$ and $V((f)) \supseteq \bigcap_\alpha V((h'_\alpha)) = V(\sum_\alpha (h'_\alpha))$ which gives $f \in \mathfrak{I}_k(V((f))) \subseteq \mathfrak{I}_k(V(\sum_\alpha (h'_\alpha)))$. By Hilbert's Nullstellensatz we thus have that $f^m = \sum_i a_i h'_i$ for some

$a_i \in A$ and a finite set $\{i\}$ of $\{\alpha\}$. The finite number of open sets $\{D(h'_i)\}_i$ thus cover $D(f)$.

Now define $g = \sum_i a_i g'_i$. For every point $x \in D(f)$ there is an index j such that $x \in D(h'_j)$. For every i we now have that $g'_j(x)h'_i(x) = g'_i(x)h'_j(x)$. Indeed, if $x \in D(h'_i)$ we have that $s(x) = \frac{g'_j(x)}{h'_j(x)} = \frac{g'_i(x)}{h'_i(x)}$ and if $x \notin D(h'_i)$ then $r_i(x) = 0$ and $g'_i(x) = h'_i(x) = 0$.

Consequently we have that $(gh'_j)(x) = \sum(a_i g'_i h'_j)(x) = (f^m g'_j)(x)$ and $\frac{g}{f^m} = \frac{g'_j}{h'_j}$ for every j such that $x \in D(h'_j)$, which proves that $s = \frac{g}{f^m}$ and thus that ψ is a surjection. □

Corollary 2.31 *Since $X = D(1)$ we have that $\mathscr{O}_X(X) = A_1 = A$. Thus $\Gamma(\mathscr{O}_X) = A$.*

Corollary 2.32 *For each $x \in X$ we have that:*

$$\mathscr{O}_{X,x} = \varinjlim_{x \in U} \mathscr{O}_X(U) = \varinjlim_{x \in D(f)} \mathscr{O}_X\big(D(f)\big) = \varinjlim_{f(x) \neq 0} A_f = A_{\mathfrak{j}_x}$$

where $A_{\mathfrak{j}_x}$ is the localization of A in the prime ideal $\mathfrak{j}_x$.

Proof. Since the $D(f)$ form a basis, everything except the last equality is clear. For every pair of rings A_f and A_g in the injective system, i.e. $f(x) \neq 0$ and $g(x) \neq 0$, the ring A_{fg} is also in the injective system since $(fg)(x) \neq 0$. The maps $A_f \to A_{fg}$ and $A_g \to A_{fg}$ in the injective system are given by the natural inclusions $a/f^m \mapsto ag^m/(fg)^m$ and the corresponding for A_g. Thus $\varinjlim_{f(x) \neq 0} A_f = \bigcup_{f(x) \neq 0} A_f$ which clearly is $A_{\mathfrak{j}_x}$ since $f(x) \neq 0$ if and only if $f \notin \mathfrak{j}_x$. □

Corollary 2.33 *The* generic *stalk is the direct limit of the injective system consisting of* all *non-empty open sets. If X is irreducible, the generic stalk equals the function field of X.*

Proof. As in the previous corollary it follows from the identity

$$\mathscr{O}_{X,\xi} = \varinjlim_{U \neq \emptyset} \mathscr{O}_X(U) = \varinjlim_{f \neq 0} \mathscr{O}_X\big(D(f)\big) = \varinjlim_{f \neq 0} A_f = A_{(0)} = k(X).$$

Note that this requires that $fg \neq 0$ if $f, g \neq 0$ which is only true when A is an integral domain or equivalently X is irreducible. □

Remark 2.34 By proposition 2.24 the restriction maps ρ^V_U are injective when X is irreducible. Consequently, the restriction ρ^U_ξ to the generic stalk is injective. We can thus see the regular functions of U as elements of the function field of X, i.e. *rational functions*. This makes $\mathscr{O}_X(U)$ into a subring of X and we get that

$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x}.$$

Note that this does *not* imply that every regular function f on U can be defined as $f = g/h$ for a single choice of polynomials $g, h \in A$ on U. As an example, let $A = k[x,y,u,v]/(xv - yu)$ be the coordinate ring of a variety in $\mathbb{A}^4$. Let f be the regular function on $U = D(y) \cup D(v) = \{y \neq 0\} \cup \{v \neq 0\}$ given by $f = x/y$ on $D(y)$ and $f = u/v$ on $D(v)$. This regular function is mapped onto the element $x/y = u/v$ in $k(X)$ and we may write $f = x/y$ if we see $\mathscr{O}_X(U)$, the regular functions on U, as a subring of $k(X)$. But as a *function*, there is no polynomials $g, h \in A$ such that $f = g/h$ everywhere on U.

Remark 2.35 The most difficult part when defining the structure sheaf and determining its properties is proposition 2.30. If we only define morphisms for irreducible varieties proposition 2.30 is much easier to prove, cf. [Mu, Ch. I, Prop. 4.1].

Sheaf of projective algebraic sets

Definition 2.36 Let A be a graded ring and $\mathfrak{p} \subset A$ a homogeneous prime ideal. We then define the *localization with respect to homogeneous elements* as

$$A_{\mathfrak{p}} = \{ f/g \ : \ f, g \in A, \ g \notin \mathfrak{p}, \ g \text{ homogeneous} \}$$

Equivalently we define the homogeneous localization A_f for a homogeneous element $f \in A$. It is clear that $A_{\mathfrak{p}}$ and A_f are graded rings.

Definition 2.37 Let A, $\mathfrak{p}$ and f be as in the previous definition. We define $A_{(\mathfrak{p})}$ and $A_{(f)}$ to be *the zeroth homogeneous part* of the homogeneous localizations $A_{\mathfrak{p}}$ and A_f.

Definition 2.38 For any homogeneous element $f \in A$ we define $D(f) = X \setminus VP_K(\{f\})$. As in the affine case, proposition 2.28, the open sets $D(f)$ form a basis for X.

Definition 2.39 Let $X \subseteq \mathbb{P}^n$ be a projective k-variety and $A = k[X]$ its coordinate ring. For any point $x \in X$ we let $\mathfrak{j}_x = \mathfrak{I}\mathfrak{H}_k(\{x\})$ which is a homogeneous prime ideal. If x is $\bar{k}$-rational, it is maximal among those properly contained in $\mathfrak{a}^+$.

Definition 2.40 The *structure sheaf* of a projective k-variety X is the sheaf of k-algebras $\mathcal{O}_X$ in which the sections $\mathcal{O}_X(U)$ on U are regular functions on U.

Proposition 2.41 *The sections of $\mathcal{O}_X$ on the open sets $D(f)$ are $\mathcal{O}_X(D(f)) = A_{(f)}$ for any $f \in A \setminus k$.*

Proof. The proof is identical to the affine case in proposition 2.30 except that when $f \in k$, Hilbert's Nullstellensatz cannot be used to prove that $f \in \mathfrak{I}\mathfrak{H}_k(VP_K(\sum(h'_\alpha)))$ implies the existence of $a_i \in A$ such that $f^m = \sum a_i h'_i$. In fact, if $f \in k$ then $\mathfrak{r}(\sum(h'_\alpha))$ may be equal to $\mathfrak{a}^+$ in which case $\mathfrak{I}\mathfrak{H}_k(VP_K(\sum(h'_\alpha))) = A \neq \mathfrak{a}^+$. □

Proceeding as in the affine case we get the following result.

Proposition 2.42 *The stalks of $\mathcal{O}_X$ are the* zeroth graded piece *of $A_{\mathfrak{j}_x}$*

$$\mathcal{O}_{X,x} = A_{(\mathfrak{j}_x)} = \{f/g \ : \ f, g \in A_d, \ g(x) \neq 0\} .$$

If X is irreducible *the generic stalk is*

$$\mathcal{O}_{X,\xi} = \varinjlim_{U \neq \emptyset} \mathcal{O}_X(U) = A_{((0))} = k(X)$$

and the sections of an open set is a subring of $k(X)$. Further the sections on any open set U is the intersection of the stalks in U

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

Remark 2.43 If X is an irreducible projective variety and k is algebraically closed it is fairly easy (see [Ha, Chap. I, Thm 3.4a]) to prove that $\mathscr{O}_X(X) = k$. Further, the global sections of $\mathscr{O}_X$ for any variety X over an algebraically closed field k, are k^r where r is the number of *connected* components. Indeed, if V and W are two irreducible components of X which intersect, every global section is constant on $V \cup W$. And if V and W are two connected components, then V and W are open and the ring of regular functions on $V \cup W$ is the direct sum of the rings of regular functions on V and W.

Remark 2.44 If k is not algebraically closed, it is not always true that $\mathscr{O}_X(X) = k$ even when X is irreducible. As an example, consider the irreducible variety X of $\mathbb{P}^3$ defined by the prime ideal $(x^2 - 2y^2, u^2 - 2v^2, xu - 2yv, xv - yu) \subset \mathbb{Q}[\mathbb{P}^3] = k[x, y, u, v]$. Let s be the regular function on X defined by $s = \frac{x}{y}$ on $D(y)$ and by $s = \frac{u}{v}$ on $D(v)$. This defines s everywhere since $X = D(y) \cup D(v)$ and on $D(y) \cap D(v)$ we have that $\frac{x}{y} = \frac{u}{v}$. The function $s : X \to L$ is takes the values $\pm\sqrt{2}$ everywhere and is thus not a constant function with values in $\mathbb{Q}$. In fact, the coordinate ring of X is $\mathbb{Q}(\sqrt{2})$. Note that X splits into two *connected* components defined by $(x - \sqrt{2}y, u - \sqrt{2}v)$ and $(x + \sqrt{2}y, u + \sqrt{2}v)$ in $\mathbb{Q}(\sqrt{2})$.

If the dimension of X is zero, then $\mathscr{O}_X\big(D(f)\big) = \mathscr{O}_{X,\xi} = k(X) = k[X]$. As in the above case X splits into several connected components in the $\overline{k}$-Zariski topology if $k(X) \neq k$.

Remark 2.45 An analogy to the fact that $\Gamma(\mathscr{O}_X) = k[X]$ in the affine case and $\Gamma(\mathscr{O}_X) = k$ in the projective when k is algebraically closed, is analytical functions. On $\mathbb{A}^1(\mathbb{C})$ there are many analytical functions, but on $\mathbb{P}^1(\mathbb{C})$ only the constant functions.

Quasi-varieties

Definition 2.46 A non-empty open subset of an affine or projective k-variety is called a *k-quasi-variety*.

Definition 2.47 The structure sheaf of a k-quasi-variety V is the restriction $\mathscr{O}_{\overline{V}}|_V$ of the structure sheaf of its closure.

Remark 2.48 Let V be an *irreducible* k-quasi-variety V. Then $\overline{V}$ is irreducible. A function on V is rational if and only if it is rational on its closure. Indeed, if $f = g/h$ is a rational function defined on an open subset U of $\overline{V}$, it is also a rational function on V defined on $V \cap U$ which is non-empty since V is irreducible. Thus the function field of V is $k(V) = k(\overline{V})$.

Morphisms

Definition 2.49 A k-morphism is a continuous map $\psi : X \to Y$ between projective or affine k-(quasi-)varieties X and Y such that $\theta_U : \mathscr{O}_Y(U) \to \mathscr{O}_X\big(\psi^{-1}(U)\big)$ defined by $f \mapsto f \circ \psi|_{\psi^{-1}(U)}$ is a well-defined k-algebra homomorphism for every open $U \subseteq Y$.

Proposition 2.50 *Every k-morphism $\psi : X \to Y$ gives a morphism of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. The morphism of sheaves of rings $\theta : \mathscr{O}_Y \to \psi_*\mathscr{O}_X$, is given by the k-algebra homomorphisms $\theta_U : \mathscr{O}_Y(U) \to \mathscr{O}_X\big(\psi^{-1}(U)\big)$ defined by $f \mapsto f \circ \psi|_{\psi^{-1}(U)}$.*

Proof. By the definition θ_U is well-defined and we only need to show that the diagram in definition 2.13 commutes. Let $U \subseteq V$ be open subsets of X and let f be an element of $\mathscr{O}_Y(V)$. Then

$$\begin{aligned}\theta_U\big((\rho_{\mathscr{O}_Y})^V_U(f)\big) = (\rho_{\mathscr{O}_Y})^V_U(f) \circ \psi|_{\psi^{-1}(U)} = (\rho_{\mathscr{O}_X})^{\psi^{-1}(V)}_{\psi^{-1}(U)}\big(f \circ \psi|_{\psi^{-1}(V)}\big) = \\ = (\rho_{\mathscr{O}_X})^{\psi^{-1}(V)}_{\psi^{-1}(U)}\big(\theta_V(f)\big)\end{aligned}$$

which proves that θ is a morphism of sheaves. □

Proposition 2.51 *Every* dominant *k-morphism $f : X \to Y$ between irreducible k-(quasi-)-varieties induces an inclusion of fields $k(Y) \hookrightarrow k(X)$.*

Proof. In fact, by proposition 2.14 the morphism of ringed spaces (ψ, θ) induces a k-algebra homomorphism $\theta^\sharp_\xi$ on the generic stalks. By corollary 2.33 and proposition 2.42 the generic stalks are the function fields $k(Y)$ and $k(X)$. □

Theorem 2.52 *A k-morphism $\psi : X \to Y$ from a projective or affine k-(quasi-)variety to an* affine *k-variety Y is determined by $\Gamma(\theta)$ where θ is the associated morphism of sheaves. Moreover, every k-algebra homomorphism $\varphi : \Gamma(\mathscr{O}_Y) \to \Gamma(\mathscr{O}_X)$ determines a k-morphism ψ from X to Y such that $\Gamma(\theta) = \varphi$ for its associated morphism of sheaves θ. Thus we have a bijection*

$$\mathrm{Mor}(X,Y) \simeq \mathrm{Hom}_k\big(\Gamma(\mathscr{O}_Y), \Gamma(\mathscr{O}_X)\big) = \mathrm{Hom}_k\big(k[Y], \Gamma(\mathscr{O}_X)\big).$$

Proof. A morphism of sheaves $\theta : \mathscr{O}_Y \to \psi_*\mathscr{O}_X$ is determined by the homomorphisms $\theta_{D(f)} : \mathscr{O}_Y\big(D(f)\big) \to \mathscr{O}_X\big(\psi^{-1}(D(f))\big)$. But $\mathscr{O}_Y\big(D(f)\big) = k[Y]_f$ and any homomorphism from $k[Y]_f$ is determined by its values on $k[Y]$. Since $\rho^Y_{D(f)} : k[Y] \to k[Y]_f$ is an inclusion, the homomorphism $\theta_{D(f)}$ and a fortiori θ is determined by $\Gamma(\theta)$. Further $\Gamma(\theta)$ determines ψ. In fact, let $a \in X$ and $b = \psi(a)$. Then $b_j = y_j \circ \psi(a) = \theta_Y(y_j)(a)$.

Now consider any k-algebra homomorphism $\varphi : k[Y] \to \Gamma(\mathscr{O}_X)$. Then $\varphi(y_j)$ can be seen as a function $\varphi(y_j) : X \to K$ and we can consider the map ${}^a\varphi : X \to \mathbb{A}^m$, $a \mapsto b$, defined by $b_j = \varphi(y_j)(a)$. Let $\mathfrak{a}$ be the ideal of Y in $\mathbb{A}^m$ and take any $g \in \mathfrak{a}$. The image of g in $k[Y]$ is then zero and $\varphi(g)$ is zero in $\Gamma(\mathscr{O}_X)$. But $g(b) = \varphi(g)(a) = 0$ and thus $b \in Y$. The image of ${}^a\varphi$ is thus contained in Y.

Further ${}^a\varphi : X \to Y$ is continuous. In fact, let W be a k-variety of Y with ideal $\mathfrak{a} \subseteq k[Y]$. The points a with image $b = {}^a\varphi(a)$ in W are given by $g(b) = \varphi(g)(a) = 0$ for all $g \in \mathfrak{a}$. Thus $({}^a\varphi)^{-1}(W)$ is the k-variety defined by the ideal $\varphi(\mathfrak{a})$.

The homomorphism φ induces a morphism of sheaves $\theta : \mathscr{O}_Y \to \mathscr{O}_X$ with $\Gamma(\theta) = \varphi$ and it is clear that θ_U is equal to the map $f \mapsto f \circ {}^a\varphi|_{({}^a\varphi)^{-1}(U)}$. Thus ${}^a\varphi$ is a morphism of varieties and we have shown that there is a bijection $\mathrm{Mor}(X,Y) \simeq \mathrm{Hom}\big(k[Y], \Gamma(\mathscr{O}_X)\big)$ given by $\psi \mapsto \Gamma(\theta)$ and $\varphi \mapsto {}^a\varphi$. □

Remark 2.53 The above proof also implies that all morphisms $X \to Y$ can be extended (but not necessarily uniquely) to the ambient space of X if Y is affine. This is not the case when Y is projective (see example 3.8).

Corollary 2.54 *The map $\Gamma : V \mapsto \Gamma(V) = k[V]$, which takes affine varieties to coordinate rings, extends to a* contravariant functor *between the category of affine k-varieties and the category of finitely generated reduced k-algebras with k-algebra homomorphisms, which is an equivalence of categories. Further it also induces an equivalence between the category of affine irreducible k-varieties and the category of finitely generated integral domains over k.*

Proof. Follows immediately from theorem 2.52. □

Remark 2.55 Due to corollary 2.54 we can speak of a finitely generated reduced k-algebra as an affine variety without referring to an embedding into affine space. In fact, any choice of embedding gives isomorphic varieties and there is always an embedding.

Remark 2.56 The projective varieties are *not* equivalent to the category of finitely generated graded reduced k-algebras. In fact the projection from $(0:0:1)$ of the parabola $x^2 - yz$ in $\mathbb{P}^2$ onto the infinity line $z = 0$, given by $(u : v : w) = (x : y : 0)$ on $y \neq 0$ and by $(u : v : w) = (z : x : 0)$ on $x \neq 0$, is an isomorphism but the rings $k[x, y, z]/(x^2 - yz)$ and $k[u, v, w]/(w)$ are not isomorphic rings.

Theorem 2.57 *Let X be a projective variety and $f : X \to Y$ a morphism. Then f is closed, i.e. the set-theoretic image of a variety is a variety.*

Proof. This is a corollary to the main result in elimination theory that $\mathbb{P}^n$ is complete, i.e. that the projection morphism $X \times Y \to Y$ is a closed map (cf. theorem 4.11). For a proof, see [Mu, Ch. I, Thm 9.1]. □

Chapter 3

Morphisms

Characterization of morphisms

Proposition 3.1 *Let V be an irreducible k-variety (affine or projective) and let $W \subseteq \mathbb{A}^m$ be an affine irreducible k-variety. Every k-morphism $V \to W$ is given by a continuous map $f : V \to W$ defined by polynomials, i.e. $f(a) = \big(f_1(a), f_2(a), \ldots, f_m(a)\big)$ where $f_j : V \to K$ are maps given by elements in $\Gamma(\mathscr{O}_V)$ and conversely every such map uniquely determines a morphism.*

Proof. The proposition follows immediately from proposition 2.52 since every polynomial map corresponds to a ring homomorphism $\theta : k[W] \to \Gamma(\mathscr{O}_V)$ given by $\theta(w_j) = f_j$ and vice versa, where w_j is the image of $y_j \in k[y] = k[\mathbb{A}^m]$ in $k[W]$. $\square$

Definition 3.2 Let $f : V \to W$ be a k-morphism. The *set-theoretic image* of a k-variety $H \subseteq V$ by f is the image $f(H)$ as a set. The *image* of H is the closure $\overline{f(H)}$ of the set-theoretic image in the k-Zariski topology.

Proposition 3.3 *The image of an irreducible k-variety $H \subseteq V$ by a k-morphism $f : V \to W$ is an irreducible k-variety.*

Proof. The image of an irreducible set is irreducible and the closure of an irreducible set is irreducible. Thus $\overline{f(H)}$ is an irreducible closed set, i.e. an irreducible k-variety. $\square$

Remark 3.4 Since $f(V)$ is dense in the image $\overline{f(V)}$, we have by proposition 2.51 an inclusion of fields $k\big(\overline{f(V)}\big) \hookrightarrow k(V)$ if V is irreducible.

Proposition 3.5 *Let $f : V \to W$ be a k-morphism between affine k-varieties. The image $\overline{f(H)}$ of a k-subvariety $H \subseteq V$ is given by the ideal $\mathfrak{I}_k\big(\overline{f(H)}\big) = \mathfrak{I}_k\big(f(H)\big) = \theta^{-1}\big(\mathfrak{I}_k(H)\big)$ where θ is the k-algebra homomorphism $\theta : k[W] \to k[V]$ which corresponds to f by the correspondence in 2.52*

Proof. An element $g \in k[W]$ is such that $g(a) = 0$ for every point $a \in f(H)$ if and only if $g \circ f \in \mathfrak{I}_k(H)$. Since $g \circ f = \theta(g)$ the proposition follows. $\square$

Definition 3.6 A k-morphism $f : V \to W$ is an *isomorphism* if there is a k-morphism $g : W \to V$ such that $g \circ f = id_V$ and $f \circ g = id_W$.

Remark 3.7 By proposition 2.52 a k-isomorphism of affine varieties is associated to a k-isomorphism of rings. However, a k-morphism of varieties need not be an isomorphism even though it is a bijection. In fact the k-morphism $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x = t^2$ and $y = t^3$ is a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$ in $\mathbb{A}^2$ but the associated ring homomorphism $k[t^2, t^3] \hookrightarrow k[t]$ is not an isomorphism and thus f is not an isomorphism.

Example 3.8 Let $X \subset \mathbb{P}^2$ be the irreducible k-variety defined by $y^2 - xz$. The map of X onto $\mathbb{P}^1$ given by $t/s = y/x = z/y$ is a morphism. This morphism cannot be extended to the whole $\mathbb{P}^2$. In fact there are no surjective morphisms from $\mathbb{P}^2$ to $\mathbb{P}^1$.

Definition 3.9 Two irreducible k-varieties V and W are *birationally equivalent* if $k(V) \simeq k(W)$.

Definition 3.10 A k-morphism $f : V \to W$ between irreducible varieties is *birational* if it is dominant and the induced inclusion of fields $k(W) \hookrightarrow k(V)$ given in proposition 2.51 is an isomorphism.

Remark 3.11 A birational morphism of varieties need not be an isomorphism. In fact, the ring homomorphism $k[t^2, t^3] \hookrightarrow k[t]$ of the morphism in remark 3.7 gives an isomorphism $k(t) \simeq k(t)$ and the morphism is thus birational even though it is not an isomorphism. It can however be shown that it is an isomorphism on an open subset, see [Mu, Ch. I, Thm 8.4]. The above mentioned morphism onto $y^2 = x^3$ is an isomorphism between $\mathbb{A}^1 \setminus (0,0)$ and the open subset $(x, y) \neq (0,0)$ of $V_K(y^2 - x^3) \subset \mathbb{A}^2$.

Affine projections

Definition 3.12 A k-morphism $f : V \to W$ between varieties of affine spaces $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$, is called a *k-projection* if the f_j:s of proposition 3.1 are linear, i.e. $f_j = f_{j0} + f_{j1}v_1 + \cdots + f_{jn}v_n$, $j = 1, 2, \ldots, m$ with $f_{ji} \in k$ and where v_i is the images of $x_i \in k[\mathbb{A}^n]$ in $k[V]$.

Remark 3.13 Every projection can be extended to a projection of $\mathbb{A}^n$ to $\mathbb{A}^m$ by taking the same f_{ji}'s. We will therefore only consider projections from $\mathbb{A}^n$ to $\mathbb{A}^m$. A k-projection is thus a linear transformation of $\mathbb{A}^n$ onto a linear subspace of $\mathbb{A}^m$ and corresponds to a matrix with coefficients in k.

Remark 3.14 The affine projections are *not* projections from a point but projections onto a linear space from the infinity.

Definition 3.15 The linear subspace $\ker(f) = \{a \in \mathbb{A}^n : f_1(a) = f_2(a) = \cdots = f_m(a) = 0\}$ of a k-projection $f : \mathbb{A}^n \to \mathbb{A}^m$, which is an irreducible k-variety, is called the *direction* of the projection.

Definition 3.16 Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a k-projection. The image $W = \overline{f(V)}$ of a k-variety $V \subseteq \mathbb{A}^n$ is called the *projection* of V by f and is by definition a k-variety.

Remark 3.17 Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a k-projection. We can then among the f_i's choose a maximum number r of linearly independent elements over k, say $f_1, \ldots, f_r$. These elements are then also algebraically independent over k and the other elements $f_{r+1}, \ldots, f_m$ are linearly dependent on $f_1, \ldots, f_r$. The ring $k[f_1, f_2, \ldots, f_m]$, which is the coordinate ring of $f(\mathbb{A}^n)$, is thus a polynomial ring in r variables. The image $\overline{f(\mathbb{A}^n)} = f(\mathbb{A}^n)$ is consequently a k-linear variety isomorphic to $\mathbb{A}^r$.

Remark 3.18 Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a surjective k-projection and take a k-subvariety V of $\mathbb{A}^n$. Then by proposition 3.5 we have that $\mathfrak{I}_k\big(f(V)\big) = \mathfrak{I}_k(V) \cap k[f_1, f_2, \ldots, f_m]$. Further $k\big[f(V)\big] = k[f_1, f_2, \ldots, f_m]/\mathfrak{I}_k\big(f(V)\big) = \big(k[\mathbb{A}^n]/\mathfrak{I}_k(V)\big) \cap k[f_1, \ldots, f_m] = k[V] \cap k[f_1, \ldots, f_m]$.

Example 3.19 Let $H = V_K(x_1x_2 - 1) \subset \mathbb{A}^2$ be a k-variety and define the k-projection $f : \mathbb{A}^2 \to \mathbb{A}^1$ by $(a_1, a_2) \mapsto (a_1)$. The set-wise image $f(H)$ is $\{a_1 \neq 0\}$ which is not a k-variety of $\mathbb{A}^1$. The projection of H is $\overline{f(H)} = V_K\big((x_1x_2 - 1) \bigcap k[x_1]\big) = V_K(0) = \mathbb{A}^1$.

Projective projections

Definition 3.20 A k-projection from $\mathbb{P}^n$ to $\mathbb{P}^m$ is a linear transformation $f : k^{n+1} \to k^{m+1}$ given by $y_j = \sum_{i=0}^{n} f_{ji}x_i$, $j = 0, 1, \ldots, m$ with coefficients in k, i.e. a $(m+1) \times (n+1)$ matrix with coefficients in k.

Definition 3.21 The kernel $\{a \in \mathbb{P}^n : f_0(a) = f_1(a) = \cdots = f_m(a) = 0\}$ of a k-projection f, which is a linear k-variety of $\mathbb{P}^n$, is called the *center* of f.

Remark 3.22 If f is a k-projection from $\mathbb{P}^n$ to $\mathbb{P}^m$ and $V \subseteq \mathbb{P}^n$ is a k-variety which does not intersect the center D, then the projection f defines a k-morphism from V to $\mathbb{P}^m$. We say that f is a projection from V to $\mathbb{P}^m$.

Remark 3.23 A projection f does *not* give rise to a k-morphism defined on the whole space $X = \mathbb{P}^n$, as in the affine case, unless $D = \emptyset$. In that case the projection is an automorphism, corresponding to an element of $\mathrm{PGL}(n) = \mathrm{GL}(n+1)/k^*$.

Remark 3.24 The problem in the affine case with the set-theoretic image $f(H)$ not being a k-variety as seen in example 3.19 disappears when dealing with projective projections. In fact theorem 2.57 ensures that the image of a k-variety is a k-variety.

Remark 3.25 (Elimination) A set of homogeneous equations in $\mathbb{P}^n$ corresponds to a variety V in $\mathbb{P}^n$. To eliminate some variables $x_{k+1}, \ldots, x_n$, is the same as projecting $\mathbb{P}^n$ onto $\mathbb{P}^k$ using $y_s = x_s$, $s = 0, \ldots, k$. If this defines a projection from V to $\mathbb{P}^k$, i.e. there are no points $a \in V$ such that $a_0 = a_1 = \cdots = a_k = 0$, then the elimination results in equations defining the projection of V since this is a variety by theorem 2.57. For more on elimination see corollary 4.12.

Remark 3.26 Every projective projection $f : \mathbb{P}^n \to \mathbb{P}^m$ induces an affine projection $f_a : \mathbb{A}^{n+1} \to \mathbb{A}^{m+1}$ which maps the origin to the origin. Further, if V is a projective variety which does not intersect the center of the projection f, then $C\big(f(V)\big) = f_a\big(C(V)\big)$.

Proposition 3.27 *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a projective k-projection. Then* $\dim f(V) = \dim V$ *for any k-variety $V \subseteq \mathbb{P}^n$ which does not intersect the center of the projection.*

Proof. It is enough to prove the case when V is irreducible. By proposition 2.51, the k-morphism f induces an injective map $k\big(f(V)\big) \hookrightarrow k(V)$ and thus $\dim f(V) \leq \dim V$. By remark 1.52 and 3.26, it is thus enough to show that $\dim C\big(f(V)\big) = \dim f_a\big(C(V)\big)$ is not less than $\dim C(V)$. Further by remark 3.18 it is sufficient to show that

$k[V]$ is algebraic over $k[f_0, f_1, \ldots, f_m] \cap k[V]$ or that $k[V]/\big((f_0, f_1, \ldots, f_m) \cap k[V]\big) = k[x]/\big((f_0, f_1, \ldots, f_m) + \mathfrak{I}_k(V)\big)$ is algebraic over k. But $C(V) \cap C(f_0, f_1, \ldots, f_m) = \{0\}$ since V does not intersect the center of the projection and thus $\mathfrak{r}\big(\mathfrak{I}_k(V) + (f_0, f_1, \ldots, f_m)\big) = (x_0, \ldots, x_n)$. Consequently $k[x]/\big((f_0, f_1, \ldots, f_m) + \mathfrak{I}_k(V)\big)$ is algebraic over k and $\dim f(V) \geq \dim V$. $\square$

Remark 3.28 Proposition 3.27 does not imply that all projections are isomorphisms. As an example, the projection of $VP_K(x^2 - yz) \subset \mathbb{P}^2$ onto $\mathbb{P}^1$ by $(s, t) = (y, z)$ is not an isomorphism. In fact both (x, y, z) and $(-x, y, z)$ are mapped to the same point in $\mathbb{P}^1$.

Chapter 4

Products

AFFINE PRODUCTS

Definition 4.1 The product of $\mathbb{A}^n$ and $\mathbb{A}^m$ is the set $\mathbb{A}^n \times \mathbb{A}^m$ which is canonically isomorphic to $\mathbb{A}^{n+m}$ by the correspondence $\big((a_1, \ldots, a_n),\ (b_1, \ldots, b_m)\big) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

Proposition 4.2 *Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be two k-varieties with corresponding ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $k[x_1, \ldots, x_n]$ and $k[y_1, \ldots, y_m]$. Their product in the categorical sense is the k-variety corresponding to $\mathfrak{r}(\mathfrak{a}, \mathfrak{b})$ in $\mathbb{A}^{n+m}$.*

Proof. By corollary 2.54 the categories of varieties and finitely generated reduced k-algebras are equivalent by the contravariant functor $\Gamma : V \to k[V]$. The coproduct in the category of finitely generated k-algebras is given by the tensor product over k (see [L1, Ch. XVI, Prop. 6.1]). It is easy to show that the coproduct in the category of finitely generated reduced k-algebras is the reduced ring of the tensor product over k. Thus the product $V \times W$ is the k-variety corresponding to reduced ring of $k[V] \otimes_k k[W] = k[x]/\mathfrak{a} \otimes_k k[y]/\mathfrak{b} = k[x, y]/(\mathfrak{a}, \mathfrak{b})$, i.e. the ring $k[x, y]/\mathfrak{r}(\mathfrak{a}, \mathfrak{b})$. $\square$

Remark 4.3 The product of affine varieties is the same as the product of the varieties seen as sets.

Example 4.4 The product of two irreducible k-varieties is not necessarily an irreducible k-variety. Let $\mathfrak{a} = (x^2 + 1)$ and $\mathfrak{b} = (y^2 + 1)$ be ideals in $\mathbb{Q}[x]$ and $\mathbb{Q}[y]$, defining two irreducible k-varieties V and W. These irreducible varieties have a non-irreducible product since $\mathfrak{I}_{\mathbb{Q}}(V \times W) = \mathfrak{r}(x^2 + 1, y^2 + 1) = (x^2 + 1, y^2 + 1)$ is not a prime ideal. In fact the element $x^2 - y^2 = (x + y)(x - y)$ is in the ideal and both $x + y$ and $x - y$ are not.

Example 4.5 Even when the product of two irreducible k-varieties V and W is an irreducible k-variety it is not always true that $\mathfrak{I}_k(V \times W) = \mathfrak{r}(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{b})$ or equivalently, that $k[V \times W] = k[V] \otimes_k k[W]$. Indeed, the ring $k[V] \otimes_k k[W]$ may have nilpotent elements when k is not perfect. Let $k = \mathbb{F}_p(t) = (\mathbb{Z}/p\mathbb{Z})(t)$ and let $\mathfrak{a} = (x^p - t)$ and $\mathfrak{b} = (y^p - t)$ be ideals in $k[x]$ and $k[y]$. This defines two irreducible varieties V and W. Now $k[x, y]/\,(x^p - t, y^p - t)$ has nilpotent elements. In fact $(x - y)^p = x^p - y^p = 0$. The ideal of $V \times W$ is $(x^p - t, x - y)$.

Remark 4.6 For a *geometrically integral* k-variety, which will be defined in chapter 5, the situation is much simpler. A product of two geometrically integral k-varieties is always irreducible and $k[V \times W] = k[V] \otimes_k k[W]$.

Even if $V \times W$ is not irreducible we at least have the following result.

Proposition 4.7 *Let V and W be irreducible k-varieties of dimension d and d' respectively. Then the components of $V \times W$ have dimension $d + d'$.*

Proof. See [S, p. 20]. □

PROJECTIVE PRODUCTS

The projective case is more difficult since there is not a simple isomorphism between $\mathbb{P}^n \times \mathbb{P}^m$ and $\mathbb{P}^{n+m}$ as in the affine case. Instead we have that $\mathbb{P}^n(K) \times \mathbb{P}^m(K)$ is isomorphic, as a set, to K^{n+m+2} (basis $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m$) modulo the equivalence relation:

$$(a, b) \simeq (a', b') \quad \iff \quad a = \alpha a',\ b = \beta b',\ \alpha, \beta \in K.$$

Similarly to projective varieties we can now define *biprojective* varieties using *bihomogeneous* polynomials and ideals. A bihomogeneous polynomial is a polynomial which is homogeneous in both $x_0, \ldots, x_n$ and $y_0, \ldots, y_m$, e.g. $x_1x_2y_1 - x_3^2y_2$ is bihomogeneous but not $x_1x_2x_3 - y_1^3$. Bihomogeneous ideals are ideals generated by bihomogeneous polynomials.

We get a correspondence (Hilbert's Nullstellensatz) between biprojective varieties and bihomogeneous radical ideals which do not contain a multiple of any "irrelevant ideal". The rational functions of a biprojective variety are quotients of bihomogeneous polynomials. This tells us what regular functions are and we can define a structure sheaf, allowing us to speak of morphisms between affine or projective varieties and biprojective varieties.

Remark 4.8 The set-categorical product of two projective k-varieties $V = VP_K(\mathfrak{a})$ and $W = VP_K(\mathfrak{b})$ is the biprojective k-variety $V \times W$ given by the bihomogeneous ideal $(\mathfrak{a}, \mathfrak{b})$.

Proposition 4.9 *The product of two projective k-varieties V_1 and V_2 in the category of projective and biprojective varieties is the set-categorical product $V_1 \times V_2$.*

Proof. Let T be a k-variety (projective or biprojective) and let $\varphi_1 : T \to V_1$ and $\varphi_2 : T \to V_2$ be k-morphisms. Since $V \times V_2$ is the set-categorical product, there is a unique map $\varphi : T \to V_1 \times V_2$ such that $\varphi_1 = p_1 \circ \varphi$ and $\varphi_2 = p_2 \circ \varphi$, where $p_1 : V_1 \times V_2 \to V_1$ and $p_2 : V_1 \times V_2 \to V_2$ are the projection morphisms. To show that $V_1 \times V_2$ is the product in the category of varieties we thus only need to show that p_1, p_2 and φ are k-morphisms.

The projection morphisms p_1 and p_2 are k-morphisms. Indeed, it is enough to show that a regular function f on V_1 or V_2 is mapped to a regular function $f \circ p_1$ or $f \circ p_2$ on $V_1 \times V_2$ which is trivial since a rational function g/h, $g, h \in k[V_1]$ on V_1 is mapped to the same function g/h, $g, h \in k[V_1] \subset k[V_1 \times V_2]$ on $V_1 \times V_2$ and similarly for V_2.

Explicitly φ is given by $\varphi_1 \times \varphi_2$, i.e. $\varphi(t) = \big(\varphi_1(t), \varphi_2(t)\big)$. Being a k-morphism is a local property and thus it is enough to show that φ is a k-morphism on the inverse image of every open of a covering of $V_1 \times V_2$. The open sets $Z_{U_1,U_2} = p_1^{-1}(U_1) \cap p_2^{-1}(U_2)$ for all open *affine* $U_1 \subset V_1$ and $U_2 \subset V_2$ is an open covering of $V_1 \times V_2$ and it is thus enough to show that $\varphi|_{\varphi^{-1}(Z_{U_1,U_2})}$ is a k-morphism.

The open subset $Z = p_1^{-1}(U_1) \cap p_2^{-1}(U_2)$ of $V_1 \times V_2$ is canonically isomorphic to the affine variety $U_1 \times U_2$. Since $\varphi_1(\varphi^{-1}(Z)) \subseteq U_1$ and $\varphi_2(\varphi^{-1}(Z)) \subseteq U_2$, the restricted morphism $\varphi|_{\varphi^{-1}(Z)}$ is thus a k-morphism which proves that φ is a k-morphism and concludes the proof. □

Remark 4.10 Choosing two hyperplanes in $\mathbb{P}^n$ and $\mathbb{P}^m$ we can identify the affine space $\mathbb{A}^n \times \mathbb{A}^m$ as an open subset of $\mathbb{P}^n \times \mathbb{P}^m$. There is also a canonical embedding, $h : \mathbb{A}^n \times \mathbb{A}^m \to \mathbb{P}^n \times \mathbb{P}^m$, given by the hyperplanes $x_0 = y_0 = 0$, defined by $h(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = (1 : x_1 : \cdots : x_n, 1 : y_1 : \cdots : y_m)$.

Theorem 4.11 (Main theorem of elimination theory) *$\mathbb{P}^n$ is complete, i.e. the projection morphism $\mathbb{P}^n \times Y \to Y$ is a closed map for all affine or projective varieties Y.*

Proof. See [Mu, Ch. I, Thm 9.1]. □

Corollary 4.12 (Elimination) *Let $\mathfrak{F}$ be a finite set of polynomial equations in the variables $x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_m$ which are homogeneous in $x_0, x_1, \ldots, x_n$. The elimination of $y_1, y_2, \ldots, y_m$ then gives a homogeneous set of polynomial equations in $x_0, x_1, \ldots, x_n$.*

Proof. The set of polynomials $\mathfrak{F}$ defines a variety V of $\mathbb{P}^n \times \mathbb{A}^m$. By the main theorem of elimination theory, the projection of $\mathbb{P}^n \times \mathbb{A}^m$ on $\mathbb{P}^n$ is closed and thus it induces a morphism $f : V \to \mathbb{P}^n$ of varieties. The equations of the image $f(V)$ is the equations after eliminating $y_1, y_2, \ldots, y_m$. □

SEGRE EMBEDDING

It would be very unsatisfactory if the biprojective varieties were not projective varieties. Indeed, proposition 4.13 shows that every biprojective variety is isomorphic to a projective variety.

Proposition 4.13 (Segre embedding) *The map $\iota : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ defined by $(a_0, a_1, \ldots, a_n,\ b_0, b_1, \ldots, b_m) \mapsto (a_0b_0 : a_0b_1 : \cdots : a_nb_m)$ is a k-isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a projective subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$.*

Proof. See [Mu, Ch. I, Thm 6.3]. □

Remark 4.14 From the proof of proposition 4.13, the ideal of the image of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$ is generated by $x_{ij}x_{i'j'} - x_{ij'}x_{i'j}$ where x_{ij} is the coordinate corresponding to x_i and y_j, i.e. $x_{ij} = x_iy_j$.

Remark 4.15 The coordinate ring of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ can be written as $k[x_0y_0, x_0y_1, \ldots, x_ny_m]$ where the x_iy_j has degree one.

Example 4.16 The simplest Segre embedding is the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ which identifies $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric $(x_{00}x_{11} - x_{01}x_{10}) \subset k[x_{00}, x_{01}, x_{10}, x_{11}]$ of $\mathbb{P}^3$.

Veronese embedding

An important subvariety of $\mathbb{P}^n \times \mathbb{P}^n$ is the *diagonal* which consists of the points (a, a) and corresponds to the ideal $(x_0 - y_0, \dots, x_n - y_n) \in k[\mathbb{P}^n \times \mathbb{P}^n]$. It is isomorphic to $\mathbb{P}^n$ and given by the image of the morphism $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ defined by $a \mapsto (a, a)$.

Definition 4.17 Using the Segre embedding $\iota : \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(n+1)^2-1}$ and the diagonal, we get an embedding $\iota \circ \Delta : \mathbb{P}^n \hookrightarrow \mathbb{P}^{(n+1)^2-1}$. This is called the *Veronese* embedding.

Remark 4.18 The coordinate ring of the Veronese embedding of $\mathbb{P}^n$ can be written as $k[x_0x_0, x_0x_1, \dots, x_nx_n]$ where x_ix_j has degree one.

Example 4.19 The embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ by the Veronese embedding is given by the ideal $(x_{01} - x_{10}, x_{00}x_{11} - x_{01}x_{10}) \subset k[x_{00}, x_{01}, x_{10}, x_{11}]$.

Identifying $\mathbb{P}^n$ with the subset $(a, a, \dots, a)$ of $(\mathbb{P}^n)^d$ and repeatedly using the Segre embedding, we get an isomorphism between $\mathbb{P}^n$ and a subvariety of $\mathbb{P}^{(n+1)^d-1}$, the d-uple Veronese embedding.

Chapter 5

Geometrically Integral Varieties

Everything in this chapter applies equally to both affine and projective varieties even if $V_K(\mathfrak{a})$ and $\mathfrak{I}_k(V)$ is used and not $VP_K(\mathfrak{a})$ and $\mathfrak{I}\mathfrak{H}_k(V)$.

Base extensions

So far we have only used a fixed base field k. If k'/k is a field extension contained in K and V is a k-variety, we can define a k'-variety $V_{(k')}$ by using the ideal $\mathfrak{I}_k(V)k'[x]$. Note that the varieties V and $V_{(k')}$ are identical as *sets* of $\mathbb{A}^n$ or $\mathbb{P}^n$ and consequently $\mathfrak{I}_{k'}(V) = \mathfrak{I}_{k'}(V_{(k')})$. The difference is that they have different coordinate rings and different *topologies*. Thus, even though V is irreducible, the extended variety $V_{(k')}$ may be non-irreducible as example 1.24 demonstrated.

As we also saw in example 4.4, the product of two irreducible k-varieties need not be an irreducible k-variety. Both these shortcomings disappear with the notion of geometrical integral varieties.

Notation 5.1 The algebraic closure of a field F is denoted $\overline{F}$.

Definition 5.2 A k-variety V is *geometrically irreducible* if $V_{(\overline{k})}$ is irreducible.

Remark 5.3 Note that the corresponding ideal to $V_{(k')}$ need not be $\mathfrak{I}_k(V)k'[x]$ even though it is defined by it. By Hilbert's Nullstellensatz we have that $\mathfrak{I}_{k'}\left(V_{(k')}\right) = \mathfrak{r}\big(\mathfrak{I}_k(V)k'[x]\big)$. Thus V is geometrically irreducible if and only if $\mathfrak{r}\big(\mathfrak{I}_k(V)\overline{k}\big)$ is prime, i.e. $\mathfrak{I}_k(V)\overline{k}$ is primary or equivalently that all zero divisors of $k[V] \otimes \overline{k}$ are nilpotent.

Definition 5.4 A k-variety V is *geometrically reduced* if $k[V] \otimes \overline{k}$ is reduced or equivalently $\mathfrak{I}_k(V)\overline{k}$ is a radical ideal.

Definition 5.5 A k-variety V is *geometrically integral* if $k[V] \otimes \overline{k}$ is an integral domain or equivalently $\mathfrak{I}_k(V)\overline{k}$ is a prime ideal.

To conclude, a k-variety V is geometrically irreducible (reduced, integral) if $\mathfrak{I}_k(V)\overline{k}$ is a primary (radical, prime) ideal. Note that a both geometrically irreducible and reduced variety is geometrically integral.

Further V is geometrically irreducible (reduced, integral) if and only if $\mathfrak{I}_k(V)k'$ is a primary (radical, prime) ideal for all extensions k'/k.

Proposition 5.6 *If k is* perfect, *then every k-variety V is geometrically reduced and hence every geometrically irreducible k-variety is geometrically integral.*

Proof. We want to show that $k[V] \otimes_k \overline{k}$ is reduced. Since $k[V]$ is a finitely generated reduced k-algebra it is semisimple. The tensor product of a semisimple algebra and a separable extension over k is semisimple by [L1, Ch. XVII, Thm 6.2] and thus $k[V] \otimes_k \overline{k}$ is reduced. $\square$

Example 5.7 Let $k = \mathbb{F}_p(t^p)$ and $k' = \mathbb{F}_p(t)$. Consider the k-variety $V_K(x^p - t^p)$ and its extension $V_{(k')}$. The corresponding ideals to V and $V_{(k')}$ is $x^p - t^p$ and $x - t$ respectively. It is clear that V is geometrically irreducible but not integral and that $V_{(k')}$ is geometrically integral.

BASE RESTRICTION

If k'/k is a field extension and V a k'-variety, we can restrict V to a k-variety $V_{[k]}$ by restricting the ideal $\mathfrak{I}_{k'}(V) \subseteq k'[x]$ to $k[x]$. Thus $V_{[k]}$ corresponds to $\mathfrak{I}_{k'}(V) \cap k[x]$ which is a radical ideal.

A base extension never changes the variety as a *set* of $\mathbb{A}^n$. A base restriction may however result in a bigger set. In fact, the restriction $V_{[k]}$ is the *closure* of V in the k-Zariski topology and if k'/k is an algebraic extension, it consists of V and all its conjugates over k as we will see in proposition 5.42.

Even though V and $V_{[k]}$ are not necessarily equal as sets, we have that $\mathfrak{I}_k(V) = \mathfrak{I}_k(V_{[k]})$ since the ideal of a set and its closure are equal.

LINEAR DISJOINTNESS

Definition 5.8 Let F be a field and A and B integral domains over F. A field extension Ω/F is a *common extension* for A and B if there exists injective F-algebra homomorphisms from A and B to Ω.

Definition 5.9 Let A and B be integral domains over F. We say that A is *linearly disjoint* from B over F if there is a common extension Ω such that every set of elements in A which are linearly independent over F also are linearly independent over B in Ω.

Remark 5.10 Note that the choice of Ω is important. If $A = F(x)$ and $B = F(y)$ we can either let $\Omega = F(x, y)$ or $\Omega = F(x)$ with $A = B$ in Ω. In the first case a linearly independent set of elements in A over F remains linearly independent in Ω over B, but not in the latter.

Proposition 5.11 *Let A and B be integral domains over F. Then the following conditions are equivalent.*

(i) *A is linearly disjoint from B over F.*

(ii) *The canonical map $A \otimes_F B \to \Omega$ defined by $a \otimes_F b \mapsto ab$ is injective for some common extension Ω of A and B.*

(iii) *$A \otimes_F B$ is an integral domain.*

Proof. (i) $\Longrightarrow$ (ii): Suppose that A is linearly disjoint from B over F in some common extension Ω. Let $(x_\alpha)_{\alpha \in \mathscr{I}}$ be a basis for A over F. Then every element f of $A \otimes_F B$ is of the form $\sum_\alpha x_\alpha \otimes_F f_\alpha$ for some elements $f_\alpha \in B$. Since the basis (x_α) is linearly independent over F it is by the linear disjointness also linearly independent over B and the image of f by the map in (ii) is not zero and hence the map is injective.

(ii) $\Longrightarrow$ (i): Conversely assume that the map is injective for some extension Ω. Let $\{a_i\}$ be a linearly independent subset of A over F. Let b_i be elements of B such that $\sum a_i b_i = 0$ in Ω. Then the element $\sum a_i \otimes_F b_i$ is mapped to zero in Ω and thus by injectivity $\sum a_i \otimes_F b_i = 0$. Since $\{a_i\}$ are linearly independent over F all the b_i are zero and thus the a_i are linearly independent over B.

(ii) $\Longleftrightarrow$ (iii): Since Ω is a field, (ii) implies (iii). Conversely if $A \otimes_F B$ is an integral domain we can choose Ω to be the quotient field of $A \otimes_F B$ and (ii) holds. $\square$

Remark 5.12 The notion of linear disjointness is symmetric, i.e. A is linearly disjoint from B over F if and only if B is linearly disjoint from A over F. In fact, criterion (ii) of proposition 5.11 is symmetric in A and B. We will therefore say that A and B are linearly disjoint over F.

Definition 5.13 If K/F and L/F are linearly disjoint fields extensions over F we will by KL denote the quotient field of $K \otimes_F L$.

Proposition 5.14 *Let K/F and L/F be linearly disjoint field extensions of F. If K or L is algebraic over F, then $KL = K \otimes_F L$.*

Proof. Assume $L = F(\alpha)$ is a simple field extension of inseparability degree p^f and separability degree n. Let $a = \sum_{i=1}^n k_i \otimes_F l_i \in K \otimes_F L$. Then $a^{p^f} \in K \otimes_F F_{\mathrm{s}}$ where F_{s} is the separable closure of F. Further $a^{p^f} a_1 a_2 \ldots a_{n-1} \in K \otimes_F F = K$ where $a_1, a_2, \ldots, a_{n-1}$ is the conjugates of a^{p^f} over F. Thus a is invertible in $K \otimes_F L$.

Now consider any algebraic extension L over F and let $a = \sum_{i=1}^n k_i \otimes_F l_i$. Then $L' = F(l_1, l_2, \ldots, l_n)$ is a finite algebraic extension of F and $a \in K \otimes_F L'$. By induction on the number of generators for L' we have that a is invertible in $K \otimes_F L$. $\square$

Proposition 5.15 *Let A and B be integral domains over a field F and let K and L be the quotient fields. Then A and B are linearly disjoint over F if and only if K and L are linearly disjoint over F.*

Proof. It is clear that A and B are linearly disjoint over F if K and L are linearly disjoint. Assume that A and B are linearly disjoint. Let Ω be the quotient field of $A \otimes_F B$. Then Ω is a common extension for K and L. Let $\{k_i\}_{i=1}^n$ be elements of K, linearly independent over F. Assume that there are elements $\{l_i\}$ of L such that $\sum_{i=1}^n k_i l_i = 0$ in Ω. Let $a \in A$ and $b \in B$ be non-zero elements such that $ak_i \in A$ and $bl_i \in B$. Then $\{ak_i\}$ are linearly independent over F and by linear disjointness also linearly independent over L. Since $\sum_{i=1}^n (ak_i)(bl_i) = 0$ we thus have that $bl_i = 0$ and $l_i = 0$ which proves that K is linearly disjoint from L. $\square$

Regular extensions and absolute primes

Definition 5.16 If E/F is a field extension, then we say that E is a *regular extension* of F if F is algebraically closed in E and E/F is separable. Recall that E/F is separable if there exists a separating transcendence basis, i.e. a transcendence basis $\alpha_1, \ldots, \alpha_s$ such that $E/F(\alpha_1, \ldots, \alpha_s)$ is separable.

Remark 5.17 If K/F is a separable extension and E is a subfield of K containing F, then E/F is separable (cf. [L1, Ch. VIII, Cor. 4.2]). Further, if K/E and E/F are separable extensions, the composite K/F is a separable extension (cf. [L1, Ch. VIII, Cor. 4.3]). Hence it follows that a subfield of a regular extension E/F is regular and that the composite of two regular extensions is a regular extension.

Example 5.18 Let $F = \mathbb{F}_p(t^p, u^p)$ and let E be the fraction field of the integral domain $F[x,y,z]/(z^p - t^p x^p - u^p y^p)$. Then F is algebraically closed in E but E/F is not separable. That there does not exist a separating transcendence basis is implicitly shown by example 5.23.

Definition 5.19 A prime ideal $\mathfrak{p}$ of $k[x]$ is *absolutely prime* if the ideal generated by $\mathfrak{p}$ in $k'[x]$ is prime for all field extensions k'/k.

Example 5.20 The prime ideal $(x^2 - 2)$ in $\mathbb{Q}$ is not absolutely prime since it is not prime in the algebraic closure of $\mathbb{Q}$.

Example 5.21 The prime ideal $(x^p - t^p)$ in $\mathbb{F}_p(t^p)$ is not absolutely prime since it is not prime in the extension $\mathbb{F}_p(t)$.

Proposition 5.22 *Let E/F be a field extension. Then the following conditions are equivalent.*

(i) *E is a regular extension of F.*

(ii) *E and $\overline{F}$ are linearly disjoint over F.*

(iii) *E and K are linearly disjoint over F for all extensions K/F.*

(iv) *$E \otimes_F \overline{F}$ is an integral domain.*

Proof. The equivalence between (i) and (ii) follows from [L1, Ch. VIII, Lemma 4.10]. The equivalence between (ii) and (iv) follows from proposition 5.11. The implication (iii) $\Longrightarrow$ (ii) is obvious. For the reverse implication first note that if E are linear disjoint from $\overline{F}$ over F then by the definition of linear disjointness E are linear disjoint from K over F for any algebraic extensions K/F. Further if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ is a transcendence basis for K over F then clearly $E(\alpha)$ is a regular extension over $F(\alpha)$ and thus $E(\alpha) \otimes_{F(\alpha)} K$ is an integral domain and a fortiori also the subring $E \otimes_F F(\alpha) \otimes_{F(\alpha)} K = E \otimes_F K$.

Now assume that E/F is a regular extension but that E is not linearly disjoint from K or equivalently, by proposition 5.11 that $E \otimes_F K$ is not an integral domain. Then there are non-zero elements $a = \sum_i e_i \otimes_F k_i$ and $a' = \sum_i e'_i \otimes_F k'_i$ such that $aa' = 0$. Then $E \otimes_F F(k_i, k'_i)$ is also not an integral domain. But $F(k_i, k'_i)$ is finitely generated and thus there exists a transcendence basis $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ over F which is a contradiction. Thus E is linearly disjoint from K. $\square$

Example 5.23 Continuing example 5.18 we have that $t, u \in \overline{F}$. The ring $E \otimes_F \overline{F}$ is not an integral domain. In fact, $(z \otimes_F 1 - x \otimes_F t - y \otimes_F u)^p = 0$. Thus, the extension E/F is not regular.

Remark 5.24 Let V be an irreducible k-variety with prime ideal $\mathfrak{p} = \mathfrak{I}_k(V)$. Then $V_{(k')}$ is a k'-variety for any field extension k'/k and its ideal is $\mathfrak{r}(\mathfrak{p}k'[x])$. If $k[V] \otimes_k k' = k'[x]/\mathfrak{p}k'[x]$ is an integral domain, then $\mathfrak{p}k'[x]$ is a prime ideal and in particular radical. Thus $k'[V_{(k')}] = k[V] \otimes_k k'$ and $V_{(k')}$ is irreducible.

Moreover $k[V] \otimes_k k'$ is an integral domain if and only if $k(V)$ and k' are linearly disjoint over k by propositions 5.11 and 5.15. If this is the case the function field of $V_{(k')}$ is $k'(V_{(k')}) = k(V)k'$.

Theorem 5.25 *Let V be an irreducible k-variety. Then $k(V)$ is a regular extension of k if and only if $\mathfrak{I}_k(V)$ is absolutely prime.*

Proof. Let $\mathfrak{p}$ be the prime ideal corresponding to V. By remark 5.24 the ring $k[V] \otimes_k k'$ is an integral domain if and only if $k(V)$ and k' are linearly disjoint over k. Since $\mathfrak{p}$ is absolutely prime if and only if $k[V] \otimes_k k'$ is an integral domain for all k' the theorem follows by proposition 5.22. □

Geometrically integral varieties

By the definition of absolutely prime, a k-variety V is geometrically integral if and only if $\mathfrak{I}_k(V)$ is absolutely prime, or using the equivalence of theorem 5.25, if and only if $k(V)$ is a regular extension of k.

Example 5.26 Let $k = \mathbb{F}_p(t^p, u^p)$ and let V be the affine irreducible k-variety defined by the prime ideal $\mathfrak{p} = (z^p - t^p x^p - u^p y^p)$ in $k[x, y, z]$. Since $\mathfrak{p}\overline{k}[x, y, z] = \big((z - tx - uy)^p\big)$ the prime ideal $\mathfrak{p}$ is not absolutely prime. The variety V is thus not geometrically integral, but it is geometrically irreducible since $\mathfrak{r}(\mathfrak{p}\overline{k}[x, y, z]) = (z - tx - uy)$ which is prime. Note that k and $k(V)$ are the fields k and k' of examples 5.18 and 5.23 and that $k(V)$ is not a regular extension of k.

Theorem 5.27 *A k-variety V is geometrically irreducible if and only if every element in $r \in k(V)$ is transcendent over k or $r^{p^d} \in k$ for some $d \in \mathbb{N}$, i.e. every element of $k(V)$ is either in the inseparable closure $k^{p^{-\infty}}$ or transcendent over it.*

Proof. See [S, p. 32]. □

Example 5.28 With the same field $k = \mathbb{F}_p(t^p)$ as in the previous example, we have that the k-variety $V_K(x^p - t^p)$ is geometrically irreducible. In fact the function field is $\mathbb{F}_p(t)$ which is contained in the inseparable closure of k. As in the previous example, the variety is not geometrically integral.

Remark 5.29 When Weil defines varieties in [W], he starts with geometrically integral k-varieties. These are the varieties that are easiest to deal with and in some sense it is possible to only deal with geometrically integral varieties. In fact, as we will see in chapter 7, an arbitrary k-variety can be represented by a cycle of geometrically integral $\overline{k}$-varieties.

Notation 5.30 Geometrically irreducible and integral k-varieties are called absolute k-varieties and absolute varieties defined on k respectively, or simply varieties by Weil [W], Samuel [S] and other classical authors. The use of geometrically irreducible/reduced/integral is consistent with Grothendieck's terminology [EGA, Ch. IV:2, Def. 4.5.2 and Def. 4.6.2].

Remark 5.31 If V is a geometrically irreducible k-variety, then $V_{(\overline{k})}$ is geometrically integral. Further every irreducible $\overline{k}$-variety is geometrically integral. In fact, by proposition 5.22, part (iv), every extension of an algebraically closed field is regular.

Remark 5.32 Let V be a geometrically integral k-variety. Then $V_{(k')}$ is a geometrically integral k-variety for all field extensions k'/k. In fact $k(V) \otimes_k \overline{k'} = k(V) \otimes_k k' \otimes_{k'} \overline{k'}$ is an integral domain by proposition 5.22 which according to propositions 5.11 and 5.15 implies that $k'(V) = k(V)k'$ and $\overline{k'}$ are linearly disjoint over k'.

Definition 5.33 Let V be a geometrically integral k-variety. If k'/k is a field extension, or k' is a subfield of k such that $V_{[k']}$ is geometrically integral, we say that V is *defined on* k' or that k' is a *field of definition*.

Remark 5.34 If V is a k-variety and k' is a field of definition, then V is an irreducible closed set in the k'-Zariski topology. Further $\mathfrak{I}_k(V)k''[x] = \mathfrak{I}_{k'}(V)k''[x]$ for a common extension k'' of k and k'.

Proposition 5.35 *If V is a geometrically integral k-variety, there is a unique minimal subfield k_0 of k such that V is defined on k_0. Further k_0 is a finitely generated extension of the prime field.*

Proof. Let $\mathfrak{p} \in k[x]$ be the ideal of V. By [W, Ch. I, §7, Lemma 2] there is a smallest subfield k_0 of k such that $\mathfrak{p}$ is generated by elements in $k_0[x]$. Further by [W, Ch. I, §8, Thm 7], the field k_0 is a field of definition for V and finally by [W, Ch. IV §1, Cor. 3] every field of definition for V contains k_0, which thus is the smallest field of definition for V. Finally it is clear that k_0 is a finitely generated extension of the prime field. □

Definition 5.36 Given a geometrically integral variety V, we will denote the smallest field of definition $\mathrm{def}(V)$. If $\mathrm{def}(V)$ is the prime field, i.e. the smallest subfield of K which is either $\mathbb{Q}$ or $\mathbb{F}_p$, we say that the variety is *universal*.

Proposition 5.37 *If an affine variety is geometrically integral, its projective closure is geometrically integral. Similarly the cone of an geometrically integral projective variety is geometrically integral.*

Proof. The projective closure of an affine k-variety V is an irreducible k-variety if and only V is irreducible, which proves the first part, since base extensions commutes with taking the projective closure. Similarly the cone of a projective k-variety V is irreducible if and only if V is irreducible since the defining ideal is the same. □

Proposition 5.38 *If $f : V \to Y$ is a k-morphism and V a geometrically integral k-variety, then $W = \overline{f(V)}$ is a geometrically integral k-variety.*

Proof. By remark 3.4, every k-morphism between irreducible varieties gives an injection $k(W) \hookrightarrow k(V)$ which makes $k(W)$ a subfield of $k(V)$. Since a subfield of a regular extension is regular by remark 5.17, the image W is a geometrically integral k-variety. □

Proposition 5.39 *If V and W are both geometrically integral k-varieties, their product $V \times W$ is a geometrically integral k-variety.*

Proof. By proposition 5.22 part (iii) the function fields $k(V)$ and $k(W)$ are linearly disjoint over k and thus $k[V \times W] = k[V] \otimes_k k[W]$ is an integral domain and $V \times W$ is an irreducible k-variety. Note that proposition 5.22 only requires that one of V and W is geometrically integral. By remark 5.32 we have that $k(W)\big(V_{(k(W))}\big)$ is a regular extension of $k(W)$. Thus by the transitivity of regular extensions, see remark 5.17, the function field of the product $k(V \times W) = k(W)\big(V_{(k(W))}\big)$ is a regular extension of k and hence $V \times W$ is a geometrically integral k-variety. □

Definition 5.40 Let V be a k-variety. The k'-components of V are the components of the k'-variety $V_{(k')}$, i.e. they correspond to the minimal primes of $\mathfrak{I}_k(V)k'[x]$. The *geometrical* components of V are the $\bar{k}$-components which are geometrically integral varieties.

Definition 5.41 Let k'/k be a field extension. If V and W are k'-varieties we say that they are *conjugate* over k if there is a k-automorphism $s \in \mathrm{Gal}(K/k)$ of K such that $s(V) = W$ or more precisely that every point of W is of the form $\big(s(x_1), s(x_2), \ldots, s(x_n)\big)$ where $(x_1, x_2, \ldots, x_n)$ is a point of V.

Proposition 5.42 *Let k'/k be an algebraic field extension and V a k'-variety. The k-variety $V_{[k]}$ is then the union of V and its conjugates over k.*

Proof. Let $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m$ be a system of equations for V. The conjugate varieties of V over k is given by the conjugates of $\mathfrak{F}_i$ and there is a k-automorphism s of $k[x]$ such that the conjugates of V is defined by $s^j(\mathfrak{F}_1), s^j(\mathfrak{F}_2), \ldots, s^j(\mathfrak{F}_q)$, $j = 0, 1, \ldots, q-1$ where $s^q = \mathrm{id}$. Let W be the variety defined by the equations

$$\left(\sum_{\substack{(j_a)\text{ cyclic}\\ \text{perm. of }(i_a)}} s^0(\mathfrak{F}_{j_0}) s^1(\mathfrak{F}_{j_1}) \ldots s^{q-1}(\mathfrak{F}_{j_{q-1}})\right)^{p^f} = 0, \quad 1 \le i_a \le m,\ a = 0, 1, \ldots, q-1$$

where the sums are over the *different* cyclic permutations of $(i_0, i_1, \ldots, i_{q-1})$ and p^f is a power of the characteristic such that the coefficients of $\mathfrak{F}_i^{p^f}$ are separable over k. First note that all of the equations are invariant under s and thus is elements of $k[x]$. Secondly $s^j(V)$ is contained in W. Finally $W = \bigcup_{j=0}^{q-1} s^j(V)$. In fact, assume that there is a point $x \in W$ such that $x \notin s^j(V)$ for all j and choose for every a the smallest integer i_a such that $s^a(\mathfrak{F}_{i_a})[x] \neq 0$. All cyclic permutations (j_a) of (i_a) which are not equal to (i_a) has a component $j_a < i_a$ for which $s^a(\mathfrak{F}_{j_a})[x] = 0$ by the definition of the i_a. Thus the equation corresponding to (i_a) gives $s^a(\mathfrak{F}_{i_a})[x] = 0$ for all a which contradicts the existence of such i_a's.

This shows that the union of V and its conjugates is a variety W which is defined by an ideal generated by elements of k. It can thus be restricted to a k-variety $W_{[k]}$ which is the same variety as W in the sense that $\big(W_{[k]}\big)_{(k')} = W$. Further $V_{[k]} = W_{[k]}$ since $V_{[k]}$ contains all the conjugates of V and is the smallest k-variety containing V. □

Corollary 5.43 *Let V be a k-variety and k'/k a field extension. If C is a k'-component of V then every conjugate $s(C)$ is a k'-component of V. Further if V is irreducible, then all the k'-components are conjugates over k.*

Proof. The first part is trivial since the equations for V are invariant under s. If V is irreducible and C is a k'-component, then $C_{[k]}$ is an irreducible k-variety consisting of C and its conjugates over k by proposition 5.42. Since $C_{[k]} \subseteq V$ are two irreducible varieties of the same dimension we have that $V = C_{[k]}$, i.e. V consists of C and its conjugates. □

Chapter 6

Geometric Properties

INTERSECTIONS

Notation 6.1 We will often write V^r to denote a variety of dimension r. Further we will use L^r to denote a linear variety of dimension r, i.e. a intersection of $n-r$ independent hyperplanes in $\mathbb{A}^n$ or $\mathbb{P}^n$.

Theorem 6.2 (Dimension Theorem) *Let V^r and W^s be affine or projective k-varieties. Every component of $V \cap W$ has at least dimension $r+s-n$. If V and W are* projective *and $r+s-n \geq 0$, the intersection $V \cap W$ is not empty.*

Proof. See [S, p. 22-24] or [Ha, Ch. I, 7.1, 7.2]. □

Corollary 6.3 *Let $V^r \subset W^s$ be affine or projective irreducible k-varieties. Then there is a chain of varieties $W = W_0 \supset W_1 \supset \cdots \supset W_{s-r} = V$.*

Proof. We have that $\mathfrak{I}_K(W) \subset \mathfrak{I}_K(V)$. Choose an element $f \in \mathfrak{I}_K(V) \setminus \mathfrak{I}_K(W)$ and define the hypersurface $H = V_K\big(\{f\}\big)$. Then $V \subseteq H \cap W \subset W$ and we have that $\dim(H \cap W) = s-1$. In fact, the corresponding ideal of $H \cap W$ is generated by the ideal of W and the element f and the dimension of $H \cap W$ is therefore at least $\dim(W) - 1$. By theorem 1.33, the dimension of $H \cap W$ is less than the dimension of W since $H \cap W \neq W$. Now let W_1 be one of the irreducible components of $H \cap W$. The corollary then follows by induction on the dimension of W. □

Corollary 6.4 *The combinatorial dimension equals the dimension.*

Proof. Let V be a k-variety. We have already seen that $\dim_{comb}(V) \leq \dim(V)$ in corollary 1.34. The previous corollary establishes the converse inclusion using a component of maximum dimension of V and any zero-dimensional subvariety of the component. □

Lemma 6.5 *Let $V \subset \mathbb{A}^n$ or $V \subset \mathbb{P}^n$ be a proper k-variety. If V is irreducible or k is infinite, there is a k-hyperplane L^{n-1} such that L does not contain any component of V. Further, if $a \notin V$ is a k-rational point, there is a k-hyperplane passing through a which does not contain any component of V.*

Proof. Taking the projective closure we can assume we are in projective space. If a is not chosen, take any k-rational point a not in V. The requirement that V is irreducible or k is infinite guarantees that such a point exists. The ideal $\mathfrak{a} = \mathfrak{I}_k(\{a\})$ of $a = (a_0 : a_1 : \cdots : a_n)$ is generated by n elements of the form $a_j x_i - a_i x_j$. Let $\{V_i\}$ be the components of V. The degree-one part of the ideal $\mathfrak{b}_i = \mathfrak{I}_k(V_i)$ is generated by at most $n-1$ independent elements since V is proper, i.e. $\mathfrak{b}_i \neq \mathfrak{a}^+$. The possible hypersurfaces are the degree-one elements of the set $\mathfrak{a} \setminus \bigcup_i \mathfrak{b}_i$. If V is irreducible, i.e. there is only one component, or k is infinite this set is not empty since the degree-one part of $\mathfrak{a}$ is generated by more elements than the degree-one part of $\mathfrak{b}_i$. □

Proposition 6.6 *Let V^r be a projective k-variety, $a \notin V$ a k-rational point, and s a positive integer. If k is* infinite, *there is a linear k-variety L^{n-s} containing a such that* $\dim(V \cap L^{n-r}) = r - s$. *In particular $V \cap L^{n-r}$ is reduced to a finite number of points and $V \cap L^{n-r-1}$ is empty.*

Proof. The proposition follows immediately from lemma 6.5 using induction on s. In fact, if L^{n-1} does not contain any component of V^r then $V \cap L$ has dimension strictly less than r. Thus there is a L^{n-s} containing a such that $\dim(V \cap L^{n-r}) \leq r - s$ which by theorem 6.2 is an equality. □

Generic linear varieties and projections

Previously we have not used any specific properties in K other than it being algebraically closed over k. In this section and the following we will often let K include elements which are transcendental over k. We will also let k' be an extension of k by transcendental elements. In this case we will not distinguish a k-variety V from the k'-variety $V_{(k')}$ since $V_{(k')}$ is irreducible if and only if V is irreducible, even if V is not geometrically irreducible.

Remark 6.7 A hyperplane in $\mathbb{P}^n$ is defined by a homogeneous equation $\sum_{i=0}^n a_i x_i = 0$ and can thus be represented as a point $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$. Similarly a set of r hyperplanes, or equivalently a linear variety L^{n-r}, can be represented as a point in $(\mathbb{P}^n)^r = \mathbb{P}^n \times \cdots \times \mathbb{P}^n$.

Remark 6.8 A projection from $V \subseteq \mathbb{P}^n$ to $\mathbb{P}^m$ is essentially given by its center which is a linear variety of dimension $n - m - 1$. In fact, two projections with the same center are isomorphic.

Definition 6.9 An affine point $(a_1, a_2, \ldots, a_n)$ is *generic* over k if all its coordinates a_i are algebraically independent over k. A projective point $(a_0 : a_1 : \cdots : a_n)$ is generic over k if all the quotients a_i/a_j, $i = 0, 1, \ldots, n$ for some a_j, are algebraically independent. Equivalently, the point is generic over k after changing to affine coordinates. Note that all the coordinates of a generic point are non-zero.

Definition 6.10 A linear variety L^{n-r} of $\mathbb{P}^n$ is *generic* over k if it is generic over k as a point in $(\mathbb{P}^n)^r$.

Definition 6.11 A k'-projection $f : \mathbb{P}^n \to \mathbb{P}^m$ is *generic* over k if its center is generic over k as a point in $(\mathbb{P}^n)^{m+1}$.

Notation 6.12 If $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$ is a point we will use the notation $k_a = k(a) = k(\{a_i/a_j\}_{i=0}^n)$. Similarly if $u_1, u_2, \ldots, u_r$ are r points $u_s = (u_{s0} : u_{s1} : \cdots : u_{sn})$ we will write $k_u = k(u) = k(u_1, u_2, \dots, u_r) = k(u_{si}/u_{sj}) = k\left(\{u_{si}/u_{sj}\}_{s=1,\dots,r;\, i=0,\dots,n}\right)$. Note that in most cases the points u_s represent a linear variety of dimension $n - r$.

Proposition 6.13 *Let V^r be a k-variety of $\mathbb{P}^n$ and L^m a generic linear variety. Then V and L* intersect properly, *i.e. $V \cap L$ has dimension $r + m - n$ and is non-empty when $r + m = n$.*

Proof. First we prove that the intersection of a k-variety and a k-generic hyperplane L^{n-1} intersect properly, i.e. L does not contain V. Let L be given by the equation $h_0x_0 + \cdots + h_nx_n = 0$. If a is a point of L, then $h_0a_0 + \cdots + h_na_n = 0$. Since all h_i are algebraically independent over k, there is a quotient a_i/a_j which is not algebraic over k. Since by Hilbert's Nullstellensatz there are points in V with coordinates in $\overline{k}$, the hyperplane L cannot contain V. Thus $V \cap L^{n-1}$ has dimension $n - 1$ by theorem 6.2. Also see proposition 6.15.

We have that $L = L_1 \cap L_2 \cap \cdots \cap L_{n-m}$ where $L_j = VP_K(u_{j0}x_0 + \cdots + u_{jn}x_n)$ are generic hyperplanes. By the above discussion $V \cap L_1$ has dimension $r - 1$. This intersection is a $k(u_{10}, u_{11}, \dots, u_{1n})$-variety and since L is generic over k, the linear variety $L_2 \cap \cdots \cap L_{n-m}$ is generic over $k(u_{10}, u_{11}, \dots, u_{1n})$. The proposition then follows by induction on the dimension of L. □

Remark 6.14 Let V^r be a k-variety of $\mathbb{P}^n$. By proposition 6.13 a generic linear variety L^m intersects V exactly when $m \geq n - r$. Thus a generic projection from $\mathbb{P}^n$ to $\mathbb{P}^m$ induces a morphism from V^r to $\mathbb{P}^m$ if and only if $m \geq r$.

Proposition 6.15 *Let L^{n-r} be a generic linear variety over k of codimension r given by $u_{si} \in (\mathbb{P}^n)^r$. Then all points in L has at least transcendence degree r over k, that is* $\operatorname{tr.deg}\big(k(a)/k\big) \geq r$ *for all $a \in L$.*

Proof. Let a be a point of L^{n-r} and if L is projective, choose an hyperplane at infinity not containing a and use affine coordinates. This gives us the relations $u_{s0} = u_{s1}a_1 + \cdots + u_{sn}a_n$ for $s = 1, 2, \dots, r$. The r elements u_{s0} which are transcendental over $k(u_{s1}, u_{s2}, \dots, u_{sn})$ are thus in $k(u_{s1}, u_{s2}, \dots, u_{sn}, a)$ and $k(a)/k$ has at least transcendence degree r. □

Generic points

Definition 6.16 Let V^r be an affine (or projective) irreducible k-variety. A point $\xi \in V$ is a *generic point of V* if $k(\xi) = k(\xi_i)$ (or $k(\xi) = k(\xi_i/\xi_j)$ in the projective case) has transcendence degree r over k.

Remark 6.17 Let V^r be an irreducible k-variety in $\mathbb{A}^n$ or $\mathbb{P}^n$ and let as usual v_i be the image of x_i by the quotient map $k[\mathbb{A}^n] = k[x_1, x_2, \dots, x_n] \twoheadrightarrow k[V]$ or $k[\mathbb{P}^n] = k[x_0, x_1, \dots, x_n] \twoheadrightarrow k[V]$. Letting K include $k(V)$ we can thus see $v = (v_1, v_2, \dots, v_n)$ or $v = (v_0 : v_1 : \cdots : v_n)$ as a point in V. Since $k(v) = k(V)$ it is clearly a generic point of V.

Definition 6.18 The irreducible k-variety defined by the polynomials which are zero on a point a are called the variety *generated* by a and is denoted $\overline{\{a\}}$. It is the smallest k-variety containing a.

Proposition 6.19 *The irreducible k-variety $V = \overline{\{\xi\}}$ generated by a point $\xi \in \mathbb{A}^n$ or $\xi \in \mathbb{P}^n$ of transcendence degree r over k is an r-dimensional irreducible k-variety. Further ξ is a generic point of V and $k(\xi) = k(V)$.*

Proof. For any polynomial $P \in k[t_0, t_1, \ldots, t_n]$ we have that $P(\xi_0, \xi_1, \ldots, \xi_n) = 0$ if and only if $P(v_0, v_1, \ldots, v_n) = 0$ in $k[V]$. We have thus an isomorphism between $k(\xi)$ and $k(V)$ induced by $\xi_i \mapsto v_i$ which proves the statements. □

Corollary 6.20 *Let ξ be a generic point of V. Then $\mathfrak{I}\mathfrak{H}_k(V) = \mathfrak{I}\mathfrak{H}_k(\{\xi\})$. Every point a of V is thus a* specialization *of ξ, i.e. if $f(\xi) = 0$ for a polynomial $f \in k[x]$ then $f(a) = 0$.*

Proof. The variety $W = \overline{\{\xi\}} = V_K(\mathfrak{I}\mathfrak{H}_k(\{\xi\}))$ generated by ξ is clearly contained in V. But W has the same dimension as V and thus $V = W$. □

Noether's normalization lemma

Theorem 6.21 (Noether's Normalization Lemma) *Let A be a finitely generated integral domain over k. If the quotient field of A has transcendence degree r over k there exists algebraically independent elements $y_1, y_2, \ldots, y_r$ in A such that A is integral over $k[y_1, y_2, \ldots, y_r]$. If k is infinite the elements $y_1, y_2, \ldots, y_r$ may be chosen as linear combinations of a generating set of A.*

Proof. See [L1, Ch. VIII, Thm 2.1] or [Mu, Ch. I, §1] for a proof which holds even when k is finite. A simpler proof when k is infinite which also shows that y_i can be chosen as linear combinations can be found in [AM, p. 69] or [S, p. 18-19]. □

Remark 6.22 If V is an affine irreducible k-variety of dimension r, Noether's normalization lemma says that there exists algebraically independent elements $y_1, y_2, \ldots, y_r$ in $k[V]$ such that $k[V]$ is integral over $k[y_1, y_2, \ldots, y_r]$.

A more, in our case, useful version of the Normalization Lemma is the following theorem.

Theorem 6.23 *Let $V \subseteq \mathbb{A}^n$ be an irreducible k-variety of dimension r with coordinate ring $k[V] = k[v_1, v_2, \ldots, v_n]$. Let $m \geq r$ and $(u_{si})_{1 \leq s \leq m, 1 \leq i \leq n}$ be mn algebraically independent elements over k and let $k_u = k(u) = k(u_{si})$. Define the change of coordinates $y_s = \sum_{i=1}^n u_{si} v_i$, $s = 1, 2, \ldots, m$. Then $k_u[V]$ is integral over $k_u[y_1, y_2, \ldots, y_r]$.*

Proof. See [L1, Ch. VIII, Thm 2.2] and remark 6.27. □

Remark 6.24 The Normalization Lemma 6.21 and its variant 6.23 also holds for projective spaces. In fact, if V is a projective irreducible k-variety of dimension r, apply the Normalization Lemma on its cone, which has dimension $r + 1$. Since the coordinate ring of V and that of its cone are identical, the result is the same except that we need $r + 1$ elements instead of r. Note that if k is finite, the elements $y_0, y_1, \ldots, y_r$ need not be homogeneous, but if k is infinite there exists homogeneous elements $y_0, y_1, \ldots, y_r$ of degree one.

Remark 6.25 A direct consequence of theorem 6.23 in the projective case is that given a *generic* projection $f : V^r \to \mathbb{P}^m$ with coefficients u of a projective irreducible variety V into $\mathbb{P}^m$ with image $W = f(V)$, the ring $k_u[W] = k_u[w_0, w_1, \ldots, w_m]$ is integral over $k_u[w_0, w_1, \ldots, w_r]$. In fact $k_u[W]$ is a subring of $k_u[V]$ which is integral over $k_u[w_0, w_1, w_2, \ldots, w_r]$ by theorem 6.23.

Proposition 6.26 *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a k-projection and let V be a k-variety of $\mathbb{P}^n$ such that the center of f does not intersect V. Then $k[V]$ is integral over $k[f(V)]$.*

Proof. Let the projection be defined by

$$y_j = f_{j0}x_0 + f_{j1}x_1 + \cdots + f_{jn}x_n, \quad j = 0, 1, \ldots, m.$$

Removing linear dependent elements among $y_0, y_1, \ldots, y_m$, we can assume that they are linearly independent. Further with a linear change of coordinates, we can assume that $x_i = y_i$, $i = 0, \ldots, m$. Taking the images of y in $k[V] = k[v_0, v_1, \ldots, v_n]$ we get $k[f(V)] = k[v_0, v_1, \ldots, v_m]$. Now v_i for $i > m$ is integral over $k[v_0, v_1, \ldots, v_{i-1}]$. In fact, consider the projection g of $\mathbb{P}^n$ onto $\mathbb{P}^i$. Then $k[g(V)] = k[v_0, v_1, \ldots, v_i]$ and since V does not intersect $x_0 = x_1 = \cdots = x_m = 0$, it does not intersect $x_0 = x_1 = \cdots = x_{i-1} = 0$. Thus v_i is nilpotent in $k[v_0, v_1 \ldots, v_i]/(v_0, v_1, \ldots, v_{i-1})$ or equivalent v_i is integral over $k[v_0, v_1, \ldots, v_{i-1}]$. By the transitivity of integral dependence, it follows that $k[V] = k[v_0, v_1, \ldots, v_n]$ is integral over $k[f(V)] = k[v_0, v_1, \ldots, v_m]$. □

Remark 6.27 Proposition 6.26, gives an immediate proof of the projective equivalent of theorem 6.23 since the center of a generic projection from $\mathbb{P}^n$ to $\mathbb{P}^r$ does not intersect the center of an r-dimensional variety V.

DEGREE

Notation 6.28 In this section $A = k[x_1, x_2, \ldots, x_n]$ will always be a graded ring, finitely generated over k by elements of degree 1, and M a finitely generated graded A-module, e.g. a homogeneous ideal in A or a quotient of A. Further we use $\varphi_M(l) = \dim_k(M_l)$, the vector space dimension over k of the l:th graded part of M.

Theorem 6.29 (Hilbert-Serre) *There is a unique polynomial $h_M(t) \in \mathbb{Q}[t]$ such that $h_M(l) = \varphi_M(l)$ for all sufficiently large l. Furthermore the degree of h_M is the dimension of the projective k-variety in $\mathbb{P}^n$ given by the ideal* $\mathrm{Ann}(M)$ *in A.*

Proof. See [Ha, Ch. I, Thm 7.5]. □

Definition 6.30 The Hilbert polynomial of a projective k-variety $V \subseteq \mathbb{P}^n$ is the Hilbert polynomial of the coordinate ring $k[V]$.

Example 6.31 Let $M = k[\mathbb{P}^n] = k[x_0, x_1, \ldots, x_n]$. A simple calculations gives that $\varphi_M(l) = \binom{l+n}{n}$. Thus the Hilbert polynomial is $h_M(t) = \binom{t+n}{n} = \frac{1}{n!}(t+n)(t+n-1)\ldots(t+1)$ which is of degree n as expected.

Example 6.32 Let $M = k[x, y, z]/(x^2 - yz)$. A basis for the homogeneous parts M_l is: $1;\ x, y, z;\ x^2, xy, xz, y^2, z^2;\ \ldots$. The Hilbert polynomial in this case is $h_M(t) = 1 + 2t$ which is of degree 1 as expected since $\mathrm{Ann}(M) = (x^2 - yz)$ the defining ideal of a curve in $\mathbb{P}^2$.

Definition 6.33 The degree $\deg(V)$ of a projective k-variety $V^r \subseteq \mathbb{P}^n$ is $r!h_r$ where h_r is the coefficient of the t^r-term in the Hilbert polynomial of V.

Example 6.34 By the previous examples, the projective n-space $\mathbb{P}^n$ has degree 1 and $VP_K(x^2 - yz) \subset \mathbb{P}^2$ has degree 2.

Remark 6.35 The degree depends on which space we embed the k-variety in, i.e. the ring A. In fact $\mathbb{P}^1$ seen as the subspace $(x_0^2)(x_1^2) - (x_0x_1)^2$ of $\mathbb{P}^2$ by the Veronese embedding in section has degree 2.

Remark 6.36 Let $M = k[x_0^d, x_0^{d-1}x_1, \ldots, x_n^d]$ be the homogeneous coordinate ring of the d-uple Veronese embedding of $\mathbb{P}^n$. The elements x_i^d are of degree 1 and M is a quotient ring of the polynomial ring $k[y_0, y_1, \ldots, y_N]$ where $N = \binom{d+n}{n} - 1$. We have that $\varphi_M(t) = \binom{td+n}{n}$ and the leading term of the Hilbert polynomial is $d^n \frac{t^n}{n!}$. Thus the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ is d^n. Similarly it can be shown that the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^{n'}$ in $\mathbb{P}^N$ has degree $\binom{n+n'}{n}$.

Proposition 6.37 *If V is a* geometrically integral *projective k-variety, the degree of V is invariant under base extensions, i.e. the degree of V and $V_{(k')}$ is equal for all field extensions k'/k.*

Proof. Since V is geometrically integral we have that $k'[V] = k[V] \otimes_k k'$ for any field extension k'/k. Thus $\dim_{k'}\big(k'[V]_l\big) = \dim_k\big(k[V]_l\big)$ which shows that the Hilbert polynomial is the same. □

Example 6.38 If V is an arbitrary k-variety the degree of V and $V_{(k')}$ may be different. Let $V = VP_K(x^p - t^p y^p)$ be a $\mathbb{F}_p(t^p)$-variety. Its degree is p but $V_{(\mathbb{F}_p(t))}$ has degree 1. Note that V is geometrically irreducible but not geometrically integral.

Proposition 6.39 *If V is a k-variety of dimension r, the degree of V is the sum of the degrees of its components of dimension r.*

Proof. Let $V = V_1 \cup V_2$ where V_1 is a k-variety of dimension r and V_2 is an irreducible k-variety of dimension $r' \leq r$ which is not contained in V_1. Then $V_1 \cap V_2 \subset V_1$ and thus $\dim(V_1 \cap V_2) < r$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are the defining ideals of V_1 and V_2 and $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ the ideal of V we have an exact sequence

$$0 \longrightarrow k[x]/\mathfrak{a} \longrightarrow k[x]/\mathfrak{a}_1 \oplus k[x]/\mathfrak{a}_2 \longrightarrow k[x]/(\mathfrak{a}_1, \mathfrak{a}_2) \longrightarrow 0.$$

Or equivalently

$$0 \longrightarrow k[V] \longrightarrow k[V_1] \oplus k[V_2] \longrightarrow k[x]/(\mathfrak{a}_1, \mathfrak{a}_2) \longrightarrow 0.$$

The degree of the Hilbert polynomial of $k[x]/(\mathfrak{a}_1, \mathfrak{a}_2)$ is the dimension of the variety defined by $(\mathfrak{a}_1, \mathfrak{a}_2)$. But $\mathfrak{r}(\mathfrak{a}_1, \mathfrak{a}_2) = \mathfrak{I}_k(V_1 \cap V_2)$ and thus the degree is the dimension of the variety $V_1 \cap V_2$ which is less than r. The leading coefficient of the Hilbert polynomial for V, which is the one in front of t^r, is consequently the sum of the coefficients of the t^r-terms in the Hilbert polynomials for V_1 and V_2. If V_2 has dimension r, the degree is thus the sum of the degrees, and if V_2 has smaller dimension, the degree is that of V_1.

Since V_1 has a fewer number of components than V, the proposition follows by induction on the number of components. □

Proposition 6.40 *Let $\mathfrak{a}$ be a homogeneous ideal of A and $f \in A$ a homogeneous polynomial of degree m such that f is not a zero divisor in $A/\mathfrak{a}$. Then*

$$\varphi_{A/(\mathfrak{a},f)}(l) = \varphi_{A/\mathfrak{a}}(l) - \varphi_{A/\mathfrak{a}}(l-m).$$

Proof. The proposition follows immediately from the exact sequence

$$0 \longrightarrow \big(A/(\mathfrak{a})\big)_{l-m} \xrightarrow{f} \big(A/(\mathfrak{a})\big)_{l} \longrightarrow \big(A/(\mathfrak{a},f)\big)_{l} \longrightarrow 0.$$

□

Corollary 6.41 *Let $\mathfrak{a}$ be a homogeneous ideal of A and $f \in A$ a homogeneous polynomial of degree m such that f is not zero a zero divisor in $A/\mathfrak{a}$. Then the degree d' of $A/(\mathfrak{a}, f)$ is md, where d is the degree of $A/\mathfrak{a}$.*

Proof. Let r be the dimension of $V = VP_K(\mathfrak{a})$. Since f is not zero in $A/\mathfrak{a}$ it defines a hypersurface H which does not contain V and thus the dimension of $V \cap H = VP_K(\mathfrak{a}, f)$ is $r-1$. The Hilbert polynomials of $A/\mathfrak{a}$ and $A/(\mathfrak{a}, f)$ are $h(t) = d\frac{t^r}{r!} + \dots$ and $h'(t) = d'\frac{t^{r-1}}{(r-1)!} + \dots$. By proposition 6.40 we have that $h'(l) = h(l) - h(l-m)$ for sufficiently large l. Identifying the highest terms we have that $d' = md$. □

Corollary 6.42 *A hypersurface given by an irreducible homogeneous polynomial of degree m has degree m.*

Proof. This follows immediately from the fact that $\mathbb{P}^n$ has degree 1 as we have seen in the previous examples. □

Proposition 6.43 *Let A be a one-dimensional graded k-algebra and $f \in A$ a homogeneous polynomial such that f is not a zero divisor in A. Then $[A_{(f)} : k] = \deg(A)$ where $A_{(f)}$ is the degree zero part of the homogeneous localization A_f.*

Proof. Since A_{red} has projective dimension 0, the Hilbert polynomial for A is the constant polynomial $h_A(t) = d$. For sufficiently large l we thus have that $\varphi_A(l) = d$. Let $f_1, f_2, \dots, f_m$ be elements of A_l. Then $f_1, f_2, \dots, f_m$ are linearly independent over k if and only if $f_1/f^l, f_2/f^l, \dots, f_m/f^l$ are linearly independent in $A_{(f)}$. In fact, we have that $\sum_{i=1}^n \lambda_i f_i/f^l = 0$ in $A_{(f)}$ if and only if $\sum_{i=1}^n \lambda_i f_i = 0$ in A since f is not a zero divisor in A.

The dimension of $A_{(f)}$ as a vector space over k is thus at least d. Now assume that there is a basis $\big(f_i/f^{l_i}\big)_{i=1}^m$, $f_i \in A_{l_i}$ with $m > d$ elements. Let l be an integer greater than all l_i:s such that $\varphi_A(l) = d$. Then $\{f_i f^{l-l_i}/f^l\}_{i=1}^m$ are linearly independent and thus we have m linearly independent elements $f_i f^{l-l_i}$ in A_l which is a contradiction to $m > d$. □

Corollary 6.44 *Let V be a zero-dimensional irreducible k-variety. Then $[k(V) : k] = \deg(V)$.*

Proof. Let $f = v_j$ for some non-zero v_j. Then $k(V) = k(v_i/v_j) = k[v_i/v_j] = k[V]_{(f)}$. Thus by proposition 6.43 we have that $[k(V) : k] = \deg(V)$. □

Remark 6.45 Note that corollary 6.44 also implies that the degree of a projective variety V of dimension zero is independent of the embedding since $k(V)$ is independent of the embedding.

Degree and intersections with linear varieties

Proposition 6.46 *Let $V \subseteq X = \mathbb{P}^n$ be an irreducible k-variety of dimension r and let L^{n-r} be a linear k-variety defined by the equations $f_s = f_{s0}x_0 + \cdots + f_{sn}x_n \in k[x]$ for $s = 1, \dots, r$ such that $k[V_{\text{aff}}] = k[v_i/v_j]$ is integral over $k[g] = k[g_1, g_2, \dots, g_r] = k[f_s/v_j]$ where $g_s = f_s/v_j$ are the images of the equations of L_{aff} in $k[V_{\text{aff}}]$. Then*

$$[k(V) : k(f)] = [k(v_i/v_j) : k(f_s/v_j)] = [k(V_{\text{aff}}) : k(g)] = \deg(V)$$

where we also let f_s and g_s denote their images in $k[V]$ and $k[V_{\text{aff}}]$ respectively.

Proof. We can assume that v_0 is not zero and let $k[V_{\text{aff}}] = k[v_1, v_2, \dots, v_n]$, $g_s = f_{s0} + f_{s1}x_1 + \cdots + f_{sn}x_n$. We will now proceed to prove that $[k(V_{\text{aff}}) : k(g)] = [k[V_{\text{aff}}]/(g) : k]$.

Since $k[V_{\text{aff}}]$ is integral over $k[g]$ we have that $k(V_{\text{aff}})/k(g)$ is algebraic and the minimal monic polynomial $P_1(v_1)$ of v_1 over $k(g)$ has coefficients in $k[g_1, g_2, \dots, g_r]$, see [AM, Prop. 5.15]. Further, the minimal monic polynomial $P_i(v_i)$ of v_i over $k(g, v_1, v_2, \dots, v_{i-1})$ has coefficients in $k[g_1, g_2, \dots, g_r, v_1, v_2, \dots, v_{i-1}]$. If $d_1, d_2, \dots, d_n$ are the degrees of the minimal polynomials we have that

$$\{b_i\} = \{1, v_1, v_1^2, \dots, v_1^{d_1-1}, v_2, v_2^2, \dots, v_2^{d_2-1}, \dots, v_n^{d_n-1}\}$$

is a basis for $k(V_{\text{aff}})/k(g)$ and that $[k(V_{\text{aff}}) : k(g)] = \sum_{i=1}^{n}(d_i - 1) + 1$.

Now, the images of b_i in $k[V_{\text{aff}}]/(g)$ is a basis for $k[V_{\text{aff}}]/(g)$ over k. In fact, they are clearly linearly independent and the image of $P_i(v_i)$ in $k[V_{\text{aff}}]/(g)$ gives a linear dependence of $v_i^{d_n}$ over $k[b]$ which makes $\{b_i\}$ a generating set.

We have thus proved that $[k(V) : k(f)] = [k(V_{\text{aff}}) : k(g)] = [k[V_{\text{aff}}]/(g) : k]$ which according to proposition 6.43 is the degree of $k[V]/(f)$ since localizations and quotients commute. Repeatedly using corollary 6.41 for $f_1, f_2, \dots, f_r$ we have that $\deg\big(k[V]/(f)\big) = \deg(V)$. □

Corollary 6.47 *Let $V \subseteq X = \mathbb{P}^n$ be an irreducible k-variety of dimension r and let L^{n-r} be a* generic *linear k_u-variety defined by the equations $f_s = u_{s0}x_0 + \cdots + u_{sn}x_n \in k_u[x]$, $s = 1, \dots, r$. Then $[k_u(V) : k_u(f)] = [k_u(v_i/v_j) : k_u(f_s/v_j)] = \deg(V)$.*

Proof. By the generic variant of Noether's Normalization lemma, theorem 6.23, the coordinate ring $k_u[V_{\text{aff}}] = k_u[v_i/v_j]$ is integral over $k_u[f_1/v_j, f_2/v_j, \dots, f_r/v_j]$. Since the degree of V and $V_{(k_u)}$ are equal, the corollary follows from proposition 6.46. □

Proposition 6.48 *Let $V \subseteq X = \mathbb{P}^n$ be an* geometrically integral *k-variety of dimension r and let L^{n-r} be a linear variety,* generic *over k, defined by the equations $f_s = u_{s0}x_0 + \cdots + u_{sn}x_n \in k_u[x]$, $s = 1, \dots, r$. Then the intersection $V \cap L$ consists of* $\deg(V)$ *points of V, which are* conjugate *and* separable *over $k_u = k(u) = k(u_{si})$. Moreover, the points of $V \cap L$ are* generic points *of V.*

Proof. By proposition 6.13 the intersection $W = V \cap L$ is proper and thus has dimension zero. It can be shown, see [L2, Ch. VIII, Thm 7 and Prop. 12] or [S, p. 38-40], that the intersection W is a geometrically integral k_u-variety with prime ideal $(\mathfrak{I}\mathfrak{H}_k(V), f_1, f_2, \dots, f_r)$. Repeatedly using corollary 6.41 for $f_1, f_2, \dots, f_r$ shows that $\deg(W) = \deg(V) = d$ and by proposition 6.44 we have that $[k_u(W) : k_u] = \deg(W) = d$.

Since $k_u(W)$ is a regular extension of k_u it is a separable extension. Thus there are d separable points in W which are conjugates, as noted in remark 1.55.

Finally, the transcendence degree of the points over k is at least r by proposition 6.15. Since every point in an r-dimensional k-variety has at most transcendence degree r, they have exactly transcendence degree r and are thus generic points of V. □

Corollary 6.49 *Let V be an geometrically integral k-variety and L^{n-r} a linear variety generic over k. Then $V \cap L$ is non-empty and has a finite number of points if and only if V is of dimension r. In particular $V \cap L$ has a finite number of* generic *points if and only if V is of dimension r.*

Proof. Assume that V is of dimension greater than r and $V \cap L^{n-r}$ only has finite number of (generic) points. Then it is clear that the intersection $V \cap L'^{n-r-1} = V \cap L^{n-r} \cap H$ of $V \cap L^{n-r}$ and a generic hyperplane H, is empty which contradicts proposition 6.48. □

Remark 6.50 Classically, proposition 6.48 is taken as the definition of the degree, which then only is defined for geometrically integral varieties. Since the pure algebraic definition in 6.33 using Hilbert polynomials is much more clear, easier to define, more generalizable and easier to compute, it is now commonly taken as the definition of the degree. The interpretation as a degree of field extensions in proposition 6.46 is also useful.

Remark 6.51 Proposition 6.48 is a generic special case of Bézout's theorem which states that the degree of the intersection of two varieties V and W is the product of the degrees of V and W (when taking the intersection we must count with multiplicity, e.g. the intersection of $y = x^2$ and $y = 0$ has multiplicity two). In our case $W = L^{n-r}$ has degree one and due to the generic requirement, all the intersection points have degree one.

Remark 6.52 Let V^r be a irreducible k-variety of $\mathbb{P}^n$. Consider all hyperplanes, given by $u_0x_0 + u_1x_1 + \cdots + u_nx_n = 0$, which intersect V. The hyperplanes and their intersections with V are then the points of a k-variety C of $\mathbb{P}^n \times \mathbb{P}^n$ with coordinates $(x_0 : x_1 : \cdots : x_n,\ u_0 : u_1 : \cdots : u_n)$. A defining ideal for C is $\mathfrak{a} = \big(\mathfrak{I}\mathfrak{H}_k(V), u_0x_0 + u_1x_1 + \cdots + u_nx_n\big)$. It is not clear if $\mathfrak{a}$ is prime or even primary, but the variety C is irreducible. In fact, if ξ is a generic point for V then the points of C are k-specializations of (ξ, λ) where $(\lambda_0 : \lambda_1 : \cdots : \lambda_n)$ is a generic point over $k(\xi)$ satisfying $\xi_0\lambda_0 + \xi_1\lambda_1 + \cdots + \xi_n\lambda_n = 0$.

The reason that we cannot even say that the ideal $\mathfrak{a}$ is primary, is that $\mathbb{P}^n \times \mathbb{P}^n$ is a multi-projective variety in which Hilbert's Nullstellensatz gives a correspondence between the varieties and the radical ideals which do not contain a multiple of an irrelevant ideal. Take for example the $\mathbb{Q}$-variety V of $\mathbb{P}^1$ given by the ideal $(x^2 + y^2)$. Then $\mathfrak{a} = (x^2 + y^2, ux + vy)$ is not a prime ideal of $\mathbb{Q}[x, y, u, v]$. In fact, the polynomial $(u^2 + v^2)x^2$ is in $\mathfrak{a}$ but neither $u^2 + v^2$ nor x^2. The radical of $\mathfrak{a}$ is $\mathfrak{r}(\mathfrak{a}) = \big(x^2 + y^2, ux + vy, (u^2 + v^2)x, (u^2 + v^2)y\big)$. The variety C given by $\mathfrak{a}$ does not correspond to $\mathfrak{r}(\mathfrak{a})$ since it contains a product of the irrelevant ideal (x, y). It is easy to see that the ideal corresponding to C is $(x^2 + y^2, ux + vy, u^2 + v^2)$. In fact, the points of V are $(1 : \pm i)$ and the hyperplanes intersecting V are the same two points, thus the points of C are the two points $(1 : \pm i, 1 : \pm i)$.

Likewise, we can construct an irreducible k-variety C of $\mathbb{P}^n \times (\mathbb{P}^n)^m$ consisting of systems of m hyperplanes with a common intersection with V. The points of C are the k-specializations of (ξ, λ) where λ is generic over $k(\xi)$ fulfilling $\sum_{i=0}^n \xi_i \lambda_{si} = 0$, $s = 1, 2, \ldots, m$.

Degree of morphisms

Definition 6.53 Let $f : X \to Y$ be a k-morphism and V an irreducible k-variety of X. We let $W = \overline{f(V)}$ be the image of V and denote $\deg(V/W) = [V : W]$ the degree of the field extension $k(V)/k(W)$ when it is finite. If the field extension is transcendental, i.e. $\dim(W) < \dim(V)$, we let $\deg(V/W)$ be zero. We call $\deg(V/W)$ the *degree of the morphism* of V onto W.

Proposition 6.54 *Let $f : X \to Y$ be a k-morphism, $V \subseteq X$ a geometrically integral k-variety and $W \subseteq Y$ its image. Then the degree of the k-morphism* $\deg(V/W)$ *does not depend on the field of definition k, i.e.* $\deg(V/W) = \deg\big(V_{(k')}/W_{(k')}\big)$ *for all k'/k.*

Proof. First note that W is geometrically integral by proposition 5.38 and thus $k(V)$ and $k(W)$ are regular extensions of k. First assume that k'/k is a purely transcendental field extension $k' = k(t)$. Then $[k'(V) : k'(W)] = [k(V)(t) : k(W)(t)] = [k(V) : k(W)]$. If k'/k is an algebraic field extension then proposition 5.14 states that $k'(V) = k(V)k' = k(V) \otimes_k k'$ and $k'(W) = k(W)k' = k(W) \otimes_k k'$. Thus $[k'(V) : k'(W)] = [k(V) \otimes_k k' : k(W) \otimes_k k'] = [k(V) : k(W)]$. Since every field extension is a composition of purely transcendental and algebraic extensions, the proposition follows. □

Example 6.55 Let $k = \mathbb{F}_p(t^p)$ and V be the irreducible k-variety in $\mathbb{A}^1$ defined by $x^p - t^p$. The Frobenius morphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $s = x^p$ then maps V onto $W = s - t^p$. The degree of the morphism over k is $\deg_k(V/W) = [k(t) : k] = p$ and the degree over $k' = k(t) = \mathbb{F}_p(t)$ is one. Note that V is geometrically irreducible but not geometrically integral.

Proposition 6.56 *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a k-projection, $V \subseteq \mathbb{P}^n$ an irreducible k-variety which does not intersect the center of f, and $W \subseteq \mathbb{P}^m$ the image of V. Then* $\deg(V) = \deg(V/W)\deg(W)$.

Proof. Let $W = f(V)$ be the image of V and let r be the dimension of V which by proposition 3.27 also is the dimension of W. Let $g_s = u_{s0}w_0 + \cdots + u_{sn}w_m \in k_u[W]$, $s = 1, \ldots, r$ be r generic linear combinations, i.e. the point in $(\mathbb{P}^m)^r$ corresponding to the u_{sj}:s, is generic over k. By proposition 6.47 we have that $[k_u(W) : k_u(g)] = \deg(W)$. Now by proposition 6.26, the coordinate ring $k_u[V]$ is integral over $k_u[W]$, thus by transitivity $k_u[V]$ is integral over $k_u[g]$ and thus by proposition 6.46 we obtain $[k_u(V) : k_u(g)] = \deg(V)$. Thus

$$\deg(V/W) = [k_u(V) : k_u(W)] = [k_u(V) : k_u(g)]/[k_u(W) : k_u(g)] = \deg(V)/\deg(W).$$

□

Remark 6.57 In particular, proposition 6.56 tells us that if a k-projection induces a *birational* morphism, i.e. $k(W) = k(V)$, then V and W have the same degree.

Example 6.58 Define the irreducible $\mathbb{Q}$-variety $V = VP_{\overline{\mathbb{Q}}}(x^2 + xy + y^2, x + y + z)$ of $\mathbb{P}^2$. It has two points $(1 : \xi : \xi^2)$ and $(1 : \xi^2 : \xi)$ where ξ is a non-trivial third root of unity. The projection $f : \mathbb{P}^2 \to \mathbb{P}^1$ given by $s = x + y$, $t = z$ maps V onto $W = VP_{\overline{\mathbb{Q}}}(s + t)$ which has the single point $(1 : -1)$. The degree of V is two and the degree of W is one. The function field of V is $k\left[\frac{x}{y}\right] / \left(\frac{x^2}{y^2} + \frac{x}{y} + 1\right) = k(\xi)$ and $k(W) = k$. Thus $\left[k(V) : k(W)\right] = 2$ as expected.

Example 6.59 Let $V = VP_K(x^2 - yz) \subset \mathbb{P}^2$ and let $f : \mathbb{P}^2 \to \mathbb{P}^1$ be the projection defined by $(x : y : z) \mapsto (y : z)$. Then $W = f(V) = VP_K(0) = \mathbb{P}^1$ and $\deg(V/W) = \left[k\left(\frac{x}{y}\right) : k\left(\frac{y}{z}\right)\right] = 2$ since $\frac{y}{z} = \left(\frac{y}{x}\right)^2$.

Remark 6.60 The degree $\deg(V/W)$ can be described as the number of points in V which maps to the same point in W. This is true almost everywhere, but some points have to be calculated with *multiplicity* such as the points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ in the previous example.

Example 6.61 The identity $\deg(V) = \deg(V/W)\deg(W)$ does not hold for arbitrary k-morphisms. Take for an example the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then the coordinate rings of $V = \mathbb{P}^1$ and its image W is $k[V] = k[x, y]$ and $k[W] = k[x^2, xy, y^2]/\left(x^2y^2 - (xy)^2\right) = k[s, t, u]/(su - t^2)$. Thus V has degree 1 and W has degree 2, as we also noted in remark 6.35. But $\deg(V/W) = 1$. Indeed $k(V) = k(x/y)$ and $k(W) = k(s/t, u/t) = k(s/t) = k(x/y)$ since $u/t = t/s$.

This is not unexpected since $\deg(V)$ and $\deg(W)$ are dependent on the embeddings of V and W in projective spaces and $\deg(V/W)$ is an invariant.

Dense properties

Definition 6.62 If a property holds in a non-empty open subset of an irreducible k-variety V we say that the property is *dense* in V or that the property is true *almost everywhere* in V. Note that non-empty open subsets always are dense since V is irreducible. Equivalently there is a non-zero polynomial $P(v) \in k[V]$ such that the property holds for all points $a \in V$ such that $P(a) \neq 0$.

Definition 6.63 Seeing the linear varieties of dimension r in $\mathbb{P}^n$ as points (u_{si}) in the multi-projective space $(\mathbb{P}^n)^{n-r}$, we can say that a property holds for almost every r-dimensional linear variety. This means that there is a non-zero polynomial $P(c_{si}) \in k[c_{si}]$ such that the property holds for a linear variety defined by (u_{si}) if $P(u_{si}) \neq 0$.

Remark 6.64 A dense property for linear varieties is always true for a generic linear variety. In fact, if $P(c_{si}) \in k[c_{si}]$ is a polynomial which is not identically zero, then $P(u_{si}) \neq 0$ if u_{si} are generic over k. In fact the monomials in u_{si} are algebraically independent over k.

The following propositions are about properties that holds for almost every linear transformation of the generators $v_1, v_2, \ldots, v_n$ of the coordinate ring $k[V]$ for an irreducible (geometrically integral) variety V.

Proposition 6.65 *Let V be an affine irreducible k-variety of dimension r. Let c_{si} be r series of*

n variables as in theorem 6.23. Then there is a polynomial $P(c) \in k[c_{si}]$ such that if $P(u) \neq 0$ then $k_u[V]$ is integral over $k_u[y_1, y_2, \ldots, y_r]$ where $y_s = \sum_{i=1}^{n} u_{si} v_i$. Thus the property that $k_u[V]$ is integral over $k_u[y]$ is dense since the u_{si} can be seen as points in $(\mathbb{A}^n)^m$.

Proof. See [L1, Ch. VIII, Cor. 2.3]. □

Corollary 6.66 *Let V be a projective irreducible k-variety of dimension r. Let $f_s = \sum_{i=0}^{n} u_{si} v_i$, $s = 1, \ldots, r$, $u_{si} \in \mathbb{A}^n$. The property that $[k(V) : k(f)] = \deg(V)$ is dense in (u_{si}).*

Proof. Follows immediately from 6.65 and from proposition 6.46. □

Proposition 6.67 *Let $V \subseteq \mathbb{A}^n$ be a* geometrically integral *k-variety of dimension r. The property that $y_s = \sum_{i=1}^{n} u_{si} v_i$, $s = 1, \ldots, r$, $u_{si} \in \mathbb{A}^n$, is a separating transcendence basis to $k_u(V)/k_u$ is dense in (u_{si}).*

Proof. See [S, p. 36-37]. □

Proposition 6.68 *Let $V \subseteq \mathbb{A}^n$ be a* geometrically integral *k-variety of dimension r. The property that $y_s = \sum_{i=1}^{n} u_{si} v_i$, $s = 1, \ldots, r+1$, $u_{si} \in \mathbb{A}^n$, are generators for $k_u(V)$, i.e. $k_u(y_1, y_2, \ldots, y_{r+1}) = k_u(V)$, is dense in (u_{si}) as a point in the k'-variety $(\mathbb{P}^n)^{r+1}$ for some transcendental field extension k'/k.*

Proof. See [S, p. 37-38]. □

Remark 6.69 Note that the requirement that V is geometrically integral in propositions 6.67 and 6.68 and hence geometrically irreducible, assures us that $V_{(k_u)}$ is irreducible and thus that $k_u(V)$ exists. Further since V is geometrically integral it implies that $k(V)$ is separable over k, i.e. there exists a separating transcendence basis. A transcendence basis can always be extracted from the set $\{v_1, v_2, \ldots, v_n\}$.

Corollary 6.70 *Let $V \subseteq \mathbb{A}^n$ be a* geometrically integral *k-variety dimension r and $f : \mathbb{A}^n \to \mathbb{A}^{r+1}$ be a generic projection with coefficients u_{si}. The image $W = \overline{f(V_{(k_u)})}$ of $V_{(k_u)}$ is then birational to $V_{(k_u)}$, i.e. $k_u(W) = k_u(V)$.*

Proof. Follows immediately from proposition 6.68. □

Remark 6.71 If V^r is an irreducible k-variety and k is algebraically closed, then V is birational to an irreducible hypersurface of $\mathbb{A}^{r+1}$. In fact, $k(V)$ is separable over k since k is algebraically closed and we can thus find a separating transcendence basis $y_1, y_2, \ldots, y_r$. Further, since $k(V)/k(y_1, y_2, \ldots, y_r)$ is simple, there is an element y_{r+1} which is algebraic over $k(y_1, \ldots, y_r)$ such that $k(y_1, \ldots, y_r, y_{r+1}) = k(V)$. Also see [Ha, Ch. I, Prop. 4.9]. Note that the element y_{r+1} is not necessarily a linear combination of the v_i:s and that proposition 6.68 only states that there are linear combinations y_s of v_i:s with coefficients u_{si} in K such that $k_u(V) = k_u(y_1, y_2, \ldots, y_{r+1})$, not that $k(V) = k(y_1, y_2, \ldots, y_{r+1})$.

Remark 6.72 Propositions 6.67, 6.68 and corollary 6.70 have corresponding projective variants in which an extra linear equation y_0 is added. For example, the property that $y_0, y_1, \ldots, y_r$ is a separating transcendence basis, meaning that y_i/y_j is a separating transcendence basis, is dense.

Propositions 6.46 and 6.48 do not only hold for generic linear varieties. In fact, both propositions is true for almost every linear variety, i.e. a dense property. For the dense property corresponding to proposition 6.46, see [S, p. 38]. Proposition 6.48 is a special case of Bertini's Theorem.

Theorem 6.73 (Bertini's Theorem) *Let V^r be a geometrically integral k-variety. Then the intersection of almost every linear variety L^{n-r} and V consists of* $\deg(V)$ *points.*

Proof. See [S, p. 39]. □

Remark 6.74 A more general formulation of theorem 6.73 is that the intersection of a not everywhere singular projective variety V with almost every linear variety of codimension r is non-singular. A weaker theorem stating that the intersection of a non-singular variety with a *hyperplane* is non-singular can be found in [Ha, Ch. II, Thm 8.18].

Chapter 7

Cycles

Cycles

Definition 7.1 The *k-cycles* of a k-variety V are the elements of the free $\mathbb{Z}$-module over the irreducible k-subvarieties of V, denoted $\mathbb{Z}_* V$. In other words they are formal sums $\nu = \sum_{\alpha \in \mathscr{I}} m_\alpha [V_\alpha]$ where only a finite number of the multiplicities m_α are non-zero and V_α is an irreducible k-subvariety of V. The *components* of a cycle ν are the V_α with non-zero coefficients. The *support* is the union of the components V_α, which is a k-variety. If W is a k-variety such that the components of ν are all contained in W, we say that the ν is *supported* by W.

Definition 7.2 If all the components of a cycle have the same dimension, r, it is called a *homogeneous cycle of dimension* r or an *r-cycle*. The r-cycles form a $\mathbb{Z}$-module, $\mathbb{Z}_r V$.

Remark 7.3 It is clear that any cycle $\nu \in \mathbb{Z}_* V$ can be uniquely written as a sum of cycles $\nu_r \in \mathbb{Z}_r V$ and that $\mathbb{Z}_*$ is the graded $\mathbb{Z}$-module $\oplus_{r=0}^{\infty} \mathbb{Z}_r V$.

Definition 7.4 A cycle is termed *positive* (or *effective*) if all its coefficients are positive. If ν and ρ are cycles of V and $\nu - \rho$ is positive we write that $\nu \geq \rho$. The *positive* and *negative* part of ν is $\nu_+ = \sum_{m_\alpha > 0} m_\alpha [V_\alpha]$ and $\nu_- = \sum_{m_\alpha < 0} (-m_\alpha)[V_\alpha]$ respectively. It gives a *canonical* composition of ν as positive cycles $\nu = \nu_+ - \nu_-$.

Definition 7.5 Let V be $\mathbb{P}^n$ or $\mathbb{A}^n$. To a polynomial $f \in k[V]$ (homogeneous if V is projective) we associate the k-cycle $[\mathrm{div}(f)]$ defined by $[\mathrm{div}(f)] = \sum_i^r m_i \big[V_K\big((f_i)\big)\big]$ where $f = f_1^{m_1} f_2^{m_2} \dots f_r^{m_r}$ is a factorization of f in irreducible polynomials. Note that since $k[V]$ is a polynomial ring over k, it is a unique factorization domain and the cycle $[\mathrm{div}(f)]$ is well-defined.

Remark 7.6 The components of the k-cycle $[\mathrm{div}(f)]$ are the components of the k-variety corresponding to f.

Definition 7.7 For a quotient of polynomials f/g we define the cycle $[\mathrm{div}(f/g)] = [\mathrm{div}(f)] - [\mathrm{div}(g)]$. This does not depend of the choice of representatives of f/g because of the unique factorization.

Remark 7.8 Usually $[\mathrm{div}(f)]$ is only defined when f is a rational function on V. Thus, in the projective case f should be a quotient of homogeneous polynomials of the same degree. We will not make any such restrictions.

Remark 7.9 Definition 7.5 can only be used for $\mathbb{P}^n$ and $\mathbb{A}^n$. In fact, $k[V]$ is not always a UFD. For example, if $V = V_K(x^3 - y^2) \subset \mathbb{A}^2$ we have that $x^3 = x \cdot x = y \cdot y \cdot y$ in $k[V]$ and x and y are irreducible.

Definition 7.10 The homogeneous k-cycles of codimension 1, i.e. of dimension $d = n - 1$ of V where $\dim(V) = n$, are called (Weil) *divisors*. If V is $\mathbb{P}^n$ or $\mathbb{A}^n$, each divisor $\nu = \sum_i m_i[V_i]$ corresponds to a quotient of polynomials $r = f/g,\ f, g \in k[V]$ unique up to an element of k such that $\nu = [\mathrm{div}(r)]$. More specifically, $r = \prod_i f_i^{m_i}$, where f_i is the equation for the hypersurface V_i.

Definition 7.11 The *degree* of a k-cycle $\nu = \sum_i m_i[V_i]$ supported by a projective variety V is $\sum_i m_i d_i$ where d_i is the degree of V_i in V and the sum is over *the components of maximal dimension*. The degree of a divisor of $\mathbb{P}^n$ is the degree of the corresponding quotient of polynomials, since by corollary 6.42 the degree of the hypersurface corresponding to an irreducible polynomial, is the degree of the polynomial.

Definition 7.12 Let $\nu = \sum_i m_i[V_i]$ be a k-cycle of V and $f : X \to Y$ be a k-morphism. We define the image of the cycle by $f_*[V_i]$ to be $\deg(V_i/W_i)[W_i]$ where $W_i = \overline{f(V_i)}$ and $\deg(V_i/W_i)$ is the degree of the morphism of V_i. This defines $f_*\nu$ by linearity as $\sum_i m_i \deg(V_i/W_i)[W_i]$. Since $\deg(V_i/W_i)$ is zero if $\dim(W_i) < \dim(V_i)$ it is clear that f_* is a graded $\mathbb{Z}$-module homomorphism. Further if $g : Y \to Z$ is a morphism and $M_i = g(W_i) = (g \circ f)(V_i)$ we have that $[M_i : V_i] = [M_i : W_i][W_i : V_i]$ and thus $(fg)_* = f_* g_*$.

Remark 7.13 If $f : X \to Y$ is a projection, then f_* is degree preserving. In fact, $\deg[V] = \deg(V) = \deg(V/W)\deg(W) = \deg f_*[V]$.

Definition 7.14 If $\nu = \sum_i m_i[V_i]$ and $\mu = \sum_j m'_j[W_j]$ are cycles we define their *product* as, $\nu \times \mu = \sum_{i,j} m_i m'_j [V_i \times W_j]$.

Length and order

We will now extend definition 7.5 to other varieties than those with a unique factorization.

Definition 7.15 Let A be noetherian ring and M a finitely generated A-module. A composition series of M is a maximal chain of A-modules, i.e. $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} has no proper submodules.

Definition 7.16 The Jordan theorem, see e.g. [AM, Prop. 6.7], states that if there exists a composition series, every composition series has the same length. We will denote this length with $l_A(M)$.

Remark 7.17 The length of the A-module $A/\mathfrak{a}$, if it exists, is the length of a maximal chain of ideals, $A = \mathfrak{a}_0 \supset \mathfrak{a}_1 \cdots \supset \mathfrak{a}_n = \mathfrak{a}$, in A.

Notation 7.18 Let A be a graded ring and $\mathfrak{p} \subset A$ a homogeneous prime ideal. Let $A_\mathfrak{p}$ be the homogeneous localization in $\mathfrak{p}$ and let $\varphi_\mathfrak{p} : A \to A_\mathfrak{p}$ be the corresponding canonical map. For any ideals $\mathfrak{a} \in A$ and $\mathfrak{b} \in A_\mathfrak{p}$, we denote the extension $\varphi_\mathfrak{p}(\mathfrak{a})$ by $\mathfrak{a}^e$ and the contraction $\varphi_\mathfrak{p}^{-1}(\mathfrak{b})$ by $\mathfrak{b}^c$.

Definition 7.19 Let A be a graded ring, $\mathfrak{p} \subset A$ a homogeneous prime ideal and $\mathfrak{a} \subseteq A$ a homogeneous ideal. The *order of* $\mathfrak{a}$ *in* $\mathfrak{p}$, if it exists, is the length $l_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{a}^e)$ and is denoted $\operatorname{ord}_\mathfrak{p}(\mathfrak{a})$.

Remark 7.20 Note that if $\mathfrak{a} \not\subseteq \mathfrak{p}$ then $\mathfrak{a}^e = A_\mathfrak{p}$ and thus $\operatorname{ord}_\mathfrak{p}(\mathfrak{a}) = 0$. If $\mathfrak{a} \subseteq \mathfrak{p}$ however, we have that the length $l_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{a}^e)$ is the length of a maximal chain of ideals, $\mathfrak{p} = \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n = \mathfrak{a}^{ec}$ in A. In fact, the ideal $\mathfrak{a}^{ec}$ contains the kernel of $\varphi_\mathfrak{p}$ and thus there is a correspondence between ideals in $A/\mathfrak{a}^{ec}$ contained in $\mathfrak{p}/\mathfrak{a}^{ec}$ and ideals in $A_\mathfrak{p}/\mathfrak{a}^e$ contained in the maximal ideal $\mathfrak{p}^e$.

Definition 7.21 Let W be an irreducible k-variety of a projective variety V and let $\mathfrak{p} = \mathfrak{I}_k(W) \subset k[V]$ be the defining ideal of W in V. If $\mathfrak{a}$ is an ideal of $k[V]$ we will by the *order of* $\mathfrak{a}$ *in* W refer to the order $\operatorname{ord}_W(\mathfrak{a}) = \operatorname{ord}_\mathfrak{p}(\mathfrak{a})$ if it exists.

Remark 7.22 The length of an A-module M is finite if and only if M is noetherian and artinian. Since $A = k[V]$ and hence $A_\mathfrak{p}$ is noetherian, the order $\operatorname{ord}_\mathfrak{p}(\mathfrak{a})$ exists if and only if $A_\mathfrak{p}/\mathfrak{a}^e$ is artinian, or equivalently that $A_\mathfrak{p}/\mathfrak{a}^e$ has exactly one prime ideal. This is true if and only if $\mathfrak{a}^{ec}$ is $\mathfrak{p}$-primary and in particular if $\mathfrak{r}(\mathfrak{a}) = \mathfrak{p}$.

Proposition 7.23 *Let* $\mathfrak{a} \subseteq \mathfrak{p}$ *be a* $\mathfrak{p}$*-primary ideal of the finitely generated graded k-algebra A. Then* $\deg(A/\mathfrak{a}) = \operatorname{ord}_\mathfrak{p}(\mathfrak{a}) \deg(A/\mathfrak{p})$.

Proof. The order of $\mathfrak{a}$ in $\mathfrak{p}$ is the length of a maximal chain of homogeneous ideals $\mathfrak{p} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = \mathfrak{a}$ in A. Since it is maximal, the ideal $\mathfrak{a}_i$ is generated by $\mathfrak{a}_{i+1}$ and a homogeneous element f_i such that $f_i^2 \in \mathfrak{a}_{i+1}$. We have thus that $\mathfrak{p} = (\mathfrak{a}, f_1, f_2, \ldots, f_{n-1})$ and $A/\mathfrak{a} = (A/\mathfrak{p})\,[1, f_1, f_2, \ldots, f_{n-1}]$ as an $A/\mathfrak{p}$ vector space. If we let $d_1, d_2, \ldots, d_{n-1}$ be the degrees of the f_i:s, then for l greater than all d_i we have that $\dim_k(A/\mathfrak{a})_l = \dim_k(A/\mathfrak{p})_l + \dim_k(A/\mathfrak{p})_{l-d_1} + \cdots + \dim_k(A/\mathfrak{p})_{l-d_{n-1}}$ and consequently

$$h_{A/\mathfrak{a}}(t) = h_{A/\mathfrak{p}}(t) + \sum_{i=0}^{n-1} h_{A/\mathfrak{p}}(t - d_i).$$

Thus the highest coefficient of the Hilbert polynomial for $A/\mathfrak{a}$ is n times the highest coefficient of the Hilbert polynomial for $A/\mathfrak{p}$ which gives the relation $\deg(A/\mathfrak{a}) = \operatorname{ord}_\mathfrak{p}(\mathfrak{a}) \deg(A/\mathfrak{p})$. □

Remark 7.24 Let V be a k-variety and let $\mathfrak{a}$ be an ideal of $A = k[V]$. Since A is noetherian, the ideal $\mathfrak{a}$ has a primary decomposition as $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ where $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$, for all i, by the Lasker-Noether decomposition theorem. We let $\mathfrak{p}_i = \mathfrak{r}(\mathfrak{q}_i)$ be the prime ideals corresponding to the primary ideals. The irreducible components of $V_K(\mathfrak{a})$ corresponds to the minimal primes of $\{\mathfrak{p}_i\}$, and the corresponding $\mathfrak{q}_i$ are called *isolated components*. Let $\mathfrak{q}_i$ be an isolated component and consider the localization $A_{\mathfrak{p}_i}$. Then $\mathfrak{a}^e = \mathfrak{q}_i^e$ and $\mathfrak{a}^{ec} = \mathfrak{q}_i^{ec} = \mathfrak{q}_i$ by [AM, Prop. 4.9]. Since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, the order $\operatorname{ord}_{\mathfrak{p}_i}(\mathfrak{a}) = \operatorname{ord}_{\mathfrak{p}_i}(\mathfrak{a}_i)$ exists by remark 7.22. Further, by remark 7.20, it is the length of a maximal chain of ideals $\mathfrak{p}_i = \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n = \mathfrak{q}_i$ in A.

Definition 7.25 Let V be a projective k-variety and $\mathfrak{a}$ a homogeneous ideal of $k[V]$. To $\mathfrak{a}$ we associate the cycle $[\mathfrak{a}]$ defined by $[\mathfrak{a}] = \sum_i \operatorname{ord}_{W_i}(\mathfrak{a})[W_i]$ where the sum is over the components W_i of $W = V_K(\mathfrak{a}) \subseteq V$.

Remark 7.26 By remark 7.24, we see that $[\mathfrak{a}]$ can be expressed using a primary decomposition. In fact, if $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ then $[\mathfrak{a}] = \sum_{\mathfrak{p}_i \text{ min}} l_{A_{\mathfrak{p}_i}}(A_{\mathfrak{p}_i}/\mathfrak{q}_i^e)[W_i]$ where the sum is over the minimal (i.e. isolated) primes of the prime ideals $\{\mathfrak{p}_i\}$ associated to $\mathfrak{a}\bar{k}[V]$ which correspond to the components W_i. This is also proves that the order $\mathrm{ord}_{W_i}(\mathfrak{a})$ exists and that $[\mathfrak{a}]$ is defined.

Remark 7.27 The degree of $\mathfrak{a}$, i.e. the highest coefficient of the Hilbert polynomial of $k[V]/\mathfrak{a}$, is equal to the degree of the cycle $[\mathfrak{a}]$. In fact, proposition 6.39 can easily be extended to state that the degree of $\mathfrak{a}$ is the sum of the degrees of the isolated components $\mathfrak{q}_i$ of maximum dimension, and by proposition 7.23, the degree of $[\mathfrak{a}]$ is this sum.

Remark 7.28 Definition 7.25 agrees with the previous definition of $[\mathrm{div}(f)]$ with $f \in k[\mathbb{P}^n]$. In fact, the noetherian decomposition of $(f) = (f_1^{m_1} f_2^{m_2} \dots f_n^{m_n})$ is $(f) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n$ where $\mathfrak{q}_i = (f_i^{m_i})$ and $\mathfrak{p}_i = (f_i)$. Further the length of $k[x_1, x_2, \dots, x_n]_{(f_i)}/(f_i^{m_i})$ is m_i since a maximum chain of ideals is $\mathfrak{p}_i = (f_i) \subset (f_i)^2 \subset \dots \subset (f_i)^m = \mathfrak{q}_i$.

Remark 7.29 Let W be an irreducible k-variety of V defined by $\mathfrak{p} \subset k[V]$. Then $[V] = [\mathfrak{p}]$. Further if W is a k-variety of V then $[\mathfrak{I}_k(W)] = [W_1] + [W_2] + \dots + [W_k]$ where W_i are the components of W.

Proposition 7.30 *Let ν be a positive k-cycle of V^r* without any embedded components, *i.e. if $V \subset W$ are two irreducible components, then at most one of them is a component of ν, and such that the multiplicity of all components with dimension r is one. Then there is an ideal $\mathfrak{a} \subseteq k[V]$ such that $\nu = [\mathfrak{a}]$. In particular, this is the case when ν is a s-cycle with $s < r$.*

Proof. Let $\nu = \sum_{i=1}^n m_i[W_i]$ and let $\mathfrak{p}_i$ be the defining prime ideal of W_i in $k[V]$. If W_i has dimension r we let $\mathfrak{q}_i = \mathfrak{p}_i$. Otherwise $\mathfrak{p}_i$ has at least height one and $\mathfrak{p}_i^{m_i}$ has at least length m_i in $k[V]_{(\mathfrak{p}_i)}$. Thus we can choose a subideal $\mathfrak{q}_i$ of $\mathfrak{p}_i^{m_i}$ such that $\mathfrak{q}_i$ has length m_i.

The ideal $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n$ then satisfies $\nu = [\mathfrak{a}]$. Note that $\mathfrak{q}_i$ may have embedded components and is not necessarily $\mathfrak{p}_i$-primary, but the $\mathfrak{p}_i$-primary component has the correct length. □

Remark 7.31 There are several ideals that gives the same cycle. For example, the ideals (x^2, y) and (x, y^2) of $k[x, y]$ both give the cycle $2[V_K(x, y)]$. Also, the *embedded components* of the ideal does not add anything to the cycle. For example $(xy, x^2) = (x) \cap (x^2, y)$ has the same cycle $[(x)]$ as (x).

Base extensions and absolute cycles

Definition 7.32 If $\nu = \sum_i m_i[V_i]$ is a k-cycle and k'/k a field extension, we define the k'-cycle $\nu_{(k')} = \sum_i m_i \left[\mathfrak{I}_k(V_i)k'[x]\right]$.

Remark 7.33 Note that the cycles ν and $\nu_{(k')}$ have the same degree since the degree of $\left[\mathfrak{I}_k(V_i)k'[x]\right]$ and V_i are equal by remark 7.27. Also, if $\nu = [\mathfrak{a}]$ then $\nu_{(k')} = \left[\mathfrak{a}k'[x]\right]$.

Definition 7.34 The *absolute cycles* of a variety V (defined on any subfield of K) are the elements of the free $\mathbb{Z}$-module over the geometrically integral subvarieties of V (defined on any subfield of K).

Remark 7.35 If k is algebraically closed, a k-cycle is an absolute cycle since all irreducible k-varieties are geometrically integral.

Remark 7.36 Every k-variety W gives an absolute cycle $[\mathfrak{I}_k(W)]_{(\bar{k})}$. If k is perfect, the cycle will be the sum $[W_1] + [W_2] + \cdots + [W_k]$ of the geometrical components of W with multiplicities 1. In fact, the ideal $\mathfrak{I}_k(W)\bar{k}[V]$ is radical and is thus the intersection of prime ideals. If k is not perfect, there may be multiplicities coming from the inseparability of W, e.g. if $W = V_K(x^p - t^p)$ in $\mathbb{F}_p(t^p)$ then $[\mathfrak{I}_k(W)] = [(x^p - t^p)] = [(x-t)^p] = p[V_K(x-t)]$. This is investigated further in the next section.

Example 7.37 Let $\mathfrak{a}$ be the prime ideal $(y^p - x^p t^p, z^p - y^p u^p)$ in $k[x,y,z]$ where $k = \mathbb{F}_p(t^p, u^p)$. In $\bar{k}[x,y,z]$, the ideal is $\mathfrak{q} = \mathfrak{a}\bar{k}[x,y,z] = \big((y-xt)^p, (z-xu)^p\big)$ and its radical is the prime ideal $\mathfrak{p} = (y - xt, z - xu)$. Let $A = \bar{k}[x,y,z]$. The length $l_{A_\mathfrak{p}}/(\mathfrak{q}A_\mathfrak{p})$ is p^2. In fact, a maximal chain of ideals is

$$\begin{aligned} &\big(a,b\big) \supset \big(a,b^2\big) \supset \cdots \supset \big(a,b^p\big) \supset \\ \supset &\big(a^2,ab,b^p\big) \supset \big(a^2,ab^2,b^p\big) \supset \cdots \supset \big(a^2,b^p\big) \supset \\ \supset &\big(a^p,ab,b^p\big) \supset \cdots \supset \big(a^p,b^p\big) \end{aligned}$$

where $a = y - xt$ and $b = z - xu$.

Proposition 7.38 *Base extensions and morphisms of cycles commute. Thus if $f : X \to Y$ is a k-morphism, ν is a k-cycle of X and k'/k an extension, then $f_*(\nu)_{(k')} = f_*(\nu_{(k')})$.*

Proof. By linearity we can assume that $\nu = [V]$. Since morphisms and base extensions of varieties commute we have that the supports of $f_*(\nu)_{(k')}$ and $f_*(\nu_{(k')})$ are equal and hence also their components, which are equidimensional. Thus it is enough to check that the multiplicities equals. Further, it is enough to prove the case when k' is algebraically closed.

Let k_0 be the common minimal field of definition for all the geometrical components of V. Then we only need to prove the proposition for $k' = kk_0 \subseteq \bar{k}$. In fact, the components of $\nu_{(kk_0)}$ are geometrically integral and thus the indices $\deg\big(f(W)/W\big)$ and $\deg\big(f(W_{(k')})/W_{(k')}\big)$ are equal for any component W of $\nu_{(kk_0)}$ and any extension k'/kk_0 by proposition 6.54.

Since V has a finite number of geometric components and the minimal field of definition for each of these components are finitely generated according to proposition 5.35, the common minimal field k_0 is finitely generated and thus kk_0 is a finitely generated field extension of k and it is enough to show the case when $k' = k(f)$.

If f is transcendental, then trivially $\deg\big(f(V)/V\big) = \deg\big(f(V_{(k')})/V_{(k')}\big)$.

Assume that f is algebraic and separable over k. The variety V splits if and only if $k(V) \otimes_k k'$ is not a field. Further $k(V) \otimes_k k'$ is a field if and only if $f \notin k(V)$ and if V splits then it splits into $[k' : k]$ conjugate components. Equivalently $W = f(V)$ splits if and only if $f \notin k(W)$.

If $f \in k(W)$, then both V and W splits since we have an injection $k(W) \hookrightarrow k(V)$. They both split into $[k(f) : k]$ conjugate varieties V_i and W_i and for each pair we have that $[k'(W_i) : k'(V_i)] = [k(W) : k(V)]$. If $f \notin k(W)$ but $f \in k(V)$ we have that V splits into the varieties V_i and that $[k'(V_i) : k'(W_{(k')})] = [k(V) : k(W) \otimes_k k'] = [k' : k][k(V) : k(W)]$. If neither V nor W splits, we have that $[k'(V_{(k')}) : k'(W_{(k')})] = [k(V) : k(W)]$.

Now assume instead that f is inseparable over k. In that case neither V nor W splits but we may get a multiplicity. If $f \in k(V)$ then $f \in k[V]$ and $k[V] \otimes_k k'$ is not reduced. The nilradical $\mathfrak{p}$ of $k[V] \otimes_k k'$ is generated by $f \otimes_k 1 - 1 \otimes_k f$ and the order of (0) in $\mathfrak{p}$ is $[k(f) : k]$. The reduced ring $k'[V] = \big(k[V] \otimes_k k'\big)/\mathfrak{p}$ is equal to $k[V]$.

If $f \in k(W)$, then as in the separable case $f \in k[V], k[W]$ and from the above discussion $[V]_{(k')} = [k(f) : k][V_{(k')}]$, $[W]_{(k')} = [k(f) : k][W_{(k')}]$ and $[k'(V_{(k')}) : k'(W_{(k')})] = [k(V) : k(W)]$. If $f \notin k(W)$ but $f \in k(V)$ we similarly have that $[V]_{(k')} = [k(f) : k][V_{(k')}]$, $[W]_{(k')} = [W_{(k')}]$ and $[k'(V_{(k')}) : k'(W_{(k')})] = [k(V) : k(W) \otimes_k k'] = [k(V) : k(W)]/[k(f) : k]$. If $f \notin k(V), k(W)$ then $[k'(V_{(k')}) : k'(W_{(k')})] = [k(V) : k(W)]$.

In each of the above cases we have that $\big(f_*[V]\big)_{(k')} = f_*\big([V]_{(k')}\big)$. $\square$

Remark 7.39 Proposition 7.38 is a special case of a more general theorem, see [F, Prop. 1.7] that given a fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with X, Y algebraic schemes, f a proper morphism and g a flat morphism then $g^* f_* \nu = f'_* g'^* \nu$. In our case we have the fiber square

$$\begin{tikzcd} X_{(k')} \arrow[r, "f_{(k')}"] \arrow[d, "g'"'] & Y_{(k')} \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

since $X_{(k')} = X \times_Y Y_{(k')}$. Note that in the affine case f is not proper, but it works since our definition of f_* is $f_*[V] = \deg(W/V)[W]$ with $W = \overline{f(V)}$.

A proof of 7.38 can also be found in [K, Ch. I, Lemma 3.1.8].

Remark 7.40 From the proof of proposition 7.38 it also follows that if V is a geometrically irreducible k-variety and k'/k a field extension, then the degree $\deg(V/W)$ is a multiple of $\deg(V_{(k')}/W_{(k')})$.

Remark 7.41 Proposition 7.38 is trivial when f is a projection since then both f_* and base extensions preserves the degree by remark 7.13.

Rational cycles

An important issue is whether an absolute cycle with components defined over $\bar{k}$ supported by V comes from an ideal of $k[V]$. To answer this question we need to define conjugate cycles and the *order of inseparability*.

Definition 7.42 A cycle ν is algebraic over k if all its components are defined on the algebraic closure $\overline{k}$ of k. Two cycles ν and μ are *conjugate* over k if there is a k-automorphism s of $\overline{k}$ in $\mathrm{Gal}(\overline{k}/k)$ that transforms ν into μ, i.e. if $\nu = \sum_i m_i[V_i]$, then $\mu = \sum_i m_i\left[s\left(V_i\right)\right]$ where $s\left(V_i\right)$ is given by the induced $k[X]$-automorphism of $\overline{k}[X]$.

Definition 7.43 Let K/F be a field extension. The *order of inseparability* $[K : F]_\iota$ is the minimal degree $[K : L]$ of all *separable* field extensions L/F.

Remark 7.44 If p is the *characteristic exponent*, i.e. the characteristic of F except that $p = 1$ if $\mathrm{char}(F) = 0$, then the order of inseparability is a power of p. Further if $K/L/F$ are field extensions, then the order of inseparability of K over F is a multiple of the order of inseparability of L over F.

Remark 7.45 Another equivalent definition of the order of inseparability is the minimal inseparability degree $[K : L]_i$ of all purely transcendental field extensions L/F. For more on the order of inseparability, see [W, Ch. I, §8]. Grothendieck calls the order of inseparability for *radical multiplicity* [EGA, Ch. IV:2, Def. 4.7.4].

Definition 7.46 Let k'/k be a field extension and V be a k'-variety. The *order of inseparability* of V over k is $[k(\xi) : k]_\iota$ for a generic point ξ of V.

Remark 7.47 If V is an irreducible k'-variety and k'/k an algebraic field extension, there is a finite number of conjugate varieties of V over k and by proposition 5.42 the restriction $V_{[k]}$ is equal to the union of them. Since a generic point for $V_{[k]}$ is a generic point for V the order of inseparability of V over k is $[k(V_{[k]}) : k]_\iota$.

Example 7.48 Let $k = \mathbb{F}_p(t^p)$. Then the geometrically irreducible k-variety V defined by $x^p - t^p$ has the order of inseparability p over k. The absolute cycle associated to V is $p[V_{(k_0)}]$ where $k_0 = \mathbb{F}_p(t)$.

Remark 7.49 If V is a geometrically integral k-variety, the order of inseparability over k is *not* the degree $[k_0 : k]$ where $k_0 = \mathrm{def}(V)$ is the minimal field of definition for V. In fact, let $k = \mathbb{F}_p(t^p, u^p)$. The k-variety of $\mathbb{P}^2$ defined by $(z^p - t^p x^p - u^p y^p)$ has then order of inseparability p over k, but the minimal field of definition is $k_0 = \mathbb{F}_p(t, u)$ and $[k_0 : k]$ is p^2.

Proposition 7.50 *Let V be a* geometrically irreducible *k-variety and k_0 its minimal field of definition containing k. Let $\mathfrak{a} = \mathfrak{I}_k(V) \subseteq k[x]$ and $\mathfrak{p} = \mathfrak{I}_{k_0}(V_{(k_0)}) \subset k_0[x]$. Then* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a}k_0[x])$ *is the order of inseparability of V over k.*

Proof. It can be shown, see [W, Ch. VIII, §8, Thm 8], that the inseparability order $[k(V) : k]_\iota$ is the number p^f such that

$$[k(V) : k(u)]_i = p^f [k_0(V) : k_0(u)]_i$$

for all transcendence bases u. Choose a transcendence basis u which is generic over k_0. By proposition 6.46 we have that

$$[k(V) : k(u)] = \deg(V) \quad \text{and} \quad [k_0(V) : k_0(u)] = \deg\left(V_{(k_0)}\right).$$

Since $V_{(k_0)}$ is geometrically *integral* we have that $[k_0(V) : k_0(u)]_i = 1$. Further $W = V \cap L$ and $W_{(k_0)} = V_{(k_0)} \cap L$ are identical as *sets* and thus both contains $[k(V) : k(u)]_s = [k_0(V) : k_0(u)]_s$ points. Consequently

$$\deg(V) = [k(V) : k(u)] = [k(V) : k(u)]_s [k(V) : k(u)]_i = \deg\left(V_{(k_0)}\right) p^f$$

and by proposition 7.23 we have that

$$\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}k_0[x]) = \deg(V)/\deg\left(V_{(k_0)}\right) = p^f.$$

□

Definition 7.51 An absolute cycle $\nu = \sum_i m_i[V_i]$ is *rational* over k (or k-rational) if

(1) It is identical to its conjugates over k.

(2) Every m_i is a multiple of the order of inseparability of V_i over k.

Remark 7.52 If ν is rational over k it immediately follows from (1) that ν is algebraic over k. In fact, if ν is not algebraic over k there is a component V with a minimal field of definition containing a transcendental element α. There are then an infinite number of k-automorphisms s which maps α to an arbitrary power of α. These automorphisms will map V to different conjugates $s(V)$ which contradicts the fact that ν has an finite number of components. It also follows that the support of ν is a k-variety.

Remark 7.53 Every linear combination of k-rational cycles is k-rational. If a cycle is k-rational, its homogeneous components and its positive and negative parts are k-rational. Further every product of k-rational cycles is k-rational. In fact the order of inseparability of $V \times W$ divides the product of the orders of inseparability of V and W (see [W, Ch. I, Prop. 8.28]).

Remark 7.54 The set of positive k-rational cycles ordered by the relation in definition 7.4 clearly have minimal elements. The minimal elements, which are called *prime k-rational cycles*, are on the form $p^f \sum_s [s(V)]$ where V is a geometrically integral variety defined on $\overline{k}$ and $\{s(V)\}$ are all the conjugates of V over k. The order of inseparability of V over k is p^f. The prime rational cycles are homogeneous and every k-rational cycle is uniquely determined as a sum of such cycles.

Proposition 7.55 *The k-rational cycles corresponds to k-cycles. The correspondence is given by the base extension $\nu \mapsto \nu_{(\overline{k})}$ which assigns an absolute cycle to every k-cycle.*

Proof. It is clear that the map $\nu \mapsto \nu_{(\overline{k})}$ is injective and we only need to show that every k-rational cycle comes from a k-cycle. Let $\nu = p^f \sum_s [s(V)]$ be a prime k-rational cycle. By proposition 7.50, we have that $p^f \sum_s [s(V)] = [V_{[k]}]_{(\overline{k})}$ and thus ν is the extension of the k-cycle $[V_{[k]}]$. The proposition now follows by linearity since every k-rational cycle is a sum of prime k-rational cycles. □

Proposition 7.56 *A divisor of $\mathbb{P}^n$ is k-rational if and only if it is given by a quotient of polynomials with coefficients in k.*

Proof. The prime rational *divisors* over k of $\mathbb{P}^n$ comes from a single irreducible k-hypersurface and are thus given by irreducible polynomials in $k[\mathbb{P}^n]$. Thus a divisor of $\mathbb{P}^n$ is k-rational if and only if it is the quotient of products of irreducible polynomials in $k[\mathbb{P}^n]$. □

Proposition 7.57 *Let ν be a k-rational cycle supported by V and $f : V \to Y$ a k-morphism. Then $f_*\nu$ is a k-rational cycle.*

Proof. It is clear that f_* and a k-automorphism s of K commute and thus condition (1) is fulfilled. Let W be a component of ν. Since $k\big(f(W_{[k]})\big)$ is a subextension of $k(W_{[k]})$ the order of inseparability of W is a multiple of $f(W)$ by remarks 7.44 and 7.47 and condition (2) follows. $\square$

Remark 7.58 Let V be a geometrically irreducible k-variety with inseparability order p^f over k. Consider the k-rational absolute cycle $p^f[V_{(k_0)}]$ and a *k-projection* $f : V \to Y$. Then

$$f_*\big(p^f[V_{(k_0)}]\big) = p^f \deg(V_{(k_0)}/W_{(k_0)})[W_{k_0}] = \deg(V/W)p^g[W_{k_0}]$$

where $W = f(V)$ and p^g is the inseparability order of W. In fact, as noted in the proof of proposition 7.50 we have that

$$p^f = \frac{\deg(V)}{\deg(V_{(k_0)})} = \frac{\deg(W)}{\deg(W_{(k_0)})}\frac{\deg(V/W)}{\deg(V_{(k_0)}/W_{(k_0)})} = p^g\frac{\deg(V/W)}{\deg(V_{(k_0)}/W_{(k_0)})}.$$

Example 7.59 The converse of proposition 7.57 is not true unless k is perfect. Let $k = \mathbb{F}_p(t^p)$ and $V = V_K(x - ty)$. Then $[V]$ is not k-rational since it has order of inseparability p. Define the morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ by $(x', y') = (x^p, y^p)$. Then $W = f(V) = V_K(x' - t^p y')$ and $\deg(V/W) = 1$ since $K(V) = K(W) = K$ and $f_*[V] = [f(V)]$ which is a k-rational cycle.

Example 7.60 The converse of proposition 7.57 is not true even for projections. Let $k = \mathbb{F}_p(t^p, u^p)$ and let $V = V_K(tx + y, ux + z)$ be a $\bar{k}$-variety. Define the projection $f : \mathbb{P}^2 \to \mathbb{P}^1$ by $x' = x + y + z$ and $y' = x - y - z$, then $f(V) = V_K\big((1 + t + u)x' + (1 - t - u)y'\big)$. The order of inseparability of V over k is p^2 and thus $p[V]$ is not k-rational. However $f_*(p[V]) = [\operatorname{div}\big((1 + t^p + u^p)x'^p + (1 - t^p - u^p)y'^p\big)]$ and is thus k-rational.

Proposition 7.61 *Let ν be an absolute cycle and f a k-morphism. If $f_*\nu$ is a k-rational cycle and either k is perfect or f is a k-*projection *and ν a divisor, then ν is k-rational.*

Proof. If k is perfect the order of inseparability is always 1 and as in proposition 7.57, the cycle ν is k-rational since k-automorphisms of K and f_* commute. The case when f is a k-projection and ν is a divisor, is a result of W. L. Chow which can be found in [S, Ch. II, p. 104]. Note that the condition in this case is that ν is a divisor of an arbitrary variety, not only of $\mathbb{P}^n$ as in proposition 7.56. $\square$

Chapter 8

Chow Varieties

Chow coordinates

Definition 8.1 Let V^r be a projective geometrically integral k-variety in $X = \mathbb{P}^n$ of dimension r. Let $\gamma : X = \mathbb{P}^n \to Y = \mathbb{P}^{r+1}$ be a generic projection over k defined by $y_s = \gamma_s(x) = \sum_{i=0}^n u_{si}x_i$, $s = 0, \dots, r+1$. Since the center of the projection is a generic linear projective variety of dimension $n - (r+2)$, it does not intersect V^r by corollary 6.14. Thus γ defines a k_u-morphism from V to $\mathbb{P}^{r+1}$. We will refer to γ as the *generic projection* of V.

Definition 8.2 The variety $W = \gamma(V)$ has codimension one and is thus defined by a single polynomial $F(y) \in k_u[Y] = k_u[y_0, \dots, y_{r+1}]$. Multiplying F with its denominators in $k[u]$, we get a polynomial $G_V(y,u) \in k[y,u]$. When doing this we also divide with any non-constant common factor in $k[u]$. The coefficients of $G_V \in k[y,u]$ are called the *Chow coordinates* and are unique up to a constant in k.

Proposition 8.3 *The polynomial $G_V \in k[y,u]$ is homogeneous of degree d in y and homogeneous of degree d' in u, where d is the degree of V and d' satisfies the inequality $d' \geq (r+1)d$.*

Proof. The defining polynomial $F \in k_u[y]$ of the variety $W = \gamma(V)$ is homogeneous of degree $\deg(W)$ in y. But $\deg(W) = \deg(V)$ since γ is generic and V is geometrically integral. In fact, by proposition 6.70, the image W is birational to V and by proposition 6.56, they have the same degree. Thus G is homogeneous of degree $d = \deg(V)$ in y.

Since W is geometrically integral, the polynomial $F \in k_u[y]$ is geometrically integral and thus also $G \in k[y,u]$ since G has no non-constant factor in $k[u]$ by construction. The projection, and a fortiori W, is not changed by a multiplication of all the u_{si} by an element of k. Since W is geometrically integral we can make a base extension to an infinite field and thus it follows that the defining equation of $G(y,u) = 0$ is homogeneous in u.

Now by Noether's Normalization Lemma (remark 6.25) we have that y_s is integral over $y_0, \dots, y_{s-1}, y_{s+1}, \dots, y_{r+1}$ in $k_u[W] = k_u[Y]/(G)$. Thus G contains a term cy_s^d, with $c \in k[u]$, for any s. Further $G(y,u) = 0$ remains unchanged when multiplying $u_{s0}, u_{s1}, \dots, u_{sn}$ and y_s by a constant in k since by definition $y_s = \sum_{i=1}^n u_{si}x_i$. Since there is also a non-zero term $c'y_{s'}^d$ in G for any $s' \neq s$, we have that c is of at least degree d in the series of variables $u_{s'0}, u_{s'1}, \dots, u_{s'n}$. Thus c is of at least degree d in each of the series of variables $\{u_{s'i}\}_{i=1}^n$ for $s' \neq s$, i.e. at least of degree $d(r+1)$ in u. Consequently G is homogeneous in u of degree $d' \geq d(r+1)$. □

Remark 8.4 If we take any projection, $f : V \to \mathbb{P}^{r+1}$, not necessarily generic, with coefficients $c_{si} \in k$, we have by the specialization $u_{si} \mapsto c_{si}$ that $G_V(y,c) = 0$ is an equation for $W = f(V)$. Let $H \in k[y]$ be an irreducible polynomial defining W. Then clearly $G_V(y,c) = \lambda H(y)^d$ for some constant $\lambda \in k$ and integer d. Since the degree of G in y is $\deg(V)$ and the degree of H is $\deg(W)$ we have by proposition 6.56 that $d = \deg(V/W)$. Thus $G_V(y,c)$ is the polynomial associated to the divisor $f_*[V] = \deg(V/W)[W] = \big[(G_V(y,c))\big]$.

Example 8.5 Let V be the irreducible hyperplane in $\mathbb{P}^2$ given by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and let $y_s = u_{s0}x_0 + u_{s1}x_1 + u_{s2}x_2$, $s = 0,1,2$ be its generic projection γ. A straightforward method to find the equation for the hyperplane $W = \gamma(V)$ in k_u, and thus its Chow coordinates, is to use a generic point. A generic point for V is $P = (\frac{1}{a_0}\xi, \frac{\lambda}{a_1}\xi, -\frac{1+\lambda}{a_2}\xi)$, where ξ and λ are transcendental over k. The projection Q of the generic point P is given by the equations $y_s = \frac{1}{a_0}\xi u_{s0} + \frac{\lambda}{a_1}\xi u_{s1} - \frac{1+\lambda}{a_2}\xi u_{s2}$. All other points are specializations of these equations. Eliminating the two transcendental variables ξ and λ we thus get an equation for all points in $W = \gamma(V)$. A lengthy calculation gives:

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{vmatrix} y_0 + \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ a_0 & a_1 & a_2 \\ u_{20} & u_{21} & u_{22} \end{vmatrix} y_1 + \begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ a_0 & a_1 & a_2 \end{vmatrix} y_2 = 0$$

This is of course generalizable to linear hyperplanes in $\mathbb{P}^n$.

Example 8.6 The generic projection of the hyperplane $x_0 = 0$ in $\mathbb{P}^3$ is given by the equation:

$$\begin{vmatrix} y_0 & u_{01} & u_{02} \\ y_1 & u_{11} & u_{12} \\ y_2 & u_{21} & u_{22} \end{vmatrix} = 0$$

Example 8.7 Let V be the irreducible variety given by the point $(a_0 : a_1 : \cdots : a_n)$ in $\mathbb{P}^n$. A generic point for V is $(a_0\xi : a_1\xi : \cdots : a_n\xi)$. The image of the generic point by its generic projection is $y_s = \sum_{i=1}^n u_{si}a_i\xi$, $s = 0,1$. Eliminating ξ gives the equation $y_0 \sum_{i=1}^n u_{1i}a_i = y_1 \sum_{i=1}^n u_{0i}a_i$ or equivalently $\sum_{i=1}^n (y_0u_{1i} - y_1u_{0i})a_i = 0$.

Chow form

Proposition 8.8 *The sets of $r+1$ hyperplanes, which have a common intersection with a geometrically integral projective k-variety V^r, are parameterized by an irreducible hypersurface C in $(\mathbb{P}^n)^{r+1}$.*

Proof. Let the $r+1$ hyperplanes be given by the equations $\sum_{i=0}^n u_{si}x_i = 0$, $s = 0,1,\ldots,r$. The hyperplanes and their common intersection is the points of the k-variety of $\mathbb{P}^n \times (\mathbb{P}^n)^{r+1}$ given by the ideal $\big(\mathfrak{I}\mathfrak{H}_k(V), \sum_{i=0}^n u_{0i}x_i, \sum_{i=0}^n u_{1i}x_i, \ldots, \sum_{i=0}^n u_{ri}x_i\big)$. As we previously discussed in remark 6.52, this variety consists of the k-specializations of a generic point λ over $k(\xi)$ satisfying

$$\sum_{i=0}^{n} \xi_i \lambda_{si} = 0, \quad s = 0, 1, \dots, r \tag{8.1}$$

and is thus an irreducible k-variety. Consequently, the projection of this variety onto $(\mathbb{P}^n)^{r+1}$ is also an irreducible k-variety.

To calculate the dimension of C, we calculate the transcendence degree of $k(\lambda) = k\left(\{\lambda_{si}/\lambda_{sj}\}_{s=0,\dots,r;\, i=0,\dots,n}\right)$ over k, which is equal to the dimension of C since λ is a generic point of C. The transcendence degree of $k(\lambda, \xi)$ over $k(\xi)$ is $(r+1)n - (r+1)$. In fact, the $(r+1)n$ elements $\left(\lambda_{si}/\lambda_{sj}\right)_{i \neq j}$ are transcendental except for the $r+1$ algebraic dependencies in 8.1. Further ξ_i/ξ_j is in $k(\lambda)$ since it is the only point in the intersection of the $k(\lambda)$-varieties V and $\sum_{i=0}^{n} \lambda_{si} x_i = 0$, $s = 0, 1, \dots, r$. In fact, the intersection of V and any choice of r hyperplanes from $\lambda_0, \lambda_1, \dots, \lambda_r$ contains ξ and its conjugates over $k(\lambda)$ by proposition 6.48 and thus the intersection of V and all hyperplanes consists of the single point ξ.

Thus the transcendence degree of $k(\lambda, \xi) = k(\lambda)$ over k is

$$\text{tr.deg}_{k(\xi)}(\lambda) + \text{tr.deg}_k\big(k(\xi)\big) = (r+1)(n-1) + r = (r+1)n - 1$$

which proves that C is a hypersurface of $(\mathbb{P}^n)^{r+1}$. □

Definition 8.9 The defining polynomial $F_V \in k[u_0, u_1, \dots, u_r]$ of C, which is irreducible, is called the *Chow form* and is unique up to an element in k. It is also called the associated form, Cayley form or Chow-van-der-Waerden form.

Lemma 8.10 *Let $\{\mu_q\}_{1 \le q \le d}$ be d algebraic, separable and conjugate points over $\mathbb{P}^n(K)$, i.e. μ_{qi}/μ_{qj} are separably algebraic over k and for all pairs $1 \le q, r \le d$ there is a unique k-automorphism which maps μ_q to μ_r. Then $\prod_{q=1}^{d} \sum_{i=0}^{n} \mu_{qi} t_i$ is a polynomial in $k[t]$.*

Proof. Let $\{s_q\}$ be k-automorphisms on $k[\mu]$ such that $s_q(\mu_{1i}) = \mu_{qi}$. Then we can extend the automorphisms $\{s_q\}$ to $k[\mu, t]$ by $s_q(t_i) = t_i$ and $\{s_q\}$ are made into $k[t]$-automorphisms of $k[\mu, t]$. Further $s_q\left(\sum_{i=0}^{n} \mu_{1i} t_i\right) = \sum_{i=0}^{n} \mu_{qi} t_i$ which proves that $\sum_{i=0}^{n} \mu_{qi} t_i$ are conjugate over $k[t]$. But then they are the roots of an irreducible polynomial in $k[t][\mu]$ of degree d and their product is in $k[t]$. □

Proposition 8.11 *$F_V(u_0, u_1, \dots, u_r)$ is homogeneous of degree $d = \deg(V)$ in each of the u_s.*

Proof. Consider a set of r generic hyperplanes defined by $\{u_s\}_{1 \le s \le r}$. By proposition 6.48, the generic linear variety L^{n-r} they define, intersects V in d points $\{\mu_q\}_{1 \le q \le d}$, separable and conjugate over $k(u_1, u_2, \dots, u_r)$, where d is the degree of V. The hyperplanes, given by $u_{00}x_0 + u_{01}x_1 + \dots + u_{0n}x_n$, which have a common intersection with V and the r hyperplanes, intersects any of the points μ_q and are thus given by the equation $\prod_{q=1}^{d} \left(\sum_{i=0}^{n} u_{0i} \mu_{qi}\right) = 0$.

By lemma 8.10, this equation is a polynomial in u_0 with coefficients in $k(u_1, u_2, \dots, u_r)$. If we multiply this polynomial with its denominators in $k[u_1, u_2, \dots, u_r]$ we get an irreducible polynomial in $k[u_0, u_1, \dots, u_r]$ which defines the same variety C as in proposition 8.8.

The defining equation $F(u_0, u_1, \ldots, u_r)$ of C is thus homogeneous of degree d in u_0 and consequently, since it is symmetric in $u_0, u_1, \ldots, u_r$, it is multihomogeneous of degree d in each series of variables $u_{s0}, u_{s1}, \ldots, u_{sn}$. □

Example 8.12 Let V be the irreducible hyperplane in $\mathbb{P}^2$ given by $a_0x_0 + a_1x_1 + a_2x_2 = 0$. To find the Chow form, we look upon the equations for two hyperplanes which both intersects the generic point $P = (\frac{1}{a_0}\xi, \frac{\lambda}{a_0}\xi, -\frac{1+\lambda}{a_2}\xi)$. This gives us the equations

$$\frac{1}{a_0}u_{s0} + \frac{\lambda}{a_1}u_{s1} - \frac{1+\lambda}{a_2}u_{s2} = 0, \quad s = 0, 1.$$

Eliminating λ, we retrieve the Chow form

$$\begin{vmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ a_0 & a_1 & a_2 \end{vmatrix} = 0.$$

Example 8.13 The Chow form of the point $(a_0 : a_1 : \cdots : a_n)$ of $\mathbb{P}^n$ is

$$\sum_{i=1}^{n} u_{0i}a_i = 0$$

Example 8.14 Another way to calculate the Chow form for a variety V^r is to first find the d generic intersection points $\{\mu_s\}$ of a generic linear variety L^{n-r} and then calculate the form as in proposition 8.11. Let V be the second degree hypersurface $x^2 - yz$ of $\mathbb{P}^2$. The intersection points of V with $u_{10}x + u_{11}y + u_{12}z = 0$ are

$$\mu_1, \mu_2 = \left(u_{11}, -\frac{1}{2}\left(u_{10} \pm \sqrt{u_{10}^2 - 4u_{11}u_{12}}\right), \frac{u_{11}^2}{\frac{1}{2}\left(u_{10} \pm \sqrt{u_{10}^2 - 4u_{11}u_{12}}\right)} \right).$$

The Chow form is then given by

$$\begin{aligned} F(u_0) &= \left(\sum_{i=1}^{n} u_{0i}\mu_{1i}\right)\left(\sum_{i=1}^{n} u_{0i}\mu_{2i}\right) = \cdots = \\ &= \frac{u_{11}}{u_{12}}\left[(u_{01}u_{12} - u_{02}u_{11})^2 + (u_{01}u_{10} - u_{00}u_{11})(u_{02}u_{10} - u_{00}u_{12})\right] \end{aligned}$$

or normalized

$$F(u_0, u_1) = (u_{01}u_{12} - u_{02}u_{11})^2 + (u_{01}u_{10} - u_{00}u_{11})(u_{02}u_{10} - u_{00}u_{12}).$$

Equivalence of Chow coordinates

In this section we will see that there is an equivalence between the Chow coordinates of the generic projection and the coefficients of the Chow form.

Proposition 8.15 *The Chow coordinates, i.e. the coefficients of $G_V(y, c)$, are given by the coefficients of the Chow form $F_V(u)$. They can explicitly be calculated using the equation*

$$G_V(y_0, \ldots, y_r, 1,\ u) = F_V\ (y_0u_{r+1} - u_0, \ldots, y_ru_{r+1} - u_r)$$

In particular, we have that the Chow coordinates are given by linear combinations with coefficients in the prime ring (i.e. the ring generated by 1, either $\mathbb{Z}$ or $\mathbb{F}_p$) of the coefficients of the Chow form.

Proof. Consider the affine part $U = \{y_{r+1} \neq 0\}$ of $\mathbb{P}^{r+1}$ and the generic projection γ in definition 8.1. Let $y \in U$ and consider the *projectant* $\gamma^{-1}(y)$. It is the subset W of X fulfilling $\frac{\gamma_s(x)}{\gamma_{r+1}(x)} = \frac{y_s}{y_{r+1}}$ giving us the equations

$$y_s \sum_{i=0}^{n} u_{r+1,i} x_i - y_{r+1} \sum_{i=0}^{n} u_{s,i} x_i = \sum_{i=0}^{n} (y_s u_{r+1,i} - y_{r+1} u_{s,i})\, x_i = 0, \quad s = 0, 1, \ldots, r.$$

Thus $W = \gamma^{-1}(y)$ is the intersection W of $r+1$ hyperplanes defined by $\sum_{i=0}^{n} v_{si} x_i = 0$ with $v_{si} = y_s u_{r+1,i} - y_{r+1} u_{s,i}$. By the definition of the Chow form, the equation $F(v_0, \ldots, v_r) = 0$ is satisfied if and only if W intersects V. If we substitute with $v_{si} = y_s u_{r+1,i} - y_{r+1} u_{s,i}$ in $F(v)$ we get a polynomial $\tilde{F}(y, u)$ which is homogeneous of degree $(r+1)d$ in y and of degree $(r+1)d$ in u.

If $y \in U$, then $\tilde{F}(y, u) = 0$ when $W = \gamma^{-1}(y)$ intersects V or equivalently when $y \in \gamma(V)$, that is $G(y, u) = 0$. On the other hand, when $y_{r+1} = 0$ then $\tilde{F}(y, u) = 0$ if and only $\gamma_{r+1}(x) = 0$ for some $x \in V$, or equivalently that $\gamma(V)$ intersects the hyperplane H given by $y_{r+1} = 0$. Since $\gamma(V) \cap H$ has dimension $r - 1$, this is true when $r > 0$.

Hence we have that $\tilde{F}(y, u) = \alpha G(y, u)^a (y_{r+1})^b$ for an $\alpha \in k$ and $a, b \in \mathbb{N}$. But $G(y, u)$ is homogeneous of degree $d' \geq (r+1)d$ in u and of degree d in y. Thus we have the relations $(r+1)d = d'a$ and $(r+1)d = da + b$ which give us $a = 1$, $b = rd$ and $d' = (r+1)d$. The inequality $d' \geq (r+1)d$ in proposition 8.3 is thus an equality.

We have thus shown that

$$F\left(y_0 u_{r+1} - y_{r+1} u_0, \ldots, y_r u_{r+1} - y_{r+1} u_r\right) = F(v_0, \ldots, v_r) = \tilde{F}(y, u) = G(y, u) y_{r+1}^{rd}$$

where the "equality" is up to a constant of k. □

To conclude the equivalence between the Chow coordinates and the coefficients of the Chow form we have the converse.

Corollary 8.16 *The Chow form is given by the Chow coordinates. The coefficients of the Chow form are given by linear combinations of the Chow coordinates with coefficients in the prime ring.*

Proof. From proposition 8.15 we have $G_V(0, \ldots, 0, 1,\ c) = F_V(-c_0, -c_1, \ldots, -c_r)$. Thus F is determined by G. In fact, every coefficient of F is equal to a coefficient of G up to sign. □

Due to the above correspondence between the coefficients of the Chow forms and the Chow coordinates we will, in spite of the ambiguity, also call the coefficients of the Chow form for the Chow coordinates.

The Chow coordinates in the original sense, i.e. the coefficients of the polynomial $G_V(y, u)$ can be seen as a point of $\mathbb{P}^{\binom{(r+2)+d-1}{d}-1} \times \mathbb{P}^{\binom{(r+2)(n+1)+d(r+1)-1}{d(r+1)}-1}$ which using the Segre embedding is a point of $\mathbb{P}^N$ with $N = \binom{(r+2)+d-1}{d}\binom{(r+2)(n+1)+d(r+1)-1}{d(r+1)} - 1$. The Chow coordinates as the coefficients of the Chow form $F_V(u)$ is a point of $\left(\mathbb{P}^{\binom{(n+1)+d-1}{d}-1}\right)^{r+1}$ or using the Segre embedding, a point of $\mathbb{P}^N$ with $N = \binom{(n+1)+d-1}{d}^{r+1} - 1$.

CHOW COORDINATES FOR ABSOLUTE CYCLES

Eventually we will show that there exists a projective k-variety that parameterizes the k-cycles of pure dimension r supported by a k-variety V^r of $\mathbb{P}^n$. In the construction we will however look at all *absolute cycles* supported by V, thus the components of a cycle can be defined on any field k'/k contained in K. Note that we do not require that V should be irreducible.

Definition 8.17 Let $\nu = \sum_i m_i[V_i]$ be a positive absolute r-cycle. Let $F_{V_i}(u)$ be the Chow form of V_i defined in definition 8.9. We define the *Chow form* of ν to be $F_\nu(u) = \prod_i F_{V_i}(u)^{m_i}$. It is a homogeneous polynomial of degree $d(\nu)$ in each of the $r+1$ series of variables u_s. Equivalently we define $G_\nu(y,c) = \prod_i G_{V_i}(y,c)^{m_i}$ which is a homogeneous polynomial of degree $d(\nu)$ in y and $d(\nu)(r+1)$ in c. The coefficients of $G_\nu(y,c)$ are called the *Chow coordinates*.

Remark 8.18 It is easy to see that we retain the correspondence in proposition 8.15 and corollary 8.16 between the coefficients of $F_\nu(u)$ and $G_\nu(y,c)$. We can thus as before also call the coefficients of $F_\nu(u)$ the Chow coordinates.

Remark 8.19 Since γ is generic and birationally maps V_i onto $\gamma(V_i)$ we have that $\deg\big(V_i/\gamma(V_i)\big) = 1$ and $\big[\big(G_\nu(y,u)\big)\big] = \gamma_*\nu$. Following remark 8.4 we have that if $f : V \to \mathbb{P}^{r+1}$ is an arbitrary projection given by coefficients c, then $\big[\big(G_\nu(y,c)\big)\big] = f_*\nu$.

Remark 8.20 Let $r \in k[x] = k[\mathbb{P}^n]$, $n > 1$, and consider the positive divisor $\nu = [\operatorname{div}(r)]$. The generic projection is a projection of $\mathbb{P}^n$ to $\mathbb{P}^n$ with no center and is thus a linear invertible transformation of the coordinates, $y = ux$, $x = u^{-1}y$. Further $G_\nu(y,u) = r(u^{-1}y)\det(u) = r(\operatorname{adj}(u)y)$. In fact, let $r = \prod r_i^{m_i}$ and $V_i = V_K(r_i)$. The k_u-variety $\gamma(V_i)$ is defined by $V_K\big(r_i(u^{-1})\big)$ and after clearing denominators we get $G_{V_i}(y,u) = r_i(u^{-1}y)\det(u) = r_i(\operatorname{adj}(u)y)$. The polynomial for ν is thus $G_\nu(y,u) = \sum G_{V_i}(y,u)^{m_i} = r(\operatorname{adj}(u)y)$. Consequently, the Chow coordinates for ν are given by linear transformations, with coefficients in the prime ring, of the coefficients of r.

Definition 8.21 Let ν be an arbitrary r-cycle. The Chow coordinates of ν are the biprojective coordinates in $\mathbb{P}^N \times \mathbb{P}^N$ given by the coordinates for ν_+ and ν_-.

Proposition 8.22 *If ν is a k-rational r-cycle, the Chow coordinates are in k.*

Proof. By proposition 7.57, the projection $f_*\nu$ is a k-rational divisor. Thus as we have saw in proposition 7.56, the corresponding rational function $G(y,u)$ is k-rational, i.e. the Chow coordinates are in k. □

Remark 8.23 As example 7.60 the converse of proposition 8.22 is not always true if k is not perfect. It is however true when k is perfect or ν is a divisor, i.e. of codimension 1 as proposition 7.61 shows (cf. [S, p. 47]).

CHOW VARIETY

A natural question to ask is whether the set of Chow coordinates, which come from some r-cycle of V, is a variety, i.e. if there is a subvariety $\operatorname{Chow}_r(V)$ of $\mathbb{P}^N$ such that every point of $\operatorname{Chow}_r(V)$ corresponds to the Chow coordinates to an r-cycle supported by V. This is indeed the case, but first we show that a cycle is uniquely determined by its Chow coordinates, and thus that the points of $\mathrm{Chow}_r(V)$ correspond to the r-cycles of V.

Proposition 8.24 *Let ν be a r-cycle of degree d in $\mathbb{P}^n$. Then the Chow coordinates of ν uniquely determine ν.*

Proof. Factoring the Chow form in irreducible factors, we are taken to the case were $\nu = [V]$. Let $G(y, c)$ be the equation of the variety $W = f(V)$.

Let c be the coefficients of a projection $g : \mathbb{P}^n \to \mathbb{P}^{r+1}$. We define the hypercone $V_c = g^{-1}(W)$ over W, which includes the points in the center of g, i.e. those which maps to the forbidden origin. We have that V_c is defined by $G(\sum_{i=0}^n c_{si}x_i, c) = 0$. Clearly $V \subseteq V_c$ for any c.

Now choose a point $P \notin V$, then by proposition 6.6 there is a linear variety L^{n-r-1} containing P such that $V \cap L = \emptyset$. Now intersect L^{n-r-1} with any hyperplane which does not contain P. Then we get a linear variety L'^{n-r-2} which does not intersect neither V nor P and thus determines a projection g with coefficients c, defined on both V and P. It is clear that $P \notin V_c$. Thus $V = \bigcap_c V_c$, and V is uniquely determined from the Chow Coordinates. $\square$

Let $F(u_0, u_1, \ldots, u_r)$ be a homogeneous form of degree d in every series of variables u_s. We will now proceed to show that F corresponds to a cycle supported by a k-variety $V \subseteq \mathbb{P}^n$ if and only if the coefficients satisfy a system of homogeneous equations in k, i.e. the forms corresponding to cycles in V is a k-variety.

Lemma 8.25 *A homogeneous form $F(u_0, u_1, \ldots, u_r) \in k[u_0, u_1, \ldots, u_r]$ of degree d in each of the series of variables u_s, is the Chow form of a cycle ν supported by a k-variety $V \subseteq \mathbb{P}^n$ if and only if the following four properties hold.*

(C1) *In the algebraic closure $\overline{k_u}$ of $k_u = k(u_1, u_2, \ldots, u_r)$, the form $F(u_0, u_1, \ldots, u_r)$ splits into a product $F'(u_0, \mu_1, \ldots, \mu_d) = \prod_{q=1}^d \left(\sum_{i=0}^n u_{0i}\mu_{qi}\right)$, where μ_q are d points in $\mathbb{P}^n(\overline{k_u})$.*

(C2) *For each of the points μ_q and all $s = 1, \ldots, r$ we have that $\sum_{i=0}^n u_{si}\mu_{qi} = 0$.*

(C3) *Let $(v_s)_{s=0,\ldots,r}$ define $r+1$ hyperplanes. If they all pass through one of the points μ_q, then $F(v_0, v_1, \ldots, v_r) = 0$.*

(C4) *The points μ_q are in V.*

Proof. First note that we can assume that k is algebraically closed. The properties (C1), (C2) and (C4) for a product of irreducible forms $F = F_1F_2 \ldots F_n$ are clearly equivalent to the corresponding properties for each component F_i. Further it is clear that F verifies (C3) when each irreducible form F_i does. To show the converse, we need the following property which is equivalent to (C3)

(C3′) Let $(v_s)_{s=1,\ldots,r}$ define r hyperplanes. If they all pass through one of the points μ_q, then $F(u_0, v_1, \ldots, v_r)$ in $k(v_1, \ldots, v_r)[u_0]$ is a multiple of $\sum_{i=0}^n u_{0i}\mu_{qi}$.

Assume that (C3′) holds for F. It is enough to prove that (C3′) holds for F_1 for every *generic* system of hyperplanes (v_s) among those which pass through a point μ_q

corresponding to F_1. By (C3$'$) for F then $\sum_{i=0}^{n} u_{0i}\mu_{qi}$ divides $F(u_0, v_1, \ldots, v_r)$ and thus $F_2(u_0, v_1, \ldots, v_r)$ for an irreducible factor F_2 of F. By property (C2), the system of hyperplanes $(u_s)_{s=1,\ldots,r}$ passes through μ_q. Since (v_s) is generic we have a specialization from (v_s) to (u_s). Thus $\sum_{i=0}^{n} u_{0i}\mu_{qi}$ divides $F_2(u_0, u_1, \ldots, u_r)$ in $k(u_1, \ldots, u_r)[u_0]$. But according to (C1) the factor $\sum_{i=0}^{n} u_{0i}\mu_{qi}$ is in the decomposition of F_1 in $k(u_1, \ldots, u_r)[u_0]$. Since F_1 and F_2 are irreducible and contains the same factor $\sum_{i=0}^{n} u_{0i}\mu_{qi}$, they are equal. Thus $\sum_{i=0}^{n} u_{0i}\mu_{qi}$ divides $F_1(u_0, v_1, \ldots, v_r)$ which proves (C3$'$) for F_1.

Thus we have proven that (C1)-(C4) are true for F if and only if the same properties are true for each irreducible factor F_i. We will therefore assume that F is irreducible. From the definition of the Chow form of an irreducible variety it follows that the four properties are necessary for F to be a Chow form supported by V. Left to prove is that they are also sufficient.

Let F be an irreducible form fulfilling (C1)-(C4). Let W' be the irreducible k_u-variety generated by μ_1. Since F is irreducible over k_u, the points μ_q are conjugates over k_u and thus $\mu_q \in W'$. We restrict this to the irreducible k-variety $W = W'_{[k]}$ which is geometrically integral since k is algebraically closed. We want to show that W has dimension r. To do this it is enough, see corollary 6.49, to show that there is a finite number of generic points of W in $W \cap L$ where L^{n-r} is the generic linear variety given by u_{si}. By (C2) we already know that the d points (μ_q) are in the intersection and we will show that these are the only points.

Let λ be a generic point of W over k in $W \cap L$. Since λ and μ_1 are generic points of the same variety, we have an k-isomorphism between $\varphi : k(\lambda) \to k(\mu_1)$. We extend φ to a k-isomorphism $\widetilde{\varphi} : k(\lambda, u_1, \ldots, u_r) \to k(\mu_1, v_1, \ldots, v_r)$. From the equations $\sum_{i=1}^{n} u_{si}\lambda_i = 0$, $s = 1, 2, \ldots, r$, stating that $\lambda \in L$, and the isomorphism we deduce that $\sum_{i=1}^{n} v_{si}\mu_{1i} = 0$. By (C3$'$) it then follows that $F(u_0, v_1, \ldots, v_n)$ is a multiple of $\sum_{i=1}^{n} u_{0i}\mu_{1i}$. Using the isomorphism again, we have that $F(u_0, u_1, \ldots, u_n)$ is a multiple of $\sum_{i=1}^{n} u_{0i}\lambda_i$. Thus by the unique factorization of (C1) λ is one of the μ_q.

By (C4) μ_1 is in V and thus $W \subseteq V$. It is now immediately clear that F is the Chow form of W. In fact, the μ_q are the same and by (C1) they uniquely determine the Chow form. It is also clear that no other variety have the same Chow form, which proves proposition 8.24 a second time. □

Lemma 8.26 *Let V be a k-variety. The conditions* (C1)-(C4) *for a form F with coefficients ω_λ are equivalent to a system of polynomial equations $H_\alpha(\omega_\lambda)$ in ω_λ with coefficients in k.*

Proof. We consider the coefficients ω_λ and the points μ_q as variables. The form $F(\omega_\lambda, u_s) = F(u_s)$ is thus a polynomial in the prime ring. The coefficients of F is considered as a point in $\omega_\lambda \in \mathbb{P}^N$ where $N = \binom{d+(n+1)-1}{(n+1)-1}^{r+1} - 1 = \binom{d+n}{d}^{r+1} - 1$ and d is the degree of F in each of $u_0, u_1, \ldots, u_r$.

First we want to express (C3) as a system of equations. The general solution for $\sum_{i=0}^{n} v_{si}\mu_{qi} = 0$, $s = 0, \ldots, r$ for any $q = 1, \ldots, d$ is $v_{si} = \sum_{i'=0}^{n} a_{ii'}\mu_{qi}$ where $a_{ii'}$ are variables satisfying $a_{ii'} = -a_{i'i}$. Inserting this in $F(\omega_\lambda, v_0, \ldots, v_r) = 0$ for q we get an equation $P_q(a_{ii'}, \omega_\lambda, \mu_q) = 0$ in the prime ring. That this is zero for every q and every choice of $a_{ii'}$ is equivalent to setting every coefficient in the polynomials to zero, which gives us $d \cdot n(n+1)/2$ equations $P_{qk}(\omega_\lambda, \mu_q) = 0$.

We let Q_α be a system of equations for V. The four properties (C1)-(C4) for F are then fulfilled exactly when the equations

(E1) $F(\omega_\lambda, u_s) = F'(\mu_q, u_s)$

(E2) $\sum_{i=0}^{n} u_{si}\mu_{qi} = 0, s = 1, \ldots, r, q = 1, \ldots, d$

(E3) $P_{qk}(\omega_\lambda, \mu_q) = 0, k = 1, \ldots, n(n+1)/2$

(E4) $Q_\alpha(\mu_q) = 0$

are fulfilled for all u_s and some choice of μ_q. Note that the polynomial identity $F = F'$ in (C1) is equivalent to the corresponding equation (E1) since it should be fulfilled for all points u, which have coordinates in the infinite field K.

The coefficients of the polynomials in (E1)-(E3) are all in the prime ring and the polynomials in (E4) have coefficients in k. This gives us a k-variety of the space $\mathbb{P}^N \times (\mathbb{P}^n)^{r+1} \times (\mathbb{P}^n)^d$ with points $(\omega_\lambda,\ u_0, \ldots, u_r,\ \mu_1, \ldots, \mu_d)$. By corollary 4.12, the projection onto $\mathbb{P}^N \times (\mathbb{P}^n)^{r+1}$ maps this k-variety onto a k-variety C. Since the equations (E1)-(E4) should be true for all choices of u_s, we equal the coefficients of every monomial in u for each defining equation of C, to zero, giving us new equations H_β for a k-variety of $\mathbb{P}^N$. The points (ω_λ) in this k-variety corresponds to forms given by (ω_λ) which satisfy the four properties (C1)-(C4). □

Theorem 8.27 (Chow Variety) *The r-cycles of degree d supported by a k-variety* $V \subseteq \mathbb{P}^n$ *are parameterized by a projective algebraic k-variety* $\mathrm{Chow}_{r,d}(V)$ *called the* Chow variety.

Proof. By lemma 8.25 and 8.26 a form is the Chow form of a cycle supported by V if and only if its coefficients are in the k-variety given by H_α. Further proposition 8.24 shows that there is a one-to-one correspondence between cycles and their Chow forms. □

Example 8.28 Let $r = 0$ and let V be a k-variety of Pn. In this case the Chow variety $\mathrm{Chow}_{r,d}$ is easily described. In fact, both conditions (C2) and (C3′) are trivially fulfilled. Let

$$F(u_{00}, u_{01}, \ldots, u_{0n}) = \prod_{q=1}^{d}\left(\sum_{i=0}^{n} u_{0i}\mu_{qi}\right) = \sum_{i_1, i_2, \ldots, i_d} \widetilde{\omega}_{i_1 i_2 \ldots i_d} u_{0i_1} u_{0i_2} \ldots u_{0i_d}.$$

where $\widetilde{\omega}_{i_1 i_2 \ldots i_d} = \mu_{1i_1}\mu_{2i_2} \ldots \mu_{di_d}$. Equation (E4) states that $\mu_q \in V$. Thus $\widetilde{\omega}$ is the point in the Segre embedding of V^d corresponding to $(\mu_1, \mu_2, \ldots, \mu_d)$. The first equation (E1) is $\omega_{i_1 i_2 \ldots i_d} = \sum_j \widetilde{\omega}_{j_1 j_2 \ldots j_d}$ where $i_1 \leq i_2 \leq \cdots \leq i_d$ and the sum is over all permutations $j_1, j_2, \ldots, j_d$ of $i_1, i_2, \ldots, i_d$. Thus $\omega_{i_1 i_2 \ldots i_d}$ are the *multilinear symmetric polynomials* in $k[V^d]$, i.e. homogeneous of degree 1 in each μ_q.

When the characteristic of k is zero, the multilinear symmetric polynomials generate the multihomogeneous *elementary* symmetric polynomials of $k[V^d]$, see [Ne], and thus $k[V^d]^{\mathfrak{S}_d}$ which consists of the multihomogeneous symmetric polynomials. Consequently, in characteristic zero, we have that $\mathrm{Chow}_{0,d}(V) = V^d/\mathfrak{S}_d = \mathrm{Sym}^d(V)$. In positive characteristic, it is not always true that $\mathrm{Chow}_{0,d}(V) = \mathrm{Sym}^d(V)$. However, it can be shown, see [Ne] or [Na] and the discussion on page 71, that the normalization of $\mathrm{Chow}_{0,d}(V)$ is $\mathrm{Sym}^d(V)$.

Example 8.29 Let $r = n - 1$. An r-cycle ν of degree d in $\mathbb{P}^n$ is a divisor and thus corresponds to homogeneous polynomial p of degree d in $k[x_0, x_1, \ldots, x_n]$. As we noted in remark 8.20, the Chow coordinates of $\nu = [\mathrm{div}(p)]$ are given by an invertible linear transformation of the coefficients of p. The homogeneous polynomials of degree d can be seen as points of $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$. Consequently, we have an isomorphism of varieties $\mathrm{Chow}_{n-1,d}(\mathbb{P}^n) \simeq \mathbb{P}^N$.

Chow variety for k-cycles

In proposition 7.61 we showed that the projection $f_*\nu$ of a cycle ν is k-rational if and only if ν is k-rational when k is perfect or ν is a divisor. Thus when k is perfect or we look upon the cycles of codimension one, the Chow forms with coefficients in k corresponds to k-rational cycles, cf. prop. 8.22. Further by proposition 7.55, the k-rational cycles of V corresponds to the k-cycles of V. Hence we have proved the following theorem.

Theorem 8.30 *The positive k-cycles of degree d and dimension r supported by a k-variety V of $\mathbb{P}^n$ corresponds to the k-rational points of the Chow Variety* $\mathrm{Chow}_{r,d}(V)$*, if k is perfect or V is of pure dimension $r+1$.*

Corollary 8.31 *The positive k-cycles of degree d and dimension r supported by a k-quasi-variety U of $\mathbb{P}^n$ corresponds to the k-rational points of a k-variety, denoted the Chow Variety* $\mathrm{Chow}_{r,d}(U)$*, if k is perfect or $\overline{U}$ is of pure dimension $r+1$.*

Proof. Let $U = V \setminus W$ where $V = \overline{U}$ and $W \subset V$ is a k-variety. The cycles supported by U are the cycles supported by V such that no component is supported by W. The cycles which have at least one component in W are parameterized by the k-variety D with the equations (E1)-(E3) of lemma 8.26 and the equations

(E4′) $\prod_{q=1}^{d} Q_{\alpha_q}(\mu_q) = 0, \quad \alpha_1, \alpha_2, \ldots, \alpha_d \in \mathscr{I}$

where $\{Q_\alpha\}_{\alpha\in\mathscr{I}}$ is a generating set for the ideal $\mathfrak{I}\mathfrak{H}_k(W)$. The cycles supported by U are thus parameterized by the k-quasi-variety $\mathrm{Chow}_{r,d}(U) = \mathrm{Chow}_{r,d}(V) \setminus D$. □

Remark 8.32 From the definition of the Chow Variety it is immediately clear that if V is a k-variety then $\mathrm{Chow}_{r,d}(V_{(k')}) = \mathrm{Chow}_{r,d}(V)_{(k')}$.

Definition 8.33 For a k-(quasi-)variety V of $\mathbb{P}^n$ we let $\mathrm{Chow}_r(V) = \coprod_{d\in\mathbb{N}} \mathrm{Chow}_{r,d}(V)$ be the disjoint union of $\mathrm{Chow}_{r,d}(V)$ for $d = 0, 1, \ldots$.

Remark 8.34 Note that the "Chow Variety" $\mathrm{Chow}_r(V)$ is not noetherian, only locally noetherian, and thus not a variety in the strict sense. We will refer to $\mathrm{Chow}_r(V)$ as the Chow Variety and when k is perfect or V is of pure dimension $r+1$, the k-rational points of $\mathrm{Chow}_r(V)$ corresponds to the positive k-cycles of pure dimension r.

Definition 8.35 When V is of pure dimension m, then Chow Variety $\mathrm{Chow}_{m-p}(V)$ parameterizes the cycles of codimension p and we write $\mathrm{Chow}^p(V) = \mathrm{Chow}_{m-p}(V)$. Equivalently we let $\mathrm{Chow}^p_d(V) = \mathrm{Chow}_{m-p,d}(V)$.

Chapter 9

Chow Schemes

Independence of embedding

The construction of the Chow variety $\mathrm{Chow}_{r,d}(X)$ in chapter 8 is a priori dependent on the embedding of X in a projective space $\mathbb{P}^n$. Thus, it is commonly denoted $\mathrm{Chow}_{r,d}(X,\iota)$ where $\iota : X \hookrightarrow \mathbb{P}^n$ is a given embedding.

In [Na], Nagata shows that when X is a normal variety there exists an embedding ι such that $\mathrm{Chow}_{0,d}(X,\iota)$ is normal. When k has characteristic zero any embedding suffices, but when k has positive characteristic, it is not true that $\mathrm{Chow}_{0,d}(X,\iota)$ always is normal. A counter-example is $X = \mathbb{A}^2$ with $k = \mathbb{F}_2$. Thus in positive characteristic, the Chow variety is dependent on the embedding even for 0-cycles. A brief discussion on this matter and a reproduction of Nagata's example can be found in [K, Ch. I, Ex. 4.2]. Note that this problem is not directly related to the problem that the Chow forms with coefficients in k does not parameterize the k-cycles, which occur when k is not perfect.

Given two embeddings $\iota : X \hookrightarrow \mathbb{P}^n$ and $\iota' : X \hookrightarrow \mathbb{P}^{n'}$ of a variety X, there is a canonical bijection $\varphi : \mathrm{Chow}_{r,d}(X,\iota) \to \mathrm{Chow}_{r,d}(X,\iota')$. It maps a Chow form $F_\nu \in \mathrm{Chow}_{r,d}(X,\iota)$, corresponding to the cycle $\nu = \sum_{i=1}^n n_i[\iota(V_i)]$, to the Chow form $\varphi(F_\nu) = F_{\nu'}$ corresponding to $\nu' = \sum_{i=1}^n n_i[\iota'(V_i)]$. It can be shown, see [Ho], that φ is a *homeomorphism* of topological spaces. Thus $\mathrm{Chow}_{r,d}(X,\iota)$ is independent of the embedding up to homeomorphism.

Hoyt has generalized the result of Nagata. In [Ho] he shows that there exists an embedding $\iota : X \hookrightarrow \mathbb{P}^n$ such that for any embedding $\iota' : X \hookrightarrow \mathbb{P}^{n'}$, the canonical homeomorphism $\varphi : \mathrm{Chow}_{r,d}(X,\iota) \to \mathrm{Chow}_{r,d}(X,\iota')$ is a *finite morphism* of varieties. Clearly $\mathrm{Chow}_{r,d}(X,\iota)$ is independent of the choice of the embedding ι with this property and is thus a universal Chow variety for X. Further Hoyt shows that given an embedding $f : X \hookrightarrow \mathbb{P}^n$, then f^m, the composition of f with the m-fold Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$, has this property.

All these results are for algebraically closed fields, but are easily generalized to any field since $\mathrm{Chow}_{r,d}(X)_{(\bar{k})} = \mathrm{Chow}_{r,d}\left(X_{(\bar{k})}\right)$.

When X is a $\mathbb{C}$-variety of pure dimension n, Barlet [B] has shown that the Chow variety is independent on the embedding up to isomorphism of varieties. In fact, Barlet constructs an analytical space, denoted by $B_p(X)$, parameterizing the cycles of codimension p which he shows is isomorphic to the Chow variety $\mathrm{Chow}_d^p(X,\iota)$ [B, Ch. IV, Thm 7].

Families of cycles and functoriality

Definition 9.1 Let X be an S-scheme of pure dimension $N = n + p$. The cycles of X of codimension p is the free group generated by the irreducible closed subsets of X of pure codimension p over each fiber of S, and is denoted $\mathbb{Z}^p(X/S)$. The positive cycles of $\mathbb{Z}^p(X/S)$ are denoted $C^p(X/S)$.

To give the cycles of $C^p(X/S)$ an algebraic structure we look at families of cycles. A *family of cycles* parameterized by a S-scheme T is a cycle Z of $X \times_S T$.

The map $C^p_{X/S} : T \to C^p(X \times_S T)$ is a *contravariant functor*. Indeed, an S-morphism $\varphi : T' \to T$ induces a pull-back $\varphi^* : C^p(X \times_S T) \to C^p(X \times_S T')$ of cycles of codimension p defined by $\varphi^*_{(X)}\big(\sum_i n_i[Z_i]\big) = \sum_i n_i\big[\varphi^{-1}_{(X)}(Z_i)\big]$ where $\varphi_{(X)} : X \times_S T' \to X \times_S T$ is the induced morphism.

If the functor $C^p_{X/S}$ is representable by a scheme $\mathscr{C}^p(X)$, there is by definition an isomorphism between the functor $C^p_{X/S}$ and the functor $T \mapsto \mathrm{Mor}\big(T, \mathscr{C}^p(X/S)\big)$, i.e. for every scheme T there is a bijection $f_T : C^p_{X/S}(T) \to \mathrm{Mor}\big(T, \mathscr{C}^p(X/S)\big)$ such that the diagram

$$\begin{tikzcd} C^p_{X/S}(T) \arrow[r, "f_T"] \arrow[d, "\varphi^*"'] & \mathrm{Mor}\big(T, \mathscr{C}^p(X/S)\big) \arrow[d, "\varphi"] \\ C^p_{X/S}(T') \arrow[r, "f_{T'}"'] & \mathrm{Mor}\big(T', \mathscr{C}^p(X/S)\big) \end{tikzcd} \tag{9.1}$$

commutes for every morphism $\varphi : T' \to T$.

To be able to represent the functor $C^p_{X/S}$ by a scheme we need regularity conditions on the cycles in $C^p_{X/S}(T)$. The families of cycles $C^p_{X/S}(T) \subseteq C^p(X \times_S T)$ which fulfill these conditions are called *algebraic* families of cycles.

Varieties

We will first look at the case for varieties, i.e. reduced schemes over a field k of finite type, and see if the Chow variety defined in chapter 8 represents the functor $C^p_{X/S}$. Further we will only look at the "trivial" case when $T = \mathrm{Spec}(k')$.

In characteristic zero or when $p = 1$, we have a bijection between the cycles of $C^p\left(X_{(k')}\right)$ and the morphisms $\mathrm{Mor}\big(k', \mathrm{Chow}^p(X/k)\big)$. In fact, to give a morphism $\mathrm{Spec}(k') \to \mathrm{Chow}^p(X)$ is equivalent to specify a k'-rational point $x \in \mathrm{Chow}^p(X)$. Since $\mathrm{Chow}^p(X)_{(k')} = \mathrm{Chow}\left(X_{(k')}\right)$ it is clear that the diagram 9.1 commutes for the functor $C^p_{X/S}(k') = C^p\left(X_{(k')}\right)$ of all families of cycles.

When k has positive characteristic and $p \neq 1$, then we know by remark 8.23 and example 7.60, that when k'/k is not perfect there is not always a bijection between $C^p(X_{(k')})$ and $\mathrm{Mor}\big(k', \mathrm{Chow}^p(X)\big)$. Thus $\mathrm{Chow}^p(X)$ does *not* represent the functor C^p_X. In fact, there are no known "reasonable" restrictions on the cycles of $C^p_{X/k}(T)$ such that $C^p_{X/k}$ becomes representable by a scheme or even an algebraic space.

General case

A natural condition for algebraic families of cycles is flatness; To the cycle Z of $C^p(X \times_S T)$ we associate a scheme structure with the correct multiplicities (similar to the representation of a cycle as an ideal in chapter 7) and require that the projection of Z on T is flat.

However, it turns out that if T is not smooth, then we loose several natural families of cycles, even families of 0-cycles.

In [B], Barlet defines, for cycles of schemes over $\mathbb{C}$, what he calls an *analytical* family of cycles by imposing the requirement that every intersection Y of a family $Z \in C^p(X \times_S T)$ with p hyperplanes such that Y_t is finite for every $t \in T$, should locally be an analytical family of cycles of dimension 0. The zero-dimensional analytical families are those corresponding to morphisms $T \to \mathrm{Sym}^d(X)$.

Angéniol [A] generalizes this to *algebraic* families of cycles. Let X be a scheme over $S = \mathrm{Spec}(k)$. A family of cycles $Z \in C^p(X \times_S T)$ is algebraic, if for any local projection of Z onto a smooth S-scheme B of relative dimension n such that Z is quasi-finite over B, the cycle Z corresponds to a morphism $B \to \mathrm{Sym}^d_B(X \times_S T)$ (which is the quotient of $(X \times_S T) \times_B (X \times_S T) \times_B \cdots \times_B (X \times_S T)$ by $\mathfrak{S}_d$). The problem is to determine when two morphisms $B \to \mathrm{Sym}^d_B(X \times_S T)$ and $B' \to \mathrm{Sym}^d_{B'}(X \times_S T)$ correspond to the same cycle. This is only easily done in the case when T is reduced and when k is algebraically closed, since $\mathrm{Sym}^d(X)$ can be considered as a d-tuple of points in X without order in that case.

To solve these problems, Angéniol uses the following approach. The morphisms $B \to \mathrm{Sym}^d_B(X \times_S T)$ corresponds to certain trace morphisms $\theta : \mathscr{O}_{X \times_S T} \to \mathscr{O}_B$. Further a class $c \in H^p_Z(X \times_S T, \Omega^p_{X \times_S T/T})$ induces for every projection onto a scheme B a morphism $\mathscr{O}_{X \times_S T} \to \mathscr{O}_B$. Two morphisms correspond to the same cycle if they come from the same class c. To represent the elements of $\mathrm{Sym}^d(X)$, Angéniol uses Newton's symmetric functions (or more precisely, a generalization of the power sums $\sum_i x_i^p$ to families of several variables), which can only be done in characteristic zero since the symmetric functions are not generated by Newton's functions otherwise. Further, only some classes, called *Chow classes*, are considered. They should be closed under exterior differentiation and satisfy some additional local conditions.

Definition 9.2 Let S be a noetherian affine scheme of *characteristic zero*. Let X be a *smooth* S-scheme of pure dimension $N = n + p$ over S. When T is a noetherian S-scheme we denote by $C^p_{X/S}(T)$ the set of pairs $(|Z|, c)$ where $|Z|$ is a closed subset of $X \times_S T$ of pure codimension p over each fiber of the projection $X \times_S T \to T$ and c is a Chow class of $H^p_{|Z|}(X \times_S T, \Omega^p_{X \times_S T/T})$ such that c is not zero on any generic point of the irreducible components of $|Z|$.

The above definition gives rise to a functor $C^p_{X/S}$, the *p'th Chow functor* of X/S. Using a theorem by Artin, Angéniol proceeds to show that this functor is representable by an algebraic space which is locally of finite type over S and separated [A, Thm 5.2.1]. This space is called the *p'th algebraic Chow-space* of X/S and is denoted $\mathscr{C}^p(X/S)$.

Further, if X is a scheme, not necessarily smooth, of pure dimension $N = n + p$ over S and there exists a closed immersion of X in a smooth scheme Y, then it is possible to define a functor $C^p_{X/S}$ which is independent on the immersion and which is represented by an algebraic space $\mathscr{C}^p(X/S)$, cf. [A, Cor. 6.3.3].

Theorem 9.3 *If $S = \mathbb{C}$ then $\mathscr{C}^p(X/S)_{\text{red}}$ is isomorphic to the analytical space $B_p(X)$ constructed by Barlet in [B].*

Proof. See [A, Thm 6.1.1]. □

Corollary 9.4 *If $S = \mathbb{C}$ and X is a projective $\mathbb{C}$-variety, then $\mathscr{C}^p(X/S)_{\text{red}}$ is isomorphic to the Chow variety* $\text{Chow}^p(\mathbb{P}^n, \iota)$ *where ι is an embedding of X in $\mathbb{P}^n$.*

Proof. Follows immediately from theorem 9.3 and the isomorphism between $B_p(X)$ and $\text{Chow}^p(X, \iota)$ given by Barlet in [B, Ch. IV, Thm 7]. □

In [K, Ch. I.3], Kollár constructs another Chow functor $C_{X/S}$. The algebraic families of cycles $C_{X/S}(T)$, are again cycles Z of $X \times_S T$ such that the fibers of the projection $Z \to T$ are of constant pure dimension and which fulfill some other regularity conditions. In characteristic zero, he shows that there is a pull-back of algebraic families and thus that $C_{X/S}$ is a contravariant functor. Further, when $S = \text{Spec}(k)$ for a field k of characteristic zero, then $C_{X/S}$ is represented by the Chow variety.

Positive characteristic

As we have seen by the example 7.60, the variety $\text{Chow}_{0,d}(X) = \text{Sym}^d(X)$ does not always parameterize the zero-cycles of X when X is a variety over an imperfect field. Thus if the functor $C^p_{X/k}$ is representable by a scheme $\mathscr{C}$ where $\text{Mor}(k, \mathscr{C}) \simeq \text{Mor}\big(k, \text{Sym}^d(X)\big)$, it does not always parameterize the zero-cycles of X. The approach taken by Barlet and Angéniol, based upon $\text{Sym}^d(X)$, is thus difficult to use to construct a functor and a representable scheme for the case when k is imperfect.

In [K, Ch. I.4], Kollár introduces the *Chow-field condition*. The Chow field $k^{ch}(V)$ for a variety V is the minimum field of definition for the coefficients of the Chow form $F_V(u_0, u_1, \ldots, u_r)$. Kollár shows that this field is independent of the embedding. Roughly, the Chow-field condition is that only the Chow forms $F \in k[u_0, u_1, \ldots, u_r]$ for which the corresponding cycle is defined over k should be considered. Then we get a correspondence between cycles and these Chow forms even when k is not perfect. Unfortunately, this does not define a functor since the pull-back φ^* of a cycle fulfilling the Chow-field condition, need not fulfill the Chow-field condition.

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