ALGEBRAIC $\mathbb{C}^*$-ACTIONS AND THE INVERSE KINEMATICS OF A GENERAL 6R MANIPULATOR

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Abstract. Let X be a smooth quadric of dimension $2m$ in $\mathbb{P}_{\mathbb{C}}^{2m+1}$ and let $Y, Z \subset X$ be subvarieties both of dimension m which intersect transversely. In this paper we give an algorithm for computing the intersection points of $Y \cap Z$ based on a homotopy method. The homotopy is constructed using a $\mathbb{C}^*$-action on X whose fixed points are isolated, which induces Bialynicki-Birula decompositions of X into locally closed invariant subsets. As an application we present a new solution to the inverse kinematics problem of a general six-revolute serial-link manipulator.

1. Introduction

This paper introduces a homotopy construction for computing numerical approximations to the intersection of two m dimensional algebraic subsets of a smooth $2m$ dimensional quadric. This new method joins the larger family of homotopy techniques, also known as continuation methods, which have proven to be effective for numerically solving systems of polynomial equations [L, SW]. These methods provide a means of constructing a homotopy function and a finite set of start points such that the paths emanating from the start points end in a finite set of endpoints that contain all isolated solutions of the equations. For efficiency, it is desirable that the number of homotopy paths is as small as possible, preferably equal to the actual number of isolated solutions.

Over the years, the pursuit of reduction in the number of homotopy paths has led to a series of homotopy constructions, each successively recognizing more of the structure of the given polynomials. Notable milestones are total degree homotopies [GZ], multihomogeneous formulations [MS1], linear set structures [VC], and polyhedral homotopies [HS, VVC]. The latter completely accounts for the sparse structure of the monomials in the system, but requires the computation of the mixed volume of the associated Newton polytopes, a combinatorial problem. Nevertheless, the approach is general and can be completely automated. Even the polyhedral homotopies may require more than the minimal number of paths, as in practice, the coefficients of a polynomial system may have interrelations that reduce the number of isolated roots compared to a system with the same monomials but general coefficients. Parameter homotopies [MS2, SW] capture the coefficient relations, but require an initial solution of a generic problem in the parameterized family, which is usually obtained by one of the aforementioned general techniques. More recently, techniques have been introduced for solving systems by introducing the equations one at a time [SVW, HSW]. This often has the effect of revealing structure at early stages of the computation, when it is inexpensive to work with fewer variables and equations, thereby reducing the number of paths and the cost in the final, most expensive, stage involving all the equations. These methods do not incur the cost of the mixed volume computation and they may take advantage of coefficient relations. For some large, sparse systems, the regeneration equation-by-equation method [HSW] outperforms the polyhedral approach even though it uses more solution paths. Several computer codes [BHSW, LLT, SMSW, V, WSMMW, WSW] are available that implement one or more of the homotopies just mentioned.

The method presented in this paper resembles the equation-by-equation approaches in that less expensive preliminary computations can reveal structure that reduces the path count, and hence the computational expense, of the final homotopy. The technique is based on the cell decomposition of a quadric induced by a multiplicative $\mathbb{C}^*$-action. This $\mathbb{C}^*$-homotopy applies when one seeks the isolated points in the intersection of two m-dimensional algebraic subsets of a $2m$-dimensional smooth quadric in $\mathbb{P}^{2m+1}_{\mathbb{C}}$. While this is not as general as the techniques previously mentioned, the situation arises often in applications, where quadrics are frequently encountered. The method has the desirable property that it subdivides the target problem in $2m+1$ dimensions into four subproblems, each in only m dimensions. The solutions to these subproblems are combined to form the start points for a final homotopy that solves the target problem. It may happen that one or more of the subproblems has fewer solutions than its total degree would suggest, in which case the final homotopy has fewer than the total degree number of paths.

This work was inspired by a geometrical problem from robotics: the inverse kinematics of a general six-revolute (6R) serial-link robot. The objective in inverse kinematics is to find all sets of joint angles that place the end-effector of a robot in a desired location. For general 6R robots, that is, for robots not having certain simplifying geometries such as intersecting wrist axes, it has been known since 1986 [P] that the problem has 16 solutions (over the complex number field). The early proofs and the related algorithms for calculating the joint angles depend on rather intricate algebraic manipulations of the defining polynomial equations. However, in 2005, Selig [S, §11.5] gave a simple, although abstract, proof based on intersection theory and a cell decomposition of the Study quadric, an elegant representation of $SE(3)$, the space of rigid-body displacements.

In the work reported here, we turn Selig's abstract proof into a concrete homotopy method for numerically solving the inverse kinematics problem using just 16 paths in the final homotopy to find the 16 solutions. As the Study quadric is fundamental to robotics, we expect that an algorithm for 6R inverse kinematics that makes strong use of the properties of the Study quadric might lead to better insight on solving other problems in robot kinematics. In fact, as outlined above, our pursuit of the 6R problem has lead to a solution algorithm that applies much more generally than to robot kinematics.

This paper is organized as follows. We begin in §2 by describing the $\mathbb{C}^*$-action on a quadric, that is central to our homotopy construction, and by presenting the cell decomposition that it induces. In doing so, we introduce the notation used throughout the paper. After a brief review, in §3, of some basic ideas in continuation, §4 presents the homotopy construction and the method of determining start points for the homotopy. The original statement of the algorithm in §4.1 is made for intersecting algebraic sets determined implicitly by polynomial equations, while in §4.2 the method is extended to cover the case where the sets are defined parametrically. In §5 we show the application of the method to the 6R inverse kinematics problem.

2. $\mathbb{C}^*$ Actions and Cell Decomposition

Let X be a smooth quadric hypersurface of even dimension $2m$ in the projective space $\mathbb{P}^{2m+1}$ over $\mathbb{C}$. Let $[q_0, \dots, q_m, p_0, \dots, p_m]$ be homogeneous coordinates[1] on $\mathbb{P}^{2m+1}$. We may assume that X is defined by the equation

$$Q(q,p) = q_0p_0 + q_1p_1 + \cdots + q_mp_m = 0. \tag{1}$$

This is because any smooth quadric is given by a polynomial of the form $x^T Ax = 0$, where A is a nonsingular, symmetric matrix. Hence, A can be written as $A = A^{1/2}A^{1/2}$, where $A^{1/2}$ is a symmetric matrix with inverse $A^{-1/2}$. Let N be the matrix

$$N = \begin{bmatrix} I & I \\ -Ii & Ii \end{bmatrix}, \tag{2}$$

and let $q = \begin{bmatrix} q_0 & \cdots & q_m \end{bmatrix}$ and $p = \begin{bmatrix} p_0 & \cdots & p_m \end{bmatrix}$ be row vectors. Then one may make the nonsingular change of coordinates $x = A^{-1/2}N[q,p]^T$, whereupon

$$x^T Ax = [q,p]N^T N[q,p]^T = 4Q(q,p).$$

So $x^T Ax = 0$ implies that $Q(q,p) = 0$.

We fix an action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on X defined as follows: for $t \in \mathbb{C}^*$

$$t[q_0, \dots, q_m, p_0, \dots, p_m] = [q_0, tq_1, \dots, t^m q_m, t^{2m}p_0, t^{2m-1}p_1, \dots, t^m p_m]. \tag{3}$$

Note that the action has an inverse given by

$$t^{-1}[q_0, \dots, q_m, p_0, \dots, p_m] = [t^{2m}q_0, t^{2m-1}q_1, \dots, t^m q_m, p_0, tp_1, \dots, t^m p_m]. \tag{4}$$

For any $Y \subset X$, we use the notations tY and $t^{-1}Y$ to mean the image of Y under the $\mathbb{C}^*$ action and its inverse.

The action has $2m+2$ fixed points:

$$F_0 = [1, 0, \dots, 0], F_1 = [0, 1, 0, \dots, 0], \dots, F_{2m+1} = [0, \dots, 0, 1].$$

The quadric X can be decomposed into locally closed invariant subsets in two different ways ([B-B]):

$$X = \bigcup_{j=0}^{2m+1} X_j^+ = \bigcup_{j=0}^{2m+1} X_j^-, \tag{5}$$

where

$$X_j^+ = \{x \in X : \lim_{t\to 0} tx = F_j\} \quad \text{and} \quad X_j^- = \{x \in X : \lim_{t\to\infty} tx = F_j\}.$$

We refer to these decompositions as the *Bialynicki-Birula decompositions* and we call $\cup_{j=0}^{2m+1} X_j^+$ the *plus-decomposition* and $\cup_{j=0}^{2m+1} X_j^-$ the *minus-decomposition.* The plus cells are, for $j = 0, 1, \dots, 2m+1$,

$$X_j^+ = \begin{cases} (q_k = 0, k < j) \cap (q_j \neq 0) \cap X, & j \le m; \\ (q = 0) \cap (p_k = 0, k > j-m-1) \cap (p_{j-m-1} \neq 0), & j > m. \end{cases}$$

[1] We use square brackets $[\dots]$ to denote homogeneous coordinates. Each point in $\mathbb{P}^n$ corresponds to a line through the origin in $\mathbb{C}^{n+1}$. For $y, z \in \mathbb{P}^n$, the equality $y = z$ means that the corresponding homogeneous coordinates in $\mathbb{C}^{n+1}$ are equal up to a nonzero scalar.

The minus cells are

$$X_j^- = \begin{cases} (q_j \neq 0) \cap (q_k = 0, k > j) \cap (p = 0), & j \leq m; \\ (p_k = 0, k < j - m - 1) \cap (p_{j-m-1} \neq 0) \cap X, & j > m. \end{cases}$$

Precisely one cell from each decomposition is dense Zariski-open in X. These are X_0^+ and X_{m+1}^- and the corresponding fixed points F_0 and F_{m+1} are called the source and the sink, respectively. Hence, for almost all $x \in X$, tx flows to the source as $t \to 0$ and to the sink as $t \to \infty$. For each $k \neq m$, each decomposition contains exactly one cell of dimension k. There are however, in both decompositions, two cells of dimension m:

$$X_m^+ = \{q_0 = \cdots = q_{m-1} = p_m = 0,\ q_m \neq 0\}, \tag{6}$$

$$X_m^- = \{p_0 = \cdots = p_m = 0,\ q_m \neq 0\}, \tag{7}$$

$$X_{2m+1}^+ = \{q_0 = \cdots = q_m = 0,\ p_m \neq 0\}, \tag{8}$$

$$X_{2m+1}^- = \{p_0 = \cdots = p_{m-1} = q_m = 0,\ p_m \neq 0\}. \tag{9}$$

Moreover, the dimensions of the cells are such that

$$\dim(X_i^+) + \dim(X_i^-) = 2m, \qquad i = 0, \ldots, 2m+1. \tag{10}$$

Throughout the remainder of this paper, X denotes the quadric in $\mathbb{P}^{2m+1}$ given by Eq. 1 with the $\mathbb{C}^*$ action given by Eq. 3 and the corresponding Bialynicki-Birula decompositions as in Eq. 5.

3. Background: Continuation

Polynomial continuation gives a method for computing isolated points in a constructible algebraic set. A continuation algorithm has three essential components: a homotopy that defines a collection of algebraic curves, a set of start points on these curves, and a prescription for how to advance along the curves from the start points to a set of endpoints that include all the isolated points in the target algebraic set. The most convenient case is when the curves are multiplicity one and the start points are nonsingular. Then the points can be easily moved along the curves by predictor/corrector methods based on Newton's method. While homotopy curves of higher multiplicity can also be handled, we limit the scope of the present discussion to the nonsingular case.

We borrow the definition of a trackable path for the algebraic, nonsingular case from [HSW], as follows.

Definition 3.1 (Nonsingularly Trackable Path)**.** [HSW] Let $H(x,t) : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ be polynomial in x and t and let x^* be a nonsingular solution of $H(x,0) = 0$. We say that x^* is *nonsingularly trackable* (or equivalently we say that we can *track* x^* nonsingularly) for $t \in [0,1)$ from $t = 0$ to $t = 1$ using $H(x,t)$ if there is a smooth map $\psi_{x^*} : [0,1) \to \mathbb{C}^N$ such that $\psi_{x^*}(0) = x^*$ and $\psi_{x^*}(t)$ is a nonsingular isolated solution of $H(x,t) = 0$ for $t \in [0,1)$. By the *path endpoint*, we mean $\lim_{t\to 1} \psi_{x^*}(t)$.

Note that in this paper, in contrast to previous papers, we construct the homotopy to track the path from 0 to 1, instead of from 1 to 0. If the endpoint of a path is singular at $t = 1$, it can be estimated accurately using an endgame algorithm [SW, Chapter 10].

Remark 3.2. Suppose that $H(x,t)$ and x^* are as in Definition 3.1. Then, by Lemma 7.1.3 ("Gamma Trick") of [SW, p. 94], the related homotopy function $\widehat{H}(x,\tau) := H(x,t(\tau))$, where $t(\tau) : [0,1] \to \mathbb{C}$ is given by

$$t(\tau) = \frac{\gamma\tau}{1+(\gamma-1)\tau}, \qquad \tau \in [0,1] \subset \mathbb{R}, \tag{11}$$

is a homotopy such that x^* is nonsingularly trackable for all $\gamma \in \mathbb{C}$ except for a finite number of one-real-dimensional rays from the origin. This implies that choosing $\gamma = e^{\sqrt{-1}\,\theta}$ for a random $\theta \in [-\pi,\pi]$ gives a path ϕ that is general with probability one. Accordingly, any nonsingular solution of $H(x,0) = 0$ is nonsingularly trackable. Moreover, if S is the set of *all* nonsingular solution points of $H(x,0) = 0$, then the set of isolated endpoints for the paths of $\widehat{H}(x,t(\tau)) = 0$ starting at S is independent of the choice of γ, although the correspondence between startpoints and endpoints may be permuted. Indeed, this is true for any general path from $t = 0$ to $t = 1$, where "general" means that the path does not include a finite set of bad points in $\mathbb{C}$. Thus, we may speak of the isolated endpoints of $H(x,t)$ for startpoints S without specifically citing what general path is to be followed.

Finally, suppose that $f(x) : \mathbb{C}^{N+1} \to \mathbb{C}^N$ is homogeneous in x. Then, x may be viewed as homogeneous coordinates for $\mathbb{P}^N$, and solutions to $f(x) = 0$ define an algebraic projective variety in $\mathbb{P}^N$. To compute on $\mathbb{P}^N$, one may select a patch $u(a,x) = a_0x_0 + a_1x_1 + \cdots + a_Nx_N - 1$ for some $a = (a_0,\ldots,a_N) \in \mathbb{C}^{N+1}$ and compute solutions to $\{f(x), u(a,x)\} = 0$ on $\mathbb{C}^{N+1}$. There exists a dense Zariski-open subset $U \subset \mathbb{C}^{N+1}$ such that for $a \in U$ the nonsingular isolated solutions of $\{f(x), u(a,x)\} = 0$ in $\mathbb{C}^{N+1}$ are identical with the nonsingular isolated solutions of $f(x) = 0$ on $\mathbb{P}^N$. We say that any $a \in U$ is a *generic* patch, and one may obtain a generic patch simply by choosing a at random in $\mathbb{C}^{N+1}$. In this way, a homotopy function $H(x,t) : \mathbb{C}^{N+1} \times \mathbb{C} \to \mathbb{C}^N$ that is homogeneous in x defines homotopy paths in $\mathbb{P}^N$, and we may presume for the purpose of computation that a generic patch is chosen at random. The minor adaptation of the definition of a trackable path to projective space, and the application of the gamma trick to such homotopies, is used throughout the rest of this paper without specific mention of a generic patch, although such a patch will be used in all computations.

4. The Method

Our basic method applies to intersecting sets in *general position*, as defined in the following Definition. Later we will remove this condition by giving a technique to move sets into general position with probability one.

Definition 4.1. Let $W \subseteq X$ be a multiplicity one algebraic set of dimension m. We say that W is in *general position* with respect to the plus decomposition (resp., minus decomposition) if

(1) for any cell X_i^+ (resp., X_i^-) of dimension m, $W \cap X_i^+$ (resp., $W \cap X_j^-$) is finite and nonsingular;
(2) for any cell X_i^+ (resp., X_i^-) of dimension less than m, $W \cap X_i^+ = \emptyset$ (resp., $W \cap X_i^- = \emptyset$).

The nonsingularity condition in definition 4.1 is the following (non standard) one. Let $L(x)$ be the set of linear equations that cut out on X one of the m dimensional cells X_m^+, X_m^-, X_{2m+1}^+, or X_{2m+1}^-, as in one of Eq. 6–9. Suppose that the polynomial system $f(x)$ defines W, that is W is a solution component of $V(f,Q)$, where Q is the quadric for X, Eq. 1. In item 1 of Definition 4.1, "nonsingular" means that the Jacobian matrix for $\{f,L,Q\}$ has full rank at the intersection points. When W is given as the image of a polynomial map, say $f(\theta)$ as in § 4.2 below, "nonsingular" means that the Jacobian matrix of $L(f(\theta))$ is full rank at the intersection points.

Our method depends on the following lemma.

Lemma 4.2. *Let $Y \subset X$ and $Z \subset X$ be multiplicity one m-dimensional algebraic subsets of X. Assume that Y in general position with respect to the plus-decomposition of X and Z in general position with respect to the minus-decomposition of X. Let S be the set of nonsingular isolated points in*

$$\left\{(Y\cap X_m^+)\times(Z\cap X_m^-)\right\}\cup\left\{(Y\cap X_{2m+1}^+)\times(Z\cap X_{2m+1}^-)\right\},$$

and for $t\in\mathbb{C}^*$ let $H(t)\subset Y\times Z$ be $H(t)=\{(y,z)\in Y\times Z|ty=t^{-1}z\}$. Then, the set S is nonsingularly trackable from $t=0$ to $t=1$ and its set of endpoints includes all nonsingular isolated points of $Y\cap Z$.

Proof. Let $H^*=\{(H(t),t)|t\in\mathbb{C}^*\}\subset Y\times Z\times\mathbb{C}^*$. The condition $ty=t^{-1}z$ is algebraic, so H^* is an algebraic set parameterized by t. Hence, for generic $t\in\mathbb{C}^*$, the number of nonsingular points in $H(t)$ is constant. Moreover, away from $t=0$, the maps t and t^{-1} preserve dimension, so the codimensions of tY and $t^{-1}Z$ are each m. Since $\mathrm{codim}(tY)+\mathrm{codim}(t^{-1}Z)=\dim(X)$, upper semicontinuity implies that every nonsingular isolated solution of $H(t)$ for $t=1$ is the limit of one or more nonsingular solutions in a neighborhood of $t=1$. What remains to be shown is that the paths beginning at S include all these solutions.

Let $\pi: H^*\to X$ be the map $(y,z,t)\mapsto ty$, which by construction is identical to the map $(y,z,t)\mapsto t^{-1}z$. As $t\to 0$, every point $x\in\pi(H(t),t)$ must approach one of the fixed points. The points in H^* such that $\pi(y,z,t)\to F_i$ as $t\to 0$ must approach $S_i=(Y\cap X_i^+)\times(Z\cap X_i^-)$. But since Y and Z are in general position and the dimensions of the cells obey Eq. 10, only S_m and S_{2m+1} are non-empty. Thus, as $t\to 0$, the paths of the nonsingular points in $H(t)$ must approach $S=S_m\cup S_{2m+1}$. Moreover, the assumption that Y and Z are in general position implies that all the points in S are nonsingular. Thus, S contains a startpoint on every nonsingular path of H and S is nonsingularly trackable. □

To use Lemma 4.2 in computations, it is convenient to define the map T as follows.

Let $T(x,t):\mathbb{C}^{m+1}\times\mathbb{C}\to\mathbb{C}^{m+1}$ be the map

$$(x_0,\ldots,x_m,t)\mapsto(t^m x_0, t^{m-1}x_1,\ldots,tx_{m-1},x_m). \tag{12}$$

Lemma 4.3. *Let $[q,p]$ be homogeneous coordinates in $\mathbb{P}^{2m+1}$ and let $y,z\in\mathbb{P}^{2m+1}$. If $y=[T^2(q,t),p]$ and $z=[q,T^2(p,t)]$ then $ty=t^{-1}z$. Conversely, if $ty=t^{-1}z$, there exists a $[q,p]$ such that $y=[T^2(q,t),p]$ and $z=[q,T^2(p,t)]$.*

Proof. The proposition follows easily from the definitions of the maps t, t^{-1}, and T (see Eqs. 3,4, and 12). □

Corollary 4.4. *If for some $\tau\in\mathbb{C}$, $y=[T(q,\tau),p]$ and $z=[q,T(p,\tau)]$, then there exists a value of t such that $ty=t^{-1}z$. Conversely, if $ty=t^{-1}z$ for some $t\in\mathbb{C}$, there exists $[q,p]$ and τ such that $y=[T(q,\tau),p]$ and $z=[q,T(p,\tau)]$. Moreover, a continuous path in either t or τ maps continuously to at least one path in the other.*

Proof. Only even powers of t appear in T^2. Let t be either branch of $\sqrt{\tau}$ for the first direction and let $\tau = t^2$ for the converse. $\square$

For convenience we will write Tx to mean $T(x,t)$. If we think of x as a $(m+1) \times 1$ column vector and T as the diagonal matrix with diagonal elements $t^m, t^{m-1}, \ldots, t, 1$, then Tx is just matrix multiplication. Corollary 4.4 allows us to replace T^2 by T in our homotopies. This is not necessary for the validity of the method, but it is numerically advantageous.

4.1. **Implicit Version.** While Lemma 4.2 is formulated for abstract algebraic sets a computational algorithm must use equations and coordinates for Y and Z. We first treat the common case when these are given implicitly as the solution sets of polynomial equations; the case where Y and Z are given parametrically is treated in § 4.2.

In the following, $(q,p) \in \mathbb{C}^{2m+2}$ are to be regarded as homogeneous coordinates for $\mathbb{P}^{2m+1}$, and a generic patch will be appended for computations as discussed in § 3.

Theorem 4.5. *Let $f : \mathbb{C}^{2m+2} \to \mathbb{C}^m$ and $g : \mathbb{C}^{2m+2} \to \mathbb{C}^m$ be systems of homogeneous polynomials, and let Y and Z be multiplicity one m dimensional components of the solution sets of $\{f(y) = 0, Q(y) = 0\}$ and $\{g(z) = 0, Q(z) = 0\}$ respectively. Assume that Y and Z are in general position with respect to the Bialynicki-Birula decomposition of X. Let $h(q,p,t) : \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} \times \mathbb{C} \to \mathbb{C}^{2m+1}$ be the system*

$$h(q,p,t) = \{f(Tq,p), g(q,Tp), q^T Tp\}.$$

Let $e_m = (0,\ldots,1) \in \mathbb{C}^{m+1}$ and let $S_m^+, S_m^-, S_{2m+1}^+, S_{2m+1}^-$ be the nonsingular solution points for the following four systems

$$\begin{aligned}
S_m^+ &: \{p \in \mathbb{C}^{m+1} | f(e_m,p) = 0, p_m = 0\} \\
S_m^- &: \{q \in \mathbb{C}^{m+1} | g(q,0) = 0, q_m = 1\} \\
S_{2m+1}^+ &: \{p \in \mathbb{C}^{m+1} | f(0,p) = 0, p_m = 1\} \\
S_{2m+1}^- &: \{q \in \mathbb{C}^{m+1} | g(q,e_m) = 0, q_m = 0\}.
\end{aligned}$$

Let $S_0 = \{S_m^- \times S_m^+\} \cup \{S_{2m+1}^- \times S_{2m+1}^+\}$. Then S_0 is nonsingularly trackable for homotopy $h(q,p,t)$, and its set of endpoints includes all the nonsingular points in $f^{-1}(0) \cap g^{-1}(0) \cap X$.

Proof. First, we must confirm that the points S_0 are solutions of $h(q,p,0) = 0$. This is achieved by noting that at $t = 0$, $Tq = (0,\ldots,0,q_m)$ and $Tp = (0,\ldots,0,p_m)$, $q^T Tp = q_m p_m$. The points in $S_m^- \times S_m^+$ satisfy $q_m p_m = 0$ with $q_m = 1, p_m = 0$. The points in $S_{2m+1}^- \times S_{2m+1}^+$ satisfy it with $q_m = 0, p_m = 1$. The rest of the theorem is a direct application of Lemma 4.2, with the observation that the points $[q,p] \in S_0$ give pairs $(y,z) = ([Tq,p],[q,Tp])$ that correspond to points called S in Lemma 4.2 and the observation that by Corollary 4.4 the paths traced out by the two homotopies correspond. Finally, by assumption, Y and Z are multiplicity one and in general position, so by choosing S_0 as the nonsingular solution points at the start, we are sure to include all the start points S for Y and Z. $\square$

The systems $\{f(y) = 0, Q(y) = 0\}$ and $\{g(z) = 0, Q(z) = 0\}$ could each have several irreducible m dimensional components. These may have multiplicities greater than one, and if so, they may be considered to be in general position if they hit the m dimensional cells transversely, they miss the lower dimensional cells, and no other component meets the m dimensional cells in the same points. With this notion of general position, the algorithm of Theorem 4.5 can be extended to find all isolated solutions in the intersection of all m dimensional components in general position, regardless of multiplicity, by expanding S_0 to include all isolated solutions. For simplicity, we have restricted to just the multiplicity one case.

4.2. **Parametric Version.** Consider two polynomial mappings $f : \mathbb{C}^m \to X$ and $g : \mathbb{C}^m \to X$. Let $Y = \mathrm{im}(f)$ and $Z = \mathrm{im}(g)$, and $\dim(Y) = \dim(Z) = m$. Assume that Y and Z are in general position with respect to the plus- and minus-decompositions of X, resp. We wish to compute the nonsingular isolated points W in $\{(\theta, \phi) \in \mathbb{C}^{2m} | f(\theta) \cong g(\phi)\}$. Here $f(\theta) \cong g(\phi)$ means $f(\theta) \neq 0$, $g(\phi) \neq 0$, and there exists some $\lambda \in \mathbb{C}^*$ such that $\lambda f(\theta) = g(\phi)$. The points in W map to $Y \cap Z$.

In the main example from § 5, the inverse kinematics of a 6R robot, we have $f, g : (\mathbb{P}^1)^3 \hookrightarrow SE(3)$. After choosing a random affine patch for each copy of $\mathbb{P}^1$ and using the Study quadric to represent $SE(3)$, f and g become polynomial mappings of the sort considered in this section.

To work with the parametric forms, it is convenient to define the projection maps from $\mathbb{C}^{2m+2} \to \mathbb{C}^{m+1}$ as $\pi_1 : (q, p) \to q$ and $\pi_2 : (q, p) \to p$.

The solution set W can be computed using the parametric version of Lemma 4.2, as follows.

Theorem 4.6. *Let* $f = (f_0, \dots, f_{2m+1}) : \mathbb{C}^m \to X$ *and* $g = (g_0, \dots, g_{2m+1}) : \mathbb{C}^m \to X$. *Let* W *be the nonsingular isolated points in* $\{(\theta, \phi) \in \mathbb{C}^{2m} | f(\theta) \cong g(\phi)\}$. *Assume that* $\mathrm{im}(f)$ *and* $\mathrm{im}(g)$ *are* m *dimensional and in general position with respect to the plus- and minus-decompositions of* X, *resp. Let* $S_m^+, S_m^-, S_{2m+1}^+, S_{2m+1}^-$ *be the nonsingular solution points for the following four systems*

$$\begin{aligned}
S_m^+ &: \{\theta \in \mathbb{C}^m | f_i(\theta) = 0, i = 0, \dots, m-1, 2m+1, f_m(\theta) \neq 0\} \\
S_m^- &: \{\phi \in \mathbb{C}^m | g_i(\phi) = 0, i = m+1, \dots, 2m+1, g_m(\phi) \neq 0\} \\
S_{2m+1}^+ &: \{\theta \in \mathbb{C}^m | f_i(\theta) = 0, i = 0, \dots, m, f_{2m+1}(\theta) \neq 0\} \\
S_{2m+1}^- &: \{\phi \in \mathbb{C}^m | g_i(\phi) = 0, i = m, \dots, 2m, g_{2m+1}(\phi) \neq 0\}.
\end{aligned}$$

Build a set of points S_0 *as follows:*

(1) *For each pair* (θ, ϕ) *with* $\theta \in S_m^+$ *and* $\phi \in S_m^-$, *append the point* $(\theta, \phi, g_m(\phi)/f_m(\theta))$ *to* S_0, *and*
(2) *For each pair* (θ, ϕ) *with* $\theta \in S_{2m+1}^+$ *and* $\phi \in S_{2m+1}^-$, *append the point* $(\theta, \phi, g_{2m+1}(\phi)/f_{2m+1}(\theta))$ *to* S_0.

Let $h : \mathbb{C}^{2m+1} \times \mathbb{C} \to \mathbb{C}^{2m+2}$ *be*

$$h((\theta, \phi, \lambda), t) = \lambda \begin{bmatrix} T\pi_1(f(\theta)) \\ \pi_2(f(\theta)) \end{bmatrix} - \begin{bmatrix} \pi_1(g(\phi)) \\ T\pi_2(g(\phi)) \end{bmatrix}.$$

Then there is a nonempty Zariski-open subset $U \subset \mathbb{C}^{(2m+1)\times(2m+2)}$ *such that for* $A \in U$, S_0 *is nonsingularly trackable for homotopy* $A \cdot h = 0$, *and the endpoints include all the points in* W.

Proof. Because $\dim(Y) = m$, the fiber $f^{-1}(y)$ in $\mathbb{C}^m$ over a generic point in $y \in Y$ is finite. The exceptional subset $\widehat{Y} \subset Y$ where the number of nonsingular points in the fiber is less than the the generic number is a proper algebraic subset. Similarly, the map g has an exceptional subset of $\widehat{Z} \subset Z$ of dimension less than m. Suppose that at least one point, say $t_* \in \mathbb{C}$, exists such that all the points in $Y \times Z$, such that $t_*Y = t_*^{-1}Z$, avoid the exceptional set $\widehat{Y} \times \widehat{Z}$. Then, $tY = t^{-1}Z$ avoids $\widehat{Y} \times \widehat{Z}$ for t in a dense Zariski-open subset $U \subseteq \mathbb{C}$. But by assumption, Y and Z are in general position, so $t_* = 0$ is such a point and S_0 is the set of startpoints prescribed by Lemma 4.2. Accordingly, a general path from $t = 0$ to $t = 1$, such as is generated using the gamma trick of Remark 3.2, induces a set of nonsingular paths in $\mathbb{C}^m \times \mathbb{C}^m$ that track the corresponding solution paths in $Y \times Z$ of Lemma 4.2. By Corollary 4.4, these paths must satisfy $h((\theta, \phi, \lambda), t) = 0$, where λ is the nonzero scaling factor establishing equality between points in homogeneous coordinates. The homotopy function h is a set of $2m+2$ polynomials on $\mathbb{C}^{2m+2}$, but since Y and Z both live in X, the polynomial system cuts out a curve. Thus, by Theorem 13.5.1 of [SW, p.243], for A in a nonempty Zariski-open subset of $\mathbb{C}^{(2m+1)\times(2m+2)}$, the system $A \cdot h$ cuts out the same curve as h and if the curve as a solution component of $V(h)$ is multiplicity 1, then it is also multiplicity one as a solution component of $V(A \cdot h)$. So $V(h)$ has a finite set of nonsingular paths whose images are the paths of Lemma 4.2 with a set of end points that contain the desired set W. □

4.3. **General Position.** Theorems 4.5 and 4.6 assume that Y and Z are in general position. In an application, this may not be the case for the initial specification of the problem. However, one can always move Y and Z to general position with an appropriate change of coordinates. The maneuver involves a random linear transformation B which preserves the quadric X. Let I denote the identity matrix and 0 the zero matrix, both square of size $m + 1$. The quadric X corresponds to a quadratic form on $\mathbb{C}^{2m+2}$ defined by the $(2m + 2) \times (2m + 2)$-matrix

$$Q_{2m+2} = \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The transformation B is an element of

$$\mathrm{SO}(Q_{2m}) = \{B \in \mathrm{SL}_{2m+2}(\mathbb{C}) : B^T Q B = Q\}.$$

Let N be the $(2m+2)\times(2m+2)$ matrix from Eq. 2 and let $R \in \mathrm{SO}(2m+2)$ be an orthogonal matrix. One can generate a random matrix in $\mathrm{SO}(2m + 2)$ by choosing a random skew symmetric matrix $C \in \mathbb{C}^{(2m+2)\times(2m+2)}$, that is, $C^T = -C$, and using Cayley's formula: $R = (I + C)(I - C)^{-1}$. Then, $B = N^{-1}RN/\sqrt{2}$ preserves the quadric X.

Proposition 4.7. *There is a non-empty Zariski open subset $U \subset \mathrm{SO}(Q_{2m})$ such that, for all $B \in U$, $B(Y)$ and $B(Z)$ are in general position.*

Proof. This follows from [K] Corollary 4 (ii). □

When working with Y and Z as solution components of $f(x) = 0$ and $g(x) = 0$, resp., one may replace these with Y' and Z' in general position with probability one by the change of coordinates $f(B^{-1}w) = 0$ and $g(B^{-1}w) = 0$. When working with Y and Z as the images of polynomial maps $x = f(\theta)$ and $g(\phi)$, general position is attained, with probability one, for the sets $Y' = \mathrm{im}(B \cdot f)$ and $Z' = \mathrm{im}(B \cdot g)$. Either way, after finding a point $w \in Y' \cap Z'$, the corresponding point in $Y \cap Z$ is $x = B^{-1}w$.

5. An application to kinematics

In this section we give a new solution to the inverse kinematics problem of a general six-revolute serial-link manipulator (6R-chain). This problem has been studied extensively, see [A] for an overview. We do not claim that our method is competitive with the most efficient algorithms devised for the 6R problem, but rather we view the inverse kinematics of the 6R-chain as a test problem for our algorithm. We believe that our treatment sheds new light on the geometry underlying the problem, and that the method is applicable to other problems of kinematics as well as to problems in other arenas that may involve a smooth quadric.

5.1. **The Study quadric.** We begin with a discussion of some general concepts, see [S] for further information. A rigid body motion in 3-space is a composition of a rotation and a translation, the former is an element of the special orthogonal group $\mathrm{SO}_3(\mathbb{R})$ and the latter a vector in $\mathbb{R}^3$. These make up the *special Euclidean group* $\mathrm{SE}_3(\mathbb{R})$. This is the semi-direct product of $\mathbb{R}^3$ and $\mathrm{SO}_3(\mathbb{R})$,

$$\mathrm{SE}_3(\mathbb{R}) = \mathrm{SO}_3(\mathbb{R}) \ltimes \mathbb{R}^3,$$

with respect to the natural homomorphism $\mathrm{SO}_3(\mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^3)$. In other words $\mathrm{SE}_3(\mathbb{R})$ is the set $\mathrm{SO}_3(\mathbb{R}) \times \mathbb{R}^3$ with group operation

$$(R_2, t_2) \circ (R_1, t_1) = (R_2R_1, R_2t_1 + t_2).$$

It is a 6-dimensional Lie group; there are 3 degrees of freedom for the translation and 3 degrees of freedom for the rotation. We will embed $\mathrm{SE}_3(\mathbb{R})$ as a Zariski open set of the so-called *Study quadric* in $\mathbb{P}^7_{\mathbb{R}}$, which is the hypersurface

$$X_{\mathbb{R}} = \{[q_0, q_1, q_2, q_3, p_0, p_1, p_2, p_3) \in \mathbb{P}^7_{\mathbb{R}} : q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3 = 0\}.$$

More precisely, the image of this embedding is

$$X'_{\mathbb{R}} = X_{\mathbb{R}} \setminus \{q_0 = q_1 = q_2 = q_3 = 0\}.$$

For $q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ and $p = (p_0, p_1, p_2, p_3) \in \mathbb{R}^4$, not both zero, we will use the notation

$$[q, p] = [q_0, q_1, q_2, q_3, p_0, p_1, p_2, p_3] \in \mathbb{P}^7_{\mathbb{R}}.$$

Consider $\mathbb{R}^4$ as the division algebra of quaternions. Let 1 denote the unit element of the algebra and let $\{i, j, k\}$ denote the quaternionic units. Then $\{1, i, j, k\}$ is a basis for $\mathbb{R}^4$ and

$$i^2 = j^2 = k^2 = ijk = -1.$$

For $(a_0, a_1, a_2, a_3) = a \in \mathbb{R}^4$, a^* denotes the conjugate of a and $|a|$ denotes the norm of a, i.e.

$$a^* = (a_0, -a_1, -a_2, -a_3) \quad \text{and} \quad |a| = \sqrt{aa^*}.$$

We identify $\mathbb{R}^3$ with the subspace

$$\{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 : a_0 = 0\} = \mathbb{R}^3 \subset \mathbb{R}^4.$$

Recall that there is an isomorphism

$$\psi : \mathbb{P}^3_{\mathbb{R}} \xrightarrow{\cong} \mathrm{SO}_3(\mathbb{R}).$$

View $\mathbb{P}^3_{\mathbb{R}}$ as the unit sphere $S^3 \subset \mathbb{R}^4$ with antipodes identified. Then, for $a \in S^3$ and $x \in \mathbb{R}^3$,

$$\psi(a)x = axa^*.$$

Finally, note that for $[q,p] \in X_{\mathbb{R}}^{'}$ the quaternion pq^* has real part

$$p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 = 0,$$

and so it lies in $\mathbb{R}^3 \subset \mathbb{R}^4$. Thus we have a map

$$X_{\mathbb{R}}^{'} \to \mathrm{SO}_3(\mathbb{R}) \times \mathbb{R}^3 : [q,p] \mapsto \left(\frac{\psi(q)}{qq^*}, \frac{pq^*}{qq^*} \right),$$

which is in fact an isomorphism. Hence $\mathrm{SE}_3(\mathbb{R}) \cong X_{\mathbb{R}}^{'} \subset X_{\mathbb{R}}$, which is the promised embedding. The induced group structure on $X_{\mathbb{R}}^{'}$ is given by

$$[q_2,p_2][q_1,p_1] = [q_2q_1, p_2q_1 + q_2p_1]. \tag{13}$$

We will however work over the complex numbers. For this purpose observe that the inclusion $\mathbb{R}^8 \subset \mathbb{C}^8$ induces an embedding of $\mathbb{P}^7_{\mathbb{R}}$ into $\mathbb{P}^7_{\mathbb{C}}$. Define the corresponding quadric

$$X = \{[q_0,q_1,q_2,q_3,p_0,p_1,p_2,p_3] \in \mathbb{P}^7_{\mathbb{C}} : q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3 = 0\},$$

and the open set

$$X^{'} = X \setminus \{q_0^2 + q_1^2 + q_2^2 + q_3^2 = 0\}.$$

Note that $X_{\mathbb{R}} \subset X$ and $X_{\mathbb{R}}^{'} = X^{'} \cap X_{\mathbb{R}}$. As above we have a $\mathbb{C}^*$-action $\mathbb{C}^* \times X \to X$ defined by

$$(t,q_0,q_1,q_2,q_3,p_0,p_1,p_2,p_3) \mapsto [q_0, tq_1, t^2q_2, t^3q_3, t^6p_0, t^5p_1, t^4p_2, t^3p_3].$$

In Section 5.3 we will represent points of X as (8×8)-matrices. For this aim we introduce the matrices

$$Id = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For $x = (x_0,x_1,x_2,x_3) \in \mathbb{C}^4$ we use $\bar{x}$ to denote the (4×4)-matrix

$$\bar{x} = x_0 Id + x_1 I + x_2 J + x_3 K.$$

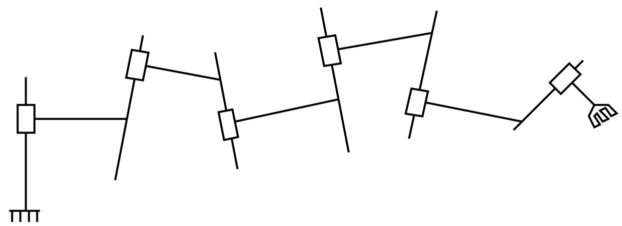
We now define an injection M from X to $\mathbb{P}^{63}_{\mathbb{C}}$, the set of non-zero (8×8)-matrices with complex entries up to multiplication by a scalar:

$$M : X \to \mathbb{P}^{63}_{\mathbb{C}} : [q,p] \mapsto \begin{pmatrix} \bar{q} & 0 \\ \bar{p} & \bar{q} \end{pmatrix}, \tag{14}$$

where 0 denotes the (4×4)-matrix with all entries equal to zero. For $[q,p] \in X$, the first column of $M([q,p])$ is $[q,p]$, so the inverse of M picks out the first column. Consider the group $\mathbb{C}^* \times \mathrm{SO}(Q_6)$ of non-degenerate linear maps on $\mathbb{C}^8$ which preserve the bi-linear form Q up to a scalar multiple. One can show that M restricted to $X^{'}$ identifies $X^{'}$ with a subgroup of $\mathbb{C}^* \times \mathrm{SO}(Q_6)$. Moreover, by (13), M restricted to $X_{\mathbb{R}}^{'}$ is a group homomorphism:

$$\begin{pmatrix} \bar{q_2} & 0 \\ \bar{p_2} & \bar{q_2} \end{pmatrix} \begin{pmatrix} \bar{q_1} & 0 \\ \bar{p_1} & \bar{q_1} \end{pmatrix} = \begin{pmatrix} \bar{q_2}\bar{q_1} & 0 \\ \bar{p_2}\bar{q_1} + \bar{q_2}\bar{p_1} & \bar{q_2}\bar{q_1} \end{pmatrix}.$$

5.2. **Inverse kinematics of general 6R-chains.** A mechanical linkage is a collection of rigid bodies connected by joints. A revolute joint is a hinge with no restriction on the rotation angle. An m-revolute serial chain linkage (mR-chain) consists of $m+1$ rigid links connected by m revolute joints. Consider a 6R-chain.



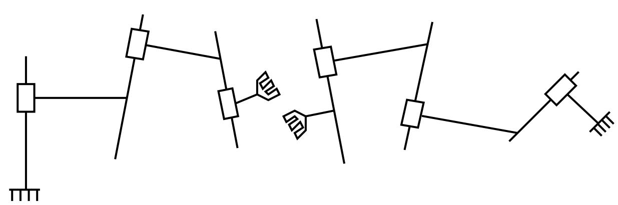
[Fig. 1]

One end of the 6R-chain is attached to the ground and at the other end there is a tool, called *the hand*, which is intended to move around in space and perform various tasks. Fix an initial pose of the mechanism with respect to which we measure the rotation angles of the joints. Place a coordinate frame in the ground link and another coordinate frame at the hand, as the mechanism takes the initial pose. The transformation from hand coordinates to ground coordinates is a function of the rotation angles. These are elements of the unit circle, which is homeomorphic to $\mathbb{P}^1_{\mathbb{R}}$. Thus we have a map

$$\Phi_{\mathbb{R}} : \underbrace{\mathbb{P}^1_{\mathbb{R}} \times \cdots \times \mathbb{P}^1_{\mathbb{R}}}_{6} \to \mathrm{SE}_3(\mathbb{R}) \subset X_{\mathbb{R}},$$

mapping a 6-tuple of angles to the corresponding transformation from hand coordinates to ground coordinates. Constructing $\Phi_{\mathbb{R}}$ is the *forward kinematics problem* and computing the fiber $\Phi_{\mathbb{R}}^{-1}(x)$, for any given $x \in X'_{\mathbb{R}}$, is the *inverse kinematics problem* (IKP).

Let a position and orientation of the hand be given. To solve the IKP of the 6R-chain we use a device introduced in [S] and explored in [HPS]. One divides the 6R-chain into two 3R-chains by cutting the middle link. The solutions to the IKP now correspond to possible poses of the 3R-chains where they reconnect to a 6R-chain.



[Fig. 2]

We will consider a general member of the family of 6R-chains (a general 6R-chain), in a sense made precise in Section 5.3. Moreover, we will allow the rotation angles to be complex numbers. This gives rise to a map

$$\Phi : \underbrace{\mathbb{P}^1_{\mathbb{C}} \times \cdots \times \mathbb{P}^1_{\mathbb{C}}}_{6} \quad \to \quad X.$$

We address the problem of computing the fiber $\Phi^{-1}(x)$ for generic $x \in X_{\mathbb{R}}$. The isolated solutions of the IKP of a particular 6R-chain for a particular position and orientation of the hand can be computed by solving the general problem and using a coefficient-parameter-homotopy to find the solutions of the special case ([SW]).

The map Φ is onto and generically 16 to 1. This corresponds to the well known fact that the generic IKP has 16 solutions ([S] 11.5.1). Fixing $x \in X_{\mathbb{R}}$ and splitting the 6R-chain gives rise to maps associated to the 3R-chains:

$$f : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \rightarrow X, \tag{15}$$

$$g : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \rightarrow X. \tag{16}$$

These maps are injective and trilinear. In Section 5.3 we give explicit descriptions of the maps f and g.

Let $Y = \operatorname{im}(f)$ and $Z = \operatorname{im}(g)$. Then Y and Z are irreducible 3-dimensional subvarieties of X which intersect in a finite number of non-singular points. The IKP is reduced to computing

$$\{(\theta_1, \theta_2) \in (\mathbb{P}^1_{\mathbb{C}})^3 \times (\mathbb{P}^1_{\mathbb{C}})^3 : f(\theta_1) = g(\theta_2)\}.$$

The algorithm of Theorem 4.6 can be used to solve this problem. We give a numerical example in Section 5.4.

5.3. **The parametric equations.** We will now give a detailed description of the maps (15) and (16).

Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$. For any coordinate $v \in \{x, y, z\}$ we let $R_v(\theta)$ denote the rotation of an angle θ about the v-axis and $T_v(d)$ the translation of a distance d in the v-direction.

Number the links of the 6R-chain 0 through 6 and number the joints so that joint i connects link $i-1$ with link i, for $i = 1, \ldots, 6$. As the mechanism is in the initial pose, place coinciding coordinate frames A_{i-1} fixed in link $i-1$ and B_i fixed in link i with origin on the rotation axis of joint i and z-direction parallel with the rotation axis of joint i ($i = 1, \ldots, 6$). Also, place a global coordinate frame B_0 in the ground link (link 0) and a hand frame A_6 in the final link (link 6). For $i = 0, \ldots, 6$, let $L_i \in \mathrm{SE}_3(\mathbb{R})$ denote the transform from A_i-coordinates to B_i-coordinates. Because A_i and B_i are both fixed in link i, L_i depends only on the particular design of the 6R-chain. The Denavit-Hartenburg convention is to place B_i where the common normal between joint axes i and $i+1$ meets joint i, with its x-axis aligned with that common normal. In case the joints are parallel, a common normal is not unique, but *any* common normal can be used. Moreover, the initial pose is chosen so that the common normals are all parallel. With these choices, the transforms L_i can be decomposed as

$$L_i = T_x(a_i) \circ R_x(\phi_i) \circ T_z(d_i),$$

where a_i, ϕ_i and d_i are geometrical constants that describe link i, known respectively as the link i's length, twist and offset.

As joint i turns by angle θ_i, the transform in $\mathrm{SE}_3(\mathbb{R})$ from B_i-coordinates to A_{i-1}-coordinates is $R_z(\theta_i)$. We can now write down the solution to the forward kinematic problem of the 6R-chain:

$$\Phi_{\mathbb{R}} : \underbrace{\mathbb{P}^1_{\mathbb{R}} \times \cdots \times \mathbb{P}^1_{\mathbb{R}}}_{6} \rightarrow \mathrm{SE}_3(\mathbb{R}) \subset X,$$

where $\Phi_{\mathbb{R}}$ maps $(\theta_1, \ldots, \theta_6)$ to

$$L_0 \circ R_z(\theta_1) \circ L_1 \circ R_z(\theta_2) \circ L_2 \circ R_z(\theta_3) \circ L_3 \circ R_z(\theta_4) \circ L_4 \circ \\ R_z(\theta_5) \circ L_5 \circ R_z(\theta_6) \circ L_6.$$

Fix a position and orientation of the hand $H \in \mathrm{SE}_3(\mathbb{R})$. We split the 6R-chain by picking any $S_1 \in \mathrm{SE}_3(\mathbb{R})$ and defining $S_2 = L_3^{-1} \circ S_1$, so that $S_1 \circ S_2^{-1} = L_3$. Observe that $R_z(\theta_i)^{-1} = R_z(-\theta_i)$. The 3R-chains give rise to maps

$$m_1, m_2 : \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \to X,$$

such that

$$\Phi_{\mathbb{R}}(\theta_1, \ldots, \theta_6) = H \;\Leftrightarrow\; m_1(\theta_1, \theta_2, \theta_3) = m_2(\theta_4, \theta_5, \theta_6).$$

These are defined by

$$m_1(\theta_1, \theta_2, \theta_3) = L_0 \circ R_z(\theta_1) \circ L_1 \circ R_z(\theta_2) \circ L_2 \circ R_z(\theta_3) \circ S_1,$$

and

$$m_2(\theta_4, \theta_5, \theta_6) = H \circ L_6^{-1} \circ R_z(-\theta_6) \circ L_5^{-1} \circ R_z(-\theta_5) \circ L_4^{-1} \circ R_z(-\theta_4) \circ S_2.$$

Note that the right-hand sides are both of the form

$$M_0 \circ R_1 \circ M_1 \circ R_2 \circ M_2 \circ R_3 \circ M_3,$$

where M_0, M_1, M_2, M_3 are constants.

By a general 6R-chain we mean one such that the L_i, $i = 0, \ldots, 6$, are general points on $X_{\mathbb{R}}$. We consider the IKP for a general 6R-chain where the fixed $H \in X_{\mathbb{R}}$ is also generic. We use the map M defined in (14) to define the maps (15) and (16) from m_1 and m_2. The rotation $R_z(\theta)$ considered as a quaternion is $\cos(\theta/2) + k\sin(\theta/2)$. Thus, as a point of X, $R_z(\theta)$ is

$$(\cos(\theta/2), 0, 0, \sin(\theta/2), 0, 0, 0, 0).$$

To simplify the notation we write

$$M(c, s) = M((c, 0, 0, s, 0, 0, 0, 0)), \quad \text{for } (c, s) \in \mathbb{P}^1_{\mathbb{C}}.$$

We generate the maps (15) and (16) as follows. For $x_0, x_1, x_2, x_3 \in X_{\mathbb{R}}$, define

$$f : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to X, \quad f((c_1, s_1), (c_2, s_2), (c_3, s_3)) =$$
$$M(x_0) \circ M(c_1, s_1) \circ M(x_1) \circ M(c_2, s_2) \circ M(x_2) \circ M(c_3, s_3) \circ x_3,$$

where the product is matrix multiplication and x_3 is considered a column vector. For generic $x_0, x_1, x_2, x_3 \in X_{\mathbb{R}}$, f is well defined and it is an embedding of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ into $\mathbb{P}^7_{\mathbb{C}}$ which is trilinear. In the same manner, for $x_4, x_5, x_6, x_7 \in X_{\mathbb{R}}$, define

$$g : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to X : g((c_4, s_4), (c_5, s_5), (c_6, s_6)) =$$
$$M(x_4) \circ M(c_4, s_4) \circ M(x_5) \circ M(c_5, s_5) \circ M(x_6) \circ M(c_6, s_6) \circ x_7.$$

Before applying Theorem 4.6 we must move $Y = \mathrm{im}(f)$ and $Z = \mathrm{im}(g)$ into general position. This is achieved by changing coordinates with the use of a random element of the group $\mathrm{SO}(Q_6)$.

5.4. **A numerical example.** We proceed to a numerical example of the algorithm of Theorem 4.6 applied to the IKP of a randomly generated 6R-chain. We have implemented it using the "user-defined homotopy" option in the Bertini program [BHSW]. Throughout we will use the notation of Theorem 4.6 and § 5.3.

The design parameters of this example are:

$$\begin{aligned} a_1 &= 0.9463, & a_2 &= 0.7637, & a_3 &= 0.5588,\\ a_4 &= 0.1838, & a_5 &= 0.4979, & a_6 &= 0.5178,\\ d_1 &= 0.9942, & d_2 &= 0.8549, & d_3 &= 0.9624,\\ d_4 &= 0.6789, & d_5 &= 0.4035, & d_6 &= 0.9350,\\ \phi_1 &= 0.4795, & \phi_2 &= 0.2318, & \phi_3 &= 0.3963,\\ \phi_4 &= 0.7051, & \phi_5 &= 0.5586, & \phi_6 &= 0.7566. \end{aligned}$$

As fixed position and orientation of the hand we have chosen

$$H = (-0.5324, -0.7003, -0.3329, -0.3397, 2.7897, 0.0405, -2.3180, -2.1845).$$

This is the value of the 6R map $\Phi_{\mathbb{R}}$ in the 6-tuple of angles

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = (1.9910, 1.9249, 1.0701, 1.9277, 0.2313, 0.1029),$$

which implies that this set of angles is among the solutions to the inverse kinematics problem for this mechanism with fixed position and orientation of the hand given by H. The computations were done using the coordinates c_i on the six affine patches $\{s_i \neq 0\} \subset \mathbb{P}^1$ (putting $s_i = 1$), $i = 1, \ldots, 6$.

The sets S_3^+, S_3^-,S_7^+ and S_7^- were computed with standard homotopy methods, and the solutions are listed in Table 1.

TABLE 1. The sets S_3^+, S_3^-, S_7^+ and S_7^-.

S_3^+	c_1	c_2	c_3
Solution 1	$0.3304 - 0.7293i$	$-0.0718 + 1.0697i$	$-0.4382 - 0.8478i$
Solution 2	$0.5640 + 1.3898i$	$0.0718 - 1.0697i$	$-0.0044 + 0.8686i$
S_3^-	c_4	c_5	c_6
Solution 1	$-0.0985 + 0.9496i$	$-0.0769 + 0.0348i$	$-0.1026 - 1.2854i$
Solution 2	$0.0518 - 1.0274i$	$0.0393 + 1.2852i$	$-0.3454 - 1.2910i$
Solution 3	$0.0465 + 0.8510i$	$0.0304 - 1.2764i$	$-0.3176 + 1.3372i$
Solution 4	$0.0206 - 0.9694i$	$-0.0652 - 0.0280i$	$-0.0695 + 1.2938i$
S_7^+	c_1	c_2	c_3
Solution 1	$0.0000 + 1.0000i$	$0.0000 - 1.0586i$	$0.0000 + 1.0000i$
Solution 2	$0.0000 - 1.0000i$	$0.0000 + 1.0586i$	$0.0000 - 1.0000i$
Solution 3	$0.0000 + 1.0000i$	$0.0000 + 0.3548i$	$0.0000 - 1.0000i$
Solution 4	$0.0000 - 1.0000i$	$0.0000 - 0.3548i$	$0.0000 + 1.0000i$
S_7^-	c_4	c_5	c_6
Solution 1	$0.0380 - 1.3648i$	$-0.0658 + 1.1880i$	$0.1816 - 1.3533i$
Solution 2	$-0.3337 + 0.5389i$	$0.0998 - 1.2287i$	$-0.8221 + 1.7786i$

Running the main homotopy with starting points according to Theorem 4.6 gives us 16 non-singular solutions to the IKP. The solutions are listed in Table 2, where $(\theta_1, \ldots, \theta_6)$ are the rotation angles of the joints given in radians.

6. CONCLUSION

The paper provides a new algorithm to compute the intersection of two m-dimensional subvarieties of a $2m$-dimensional quadric $Q \subset \mathbb{P}^{2m+1}$, taking advantage of a $\mathbb{C}^*$-action on Q. The algorithm gives a new solution to the inverse kinematics problem of a general 6R-manipulator. In future work, it would be interesting to

Table 2. The 16 solutions to the IKP.

	Solution 1	Solution 2	Solution 3
θ_1	2.2431	1.9910	$-1.7363 + 1.6583i$
θ_2	1.7050	1.9249	$2.7542 - 0.8059i$
θ_3	2.0862	1.0701	$0.0553 + 1.4781i$
θ_4	0.3765	1.9277	$2.7598 - 1.5359i$
θ_5	1.3117	0.2313	$-3.0066 - 1.6609i$
θ_6	-0.5586	0.1029	$0.1598 + 0.9059i$
	Solution 4	Solution 5	Solution 6
θ_1	$-1.7363 - 1.6583i$	$2.9220 - 0.1992i$	$2.9220 + 0.1992i$
θ_2	$2.7542 + 0.8059i$	$-2.6170 - 0.8767i$	$-2.6170 + 0.8767i$
θ_3	$0.0553 - 1.4781i$	$-2.0190 + 0.1863i$	$-2.0190 - 0.1863i$
θ_4	$2.7598 + 1.5359i$	$2.1718 + 1.4016i$	$2.1718 - 1.4016i$
θ_5	$-3.0066 + 1.6609i$	$0.6217 - 1.1119i$	$0.6217 + 1.1119i$
θ_6	$0.1598 - 0.9059i$	$-0.3667 + 0.8687i$	$-0.3667 - 0.8687i$
	Solution 7	Solution 8	Solution 9
θ_1	$1.6953 - 0.5095i$	$1.6953 + 0.5095i$	$1.2877 - 0.6327i$
θ_2	$1.9757 - 1.2651i$	$1.9757 + 1.2651i$	$2.1810 + 0.6016i$
θ_3	$3.0877 + 0.2628i$	$3.0877 - 0.2628i$	$0.3962 - 1.3499i$
θ_4	$-1.1266 + 1.9280i$	$-1.1266 - 1.9280i$	$2.9120 + 0.5212i$
θ_5	$2.4595 + 0.3528i$	$2.4595 - 0.3528i$	$0.2444 + 1.2075i$
θ_6	$-0.7309 - 0.8688i$	$-0.7309 + 0.8688i$	$0.3190 - 0.4047i$
	Solution 10	Solution 11	Solution 12
θ_1	$1.2877 + 0.6327i$	$-0.1134 - 1.6041i$	$-0.1134 + 1.6041i$
θ_2	$2.1810 - 0.6016i$	$1.4939 + 1.9975i$	$1.4939 - 1.9975i$
θ_3	$0.3962 + 1.3499i$	$0.7873 - 3.4032i$	$0.7873 + 3.4032i$
θ_4	$2.9120 - 0.5212i$	$-0.8994 + 2.6077i$	$-0.8994 - 2.6077i$
θ_5	$0.2444 - 1.2075i$	$-1.5518 - 2.4267i$	$-1.5518 + 2.4267i$
θ_6	$0.3190 + 0.4047i$	$2.7578 + 1.9916i$	$2.7578 - 1.9916i$
	Solution 13	Solution 14	Solution 15
θ_1	$-2.7292 - 0.8410i$	$-2.7292 + 0.8410i$	$-1.7811 - 2.9217i$
θ_2	$2.7231 - 0.6786i$	$2.7231 + 0.6786i$	$2.6204 - 0.7202i$
θ_3	$1.7327 + 1.6594i$	$1.7327 - 1.6594i$	$1.7095 + 2.6362i$
θ_4	$-1.6578 + 0.0624i$	$-1.6578 - 0.0624i$	$-0.1269 - 0.4056i$
θ_5	$1.3343 - 0.5607i$	$1.3343 + 0.5607i$	$-1.7362 + 2.2733i$
θ_6	$-0.5095 + 0.4606i$	$-0.5095 - 0.4606i$	$0.4227 - 0.7266i$
	Solution 16		
θ_1	$-1.7811 + 2.9217i$		
θ_2	$2.6204 + 0.7202i$		
θ_3	$1.7095 - 2.6362i$		
θ_4	$-0.1269 + 0.4056i$		
θ_5	$-1.7362 - 2.2733i$		
θ_6	$0.4227 + 0.7266i$		

consider the more general problem of intersecting two subvarieties of a rational homogeneous space X that have complementary dimensions.

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