NUMERICAL POLAR CALCULUS AND COHOMOLOGY OF LINE BUNDLES

SANDRA DI ROCCO, DAVID EKLUND, AND CHRIS PETERSON

ABSTRACT. Let $L_1,\dots,L_s$ be line bundles on a smooth variety $X\subset\mathbb{P}^r$ and let $D_1,\dots,D_s$ be divisors on X such that D_i represents L_i. We give a probabilistic algorithm for computing the degree of intersections of polar classes which are in turn used for computing the Euler characteristic of linear combinations of $L_1,\dots,L_s$. The input consists of generators for the homogeneous ideals $I_X, I_{D_i}\subset\mathbb{C}[x_0,\dots,x_r]$ defining X and D_i.

1. INTRODUCTION

Let $X\subset\mathbb{P}^r$ be a smooth n-dimensional variety with $n<r$. For $0\le j\le n$ and $V\subseteq\mathbb{P}^r$ a general linear subspace of dimension $(r-n-2)+j$, let

$$P_j(X)=\{x\in X:\dim(T_xX\cap V)\ge j-1\}.$$

If X has codimension 1 and $j=0$, then V is the empty set (rather than a linear subspace) with the convention $\dim(\emptyset)=-1$. Note that $P_j(X)$ depends on the choice of V even though we have suppressed this in the notation. However, for general V, the class $[P_j(X)]$ in the Chow ring of the variety $A_*(X)$ does not depend on V. In fact, for general V, $P_j(X)$ is either empty or of pure codimension j in X and

$$[P_j(X)]=\sum_{i=0}^{j}(-1)^i\binom{n-i+1}{j-i}H^{j-i}c_i,$$

where $H\in A_{n-1}(X)$ is the hyperplane class and c_i is the i^{th} Chern class of X. In this setting, $P_j(X)$ is called a j^{th} *polar locus* of X and $[P_j(X)]$ is called the j^{th} *polar class*. Because of the close relationship between polar classes and Chern classes, computation of the degrees of intersections of one is equivalent to computation of the degrees of intersections of the other. In particular the numerical algorithms developed in [5, 6] for computing intersection numbers of Chern classes will be used in this note in order to develop an algorithm for computing polar degrees and the degrees of intersection of polar classes, Procedure 2 and Procedure 3. We call these computations "Numerical Polar Calculus" and will denote the algorithm by **NPC**. Immediate applications of the computation include degree of the discriminant locus and the Euclidean Distance degree, as explained in Subsection 3.1. A Macaulay2 algorithm for computing polar classes of (not necessarily smooth or normal) toric varieties has been previously developed in [14]. While applications of NPC are multiple, we focus on uses of the algorithm that we regard as particularly interesting for the applied and computational algebraic geometry community. In Proposition 4.2 and Procedure 4 we illustrate how NPC can be used to develop an algorithm for computing the *Euler Characteristic* of a linear combination of divisors on X:

$$\chi(X,a_1D_1+\ldots+a_sD_s)=\sum_{i\ge 0}(-1)^i\dim(H^i(a_1D_1+\ldots+a_sD_s))$$

where the D_i are smooth divisors on X meeting properly. More precisely, we require that any $d\le n$ of the D_i meet in a subscheme every component of which has codimension d. The key ingredients are the Hirzebruch-Riemann-Roch formula and Adjunction formula as explained in Section 2.

We briefly illustrate the motivating idea. The Riemann-Roch theorem for a smooth projective curve X embedded in $\mathbb{P}^r$ by a line bundle L gives a powerful and striking link between invariants of the line bundle and invariants of the curve:

$$\dim(H^0(X,L)) - \dim(H^1(X,L)) = \chi(X,L) = \deg(L) + 1 - g$$

where $g = \dim(H^0(X,\omega_X))$ is the genus of X. It can be reinterpreted as an intersection formula involving the Chern classes of X and L:

$$\chi(X,L) = [c_1(L) + \frac{1}{2}c_1(T_X)] \cdot [X]$$

Using the Chern character approach, Hirzebruch generalized the Riemann-Roch theorem to an expression relating the Euler characteristic of a locally free sheaf $\mathscr{E}$ on a complex manifold X to the Chern character of $\mathscr{E}$ and the Todd class of X. For a line bundle L on X, the Hirzebruch-Riemann-Roch theorem states:

$$\sum_{i\geq 0}(-1)^i \dim(H^i(X,L)) = \chi(X,L) = \int_X \mathrm{ch}(L)\,\mathrm{td}(X) \tag{1}$$

where $\mathrm{ch}(L)$ is the Chern character of L and $\mathrm{td}(X)$ is the Todd class of T_X. The integral over X evaluates the top degree component of the class $\mathrm{ch}(L)\,\mathrm{td}(X)$ over X. The Chern character $\mathrm{ch}(L)$ can be expressed as a formal sum of subvarieties of codimension i in X

$$\mathrm{ch}(L) = \sum_{i\geq 0} \frac{1}{i!}[L]^i$$

The Todd class can also be given as a sum, $\mathrm{td}(X) = \mathrm{td}(T_X) = T_0 + T_1 + \ldots + T_n$, whose components can be expressed using Chern classes of T_X. The class T_k is a weighted homogeneous polynomial of weighted degree k in the Chern classes $c_1,\ldots,c_k$, provided c_i has weight i :

$$\mathrm{td}(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \ldots$$

Consequently the right-hand side of the Hirzebruch-Riemann-Roch formula (1) is computable by an algorithm similar to the ones introduced in [5, 6].

If $\dim(X) = 2$ for instance the formula leads to the expression

$$\chi(X,L) = \frac{L(L-K_X)}{2} + \chi(X,\mathscr{O}_X).$$

Notice that, for simplicity in this paper we use additive notation for the group $\mathrm{Pic}(X)$.

When the line bundle is special, i.e. $H^i(X,L) = 0$ for $i \geq 1$, the left hand side of the Hirzebruch-Riemann-Roch formula for the line bundles aL reduces to $\chi(X,aL) = \dim(H^0(X,aL))$. The graded algebra $\oplus_{a\geq 0}H^0(X,aL)$ and the Cox-ring, $\oplus_{L\in Pic(X)}H^0(X,L)$, are central in understanding the defining equations of a variety and also play an important role in birational geometry (in particular in the Minimal Model Program). For special line bundles, the algorithm gives a computational method to determine the dimension of the graded pieces of the ring $\oplus_{a\geq 0}H^0(X,aL)$. The cohomology vanishing assumption might seem strong but it is satisfied for important classes like toric varieties and Abelian varieties. Similarly the Kodaira vanishing theorem assures that, when L is an ample line bundle, $H^i(X,K_X + aL) = 0$ for $i \geq 1$.

Similar work and future directions. The degree of the j^{th} polar variety is equal to the degree of the j^{th} *coisotropic variety*. The coisotropic varieties $CH_i(X)$ are hypersurfaces of a Grassmannian variety defined and studied in [10]. Recent advances can be found in [13]. Along a similar line of approach one might consider computing the intersection formally on a Grassmannian. Consider the Gauss map $\gamma : X \to \mathbb{G}(n,r)$, which is generically birational when X is not a linear space. Instead of intersecting the polar classes on X in order to get intersections of Chern classes of X one could compute the class of $\gamma(X)$ in the Chow ring of the Grassmannian. Finally it is important to point out that the method using polar geometry was introduced in [4] where the case when the divisor is a hyperplane section is treated.

A development of theoretical tools for Polar classes of singular varieties can be found in [15, 16, 17]. Algorithms and computations of characteristic classes for singular varieties can be found in [7, 12]. The Euler characteristic algorithm relies on the Hirzebruch-Riemann-Roch formula which for singular varieties has been generalized by Baum-Fulton-MacPherson [3] and relies on localized Chern characters.

Acknowledgments. This work benefited from discussions held at Notre Dame and KTH. We are grateful to both institutions for generous support. In particular, we thank Andrew Sommese for posing the initial question that led to this note.

2. Notation and Background

Throughout this paper X denotes a smooth n-dimensional complex projective variety, T_X denotes its tangent bundle and $c_0, c_1, \dots, c_n$ denote the Chern classes of the tangent bundle. We use $c(T_X) = c_0 + c_1 + \dots + c_n$ for the total Chern class. The Todd class of T_X, also called the Todd class of X, is denoted $\mathrm{td}(X)$. For a vector bundle E on X, $\mathrm{ch}(E)$ denotes the Chern character of E.

Given a cycle class α in the intersection ring $A_*(X) = \bigoplus_{k=0}^n A_k(X)$, we use $\{\alpha\}_k$ to denote the projection of α to $A_k(X)$. For a zero-cycle class $\alpha \in A_0(X)$ represented by $\alpha = \sum_{i=1}^m n_i p_i$ with $n_i \in \mathbb{Z}$ and $p_i \in X$, we define $\int_X \alpha = \sum_{i=1}^m n_i$ (which is well defined on rational equivalence classes). This is extended to any $\alpha \in A(X)$ by putting $\int_X \alpha = \int_X \{\alpha\}_0$. In this paper, by the *degree* of a class we will mean the degree relative to an embedding of X in $\mathbb{P}^r$ in the following sense: if $H \in A_{n-1}(X)$ is the hyperplane class and $\alpha \in A_k(X)$, then $\deg(\alpha) = \int_X H^k \alpha$. For a subscheme $Z \subseteq X$, we have a corresponding cycle class $[Z] \in A_*(X)$.

Let L be a line bundle on X. The Euler characteristic of L is denoted $\chi(X, L)$ and defined as

$$\chi(X,L) = \sum_{i \geq 0} (-1)^i \dim(H^i(X,L)).$$

The Hirzebruch-Riemann-Roch (HRR) formula expresses the Euler characteristic in terms of the Todd class and the Chern character of L:

$$\chi(X,L) = \int_X \mathrm{ch}(L)\,\mathrm{td}(X).$$

Consider a smooth divisor $D \subset X$. We will use D to denote both the divisor and its class in the intersection ring of X. In order to compute the right hand side of the Hirzebruch-Riemann-Roch formula we will use the following version of the adjunction formula (see [9] Example 3.2.12):

$$f_*(c(T_D)) = c(T_X) \cdot (D - D^2 + D^3 - \ldots + (-1)^{n+1} D^n),$$

where $f : D \to X$ is the inclusion.

3. Numerical Polar calculus

In this section we present an algorithm to compute the degree of any intersection of polar classes of a non singular projective variety. A useful special case is the degree of the individual polar classes, which we state in a separate procedure. The algorithms may be implemented using symbolic software such as Macaulay2 [11] or numerical algebraic geometry software such as Bertini [2].

Let $I = (g_1, \dots, g_t) \subseteq \mathbb{C}[x_0, \dots, x_r]$ be a homogeneous ideal defining X and let $l_1, \dots, l_{n-j+2}$ be linear forms defining V. There is a scheme structure on $P_j(X)$ imposed by the sum of the ideal I and the ideal generated by the $(r-j+2) \times (r-j+2)$-minors of the Jacobian matrix of $\{g_1, \dots, g_t, l_1, \dots, l_{n-j+2}\}$. Note that the dimension of X forces all of the $(r-n+1) \times (r-n+1)$-minors of the Jacobian matrix of $\{g_1, \dots, g_t\}$ to vanish modulo the ideal I. As a consequence, to determine the scheme structure on $P_j(X)$, it is enough to take the sum of the ideal I and the ideal generated by the subcollection of $(r-j+2)\times(r-j+2)$-minors of the Jacobian matrix of $\{g_1,\dots,g_t,l_1,\dots,l_{n-j+2}\}$ that involve the final $n-j+2$ rows.

We start with a probabilistic algorithm that returns the equations defining the power of a polar class. This is then used to formulate a probabilistic algorithm to compute the degrees of the polar classes.

Remark 3.1. In practice we need to intersect general polar loci to compute the degrees of products of polar classes, so we need to know that general polar loci intersect properly. It is enough to show that for a purely k-dimensional closed subset $W\subseteq Z$ of a smooth projective variety $Z\subseteq\mathbb{P}^r$, every component of $W\cap P_j(Z)$ has dimension $k-j$ for a general polar locus $P_j(Z)$. This is done in Lemma 2.2 of [6].

Procedure 1 Power of polar class (ppc)

Input: Non-negative integers j,m,n with $j\le n$ and generators for homogeneous ideal $(g_1,\dots,g_t)\subseteq\mathbb{C}[x_0,\dots,x_r]$ defining a smooth variety $Z\subset\mathbb{P}^r$ of dimension n.
Output: An ideal J defining a subscheme representing $[P_j(Z)]^m$.
Let $S=\emptyset$.
for $1\le i\le m$ **do**
Let $l_1,\dots,l_{n-j+2}\in\mathbb{C}[x_0,\dots,x_r]$ be random linear forms.
Let S_0 be the set of $(r-j+2)\times(r-j+2)$-minors of the Jacobian matrix of $\{g_1,\dots,g_t,l_1,\dots,l_{n-j+2}\}$ that involve the last $n-j+2$ rows. Let $S=S\cup S_0$.
end for
Let J be the ideal generated by S and $g_1,\dots,g_t$.

Procedure 2 Degrees of polar classes

Input: Generators for homogeneous ideal $I\subseteq\mathbb{C}[x_0,\dots,x_r]$ defining a smooth variety $X\subset\mathbb{P}^r$ of dimension n.
Output: The degrees of the polar classes $[P_0(X)],\dots,[P_n(X)]$.
for $0\le j\le n$ **do**
Let $\deg([P_j(X)])=\deg(Y)$ where $Y\subset\mathbb{P}^r$ is the subscheme defined by ppc$(j,1,n,I)$.
end for

3.1. **The dual degree and the Euclidean distance degree.** Polar loci are defined for any variety (possibly singular):

$$P_j(X)=\overline{\{x\in X_{\mathrm{sm}}:\dim(T_xX\cap V)\ge j-1\}}.$$

The degrees of the polar classes can be used to compute two important invariants of a projective variety: the degree of the discriminant locus and the Euclidean distance degree. Recall that the irreducible variety

$$X^*=\overline{\{H\in(\mathbb{P}^r)^*\text{ such that }H\text{ is tangent to }X\text{ at a smooth point}\}}$$

is called the discriminant locus (or the dual variety) of the embedding $X\hookrightarrow\mathbb{P}^r$. Its dimension and degree is given by the polar variety as:

(1) $\operatorname{codim}(X^*)=n+1-\max\{k\mid P_k(X)\neq\emptyset\}$,
(2) If $P_n(X)\neq\emptyset$ then $\deg(P_n(X))$ is the degree of the irreducible polynomial defining the hypersurface X^*.

In the case when X is nonsingular $[P_j(X)] = c_j(J_1(L))$ where L is the line bundle defining the embedding and $J_1(L)$ denotes the first jet bundle. Thus, polar calculus gives information on invariants of jets and on differential properties of the embedding.

Another important invariant that can be expressed in terms of degrees of polar classes is the Euclidean distance degree (ED degree), see [8]. In the affine case, this is the number of critical points of the squared distance function from the variety to a given (general) point in the ambient space. For projective varieties the ED degree is defined as the ED degree of its affine cone. The ED degree of a variety X, $\mathrm{ED}(X)$, is then equal to the sum of the degrees of the polar classes of X [8].

3.2. **Intersections.** We now proceed to the general case of intersection of polar classes.

Procedure 3 Degrees of products of polar classes

Input: Generators for homogeneous ideal $I \subseteq \mathbb{C}[x_0, \dots, x_r]$ defining a smooth variety $X \subset \mathbb{P}^r$ of dimension n.

Output: An array V containing the degrees of all products

$$\prod_{j=1}^{n} [P_j(X)]^{m_j},$$

with $0 \le m_j \le n$ and $\sum_{j=1}^{n} jm_j \le n$.

Let $M \subset \mathbb{N}^n$, $M = \{(m_1, \dots, m_n) : 0 \le m_j \le n,\ \sum_{j=1}^{n} jm_j \le n\}$. Let V be an empty array.

for $(m_1, \dots, m_n) \in M$ **do**

Let $K = \sum_{j=1}^{n} \mathrm{ppc}(j, m_j, n, I)$.

Let $Y \subseteq \mathbb{P}^r$ be the subscheme defined by K. Adjoin $\deg(Y)$ to V.

end for

Example 3.2. Let M be a 2×4-matrix of general linear forms in 5 variables and consider the ideal J generated by all the 2×2-minors of M. The ideal J defines a smooth rational quartic curve $D \subset \mathbb{P}^4$. If we let I be an ideal generated by two general degree 2 elements of J, then I defines a smooth quartic surface X containing D.

We applied Procedure 3, implemented in Macaulay2 [11], to the ideal I. The result is:

$$\begin{aligned}
&\deg([P_0(X)]) = 4, \\
&\deg([P_1(X)]) = 8, \\
&\deg([P_2(X)]) = 12, \\
&\deg([P_1(X)] \cdot [P_1(X)]) = 16, \\
&\deg(X^*) = 12, \\
&\mathrm{ED}(X) = 24.
\end{aligned}$$

4. The general algorithm for HRR

We first illustrate the algorithm by working out the surface case and then present the general version.

4.1. **Surfaces.** For simplicity of notation we treat the case of two divisors. Consider the case of a smooth surface $X \subset \mathbb{P}^r$ and smooth curves $D, E \subset X$ representing line bundles L_1, L_2 (with D, E meeting in a zero-scheme). In this case, we have $\mathrm{ch}(aL_1 + bL_2) = 1 + (aD + bE) + \frac{1}{2}(a^2D^2 + 2abDE + b^2E^2)$ and $\mathrm{td}(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2)$. Therefore,

$$\{\mathrm{ch}(aL_1 + bL_2) \cdot \mathrm{td}(X)\}_0 = \frac{1}{12}(c_1^2 + c_2) + \frac{1}{2}c_1(aD + bE) + \frac{1}{2}(a^2D^2 + 2abDE + b^2E^2),$$

and the right hand side of the HRR formula is the degree of this 0-cycle class. For $j \in \{1,2\}$, let $d_j = f_*(c_{j-1}(T_D))$ and $e_j = g_*(c_{j-1}(T_E))$, where $f : D \to X, g : E \to X$ are the inclusions. We will compute the right hand side of the HRR formula by computing the degrees of the following classes on X in the given order: $1, c_1, c_2, c_1^2, d_1, e_1, c_1 d_1, c_1 e_1, d_2, e_2, d_1 e_1$. To see that this is sufficient, first observe that $d_1 = D, e_1 = E$ and hence $c_1 D = c_1 d_1, c_1 E = c_1 e_1$ and $DE = d_1 e_1$. For D^2, we apply the adjunction formula: $f_*(c(T_D)) = c(T_X) \cdot (D - D^2)$. This implies that $d_2 = c_1 d_1 - D^2$, and hence $D^2 = c_1 d_1 - d_2$. Similarly for E^2 we obtain $E^2 = c_1 e_1 - e_2$.

The degrees of the classes $1, c_1, c_2, c_1^2, d_1, e_1, c_1 d_1, c_1 e_1, d_2, e_2, d_1 e_1$ can be computed by intersecting polar classes of X, D and E. Namely, these numbers can be solved successively by computing the degrees of the following classes:

$$\begin{aligned}
&[P_0(X)] = 1, \\
&[P_1(X)] = 3H - c_1, \\
&[P_2(X)] = 3H^2 - 2Hc_1 + c_2, \\
&[P_1(X)] \cdot [P_1(X)] = 9H^2 - 6Hc_1 + c_1^2, \\
&f_*[P_0(D)] = d_1, \\
&g_*[P_0(E)] = e_1, \\
&[P_1(X)] \cdot f_*[P_0(D)] = 3Hd_1 - c_1 d_1, \\
&[P_1(X)] \cdot g_*[P_0(E)] = 3He_1 - c_1 e_1, \\
&f_*[P_1(D)] = 2Hd_1 - d_2, \\
&g_*[P_1(E)] = 2He_1 - e_2, \\
&f_*[P_0(D)] g_*[P_0(E)] = d_1 e_1,
\end{aligned}$$

where $H \in A_1(X)$ is the hyperplane class.

4.2. **The general Euler algorithm.** We now describe the general algorithm for computing the Euler characteristic of multiples of a line bundle. We have implemented it using Macaulay2 but since the problem is reduced to computing the number of solutions to a polynomial system, an alternative is to use software for the numerical solution of such systems. The input consists of homogeneous ideals $I, J_1, \ldots, J_s \subseteq \mathbb{C}[x_0, \ldots, x_r]$ where I defines a smooth scheme $X \subseteq \mathbb{P}^r$ and J_i defines a smooth divisor $D_i \subset X$. If X has dimension n then any $d \leq n$ of the D_i should meet in a subscheme every component of which has codimension d. The algorithm is probabilistic as it depends on generic choices of linear subspaces defining polar loci of X and $D_1, \ldots, D_s$. The following lemma is an application of the HRR formula and adjunction.

Lemma 4.1. *Let $f_i : D_i \to X$, $i = 1, \ldots, s$ be inclusion maps and let $d_{j,i} = f_{i*}(c_{j-1}(T_{D_i}))$, and $c_j = c_j(T_X)$ for $1 \leq j \leq n$. The Euler characteristic $\chi(X, \sum_{i=1}^s a_i D_i)$ may be expressed in terms of degrees of monomials in $c_1, \ldots, c_n, d_{j,i}$ at most linear in $d_{1,i}, \ldots, d_{n,i}$ for $i = 1, \ldots, s$.*

Proof. By the HRR formula, $\chi(X, \sum_{i=1}^s a_i D_i)$ may be expressed in terms of the degrees of monomials of the form $c \cdot D_1^{k_1} \cdots D_s^{k_s}$ where $k_i \in \mathbb{N}$ and $c \in A_{k_1 + \cdots + k_s}(X)$ is a monomial in $c_1(T_X), \ldots, c_n(T_X)$.

Furthermore, $c \cdot D_1^{k_1} \cdots D_s^{k_s}$ may be expressed in terms of monomials in $c_1, \ldots, c_n, d_{j,i}$ which are (at most) linear in $d_{1,i}, \ldots, d_{n,i}$ for $i = 1, \ldots, s$. To see this, note that the adjunction formula for D_i states that

$$\sum_{j=1}^{n} d_{j,i} = c(T_X) \cdot (D_i - D_i{}^2 + D_i{}^3 - \ldots + (-1)^{n+1} D_i{}^n),$$

which means that

$$d_{k,i} = \sum_{j=1}^{k} (-1)^{j+1} D_i^j c_{k-j}$$

for $k = 1,\dots,n$. Note that $d_{1,i} = D_i$. It follows by induction over k that for $k \geq 1$, D_i^k lies in the subgroup of $A_k(X)$ generated by elements of the form $\mu \cdot d_{j,i}$ where μ is a monomial in $c_1,\dots,c_n$. □

Proposition 4.2. *The computation of $\chi(X, \sum_{i=1}^s a_i D_i)$ may be reduced to computing degrees of monomials in the polar classes of X and $D_1,\dots,D_s$. The monomials are at most linear in the polar classes of D_i.*

Proof. By Lemma 4.1, $\chi(X, \sum_{i=1}^s a_i D_i)$ can be expressed in terms of degrees of monomials in the classes $c_1,\dots,c_n, d_{j,i}$, at most linear in $d_{1,i},\dots,d_{n,i}$. The statement is clear once we express the classes $c_1,\dots,c_n,d_{j,i}$ in terms of polar classes of X, $D_1,\dots,D_s$ and the hyperplane class $H \in A_*(X)$. Recall that for $0 \leq j \leq n$,

$$[P_j(X)] = \sum_{i=0}^{j} (-1)^i \binom{n-i+1}{j-i} H^{j-i} c_i.$$

Inverting this relationship gives, for $0 \leq j \leq n$,

$$c_j = \sum_{i=0}^{j} (-1)^i \binom{n-i+1}{j-i} H^{j-i} [P_i(X)].$$

In the same way, we get for $1 \leq i \leq s$ and $0 \leq j \leq n-1$ that

$$d_{j+1,i} = \sum_{l=0}^{j} (-1)^l \binom{n-l}{j-l} H^{j-l} f_{i*}[P_l(D_i)].$$

□

By Proposition 4.2, Remark 3.1 and the assumption regarding proper intersection of the D_i, the problem of computing the Euler characteristic can be reduced to computing the intersections of certain polar loci of X and D_i and we will state the algorithm in this form. This is a minor extension of Procedure 3 to include polar classes of the divisors D_i. To compute the degrees of these intersections we need a routine that computes the degree of a projective variety. Such a routine will be assumed given and is used as a building block in the algorithm. The main algorithm Procedure 4 will make use of the subroutine Procedure 1 (ppc) which returns the equations defining a power of a polar class.

4.3. **Complexity.** In [4], the problem of computing the Hilbert polynomial of a smooth equidimensional projective variety is reduced in polynomial time to the problem of computing the number of solutions to a polynomial system. In this paper we have a similar reduction for computing the Euler characteristic of a line bundle on a projective variety. The complexity of this reduction is not the focus of this paper but there are a few important points to make in this context.

First, computing degrees of polar varieties and their intersections by forming minors of the Jacobian matrix extended by linear forms leads to exponential size, which is unnecessary (see [4] Example 3.11).

Second, the treatment in Procedure 4 is redundant since not all monomials of the form $c \cdot D_1^{k_1} \cdots D_s^{k_s}$ (with notation as in the proof of Lemma 4.1) appear in the right hand side of the HRR formula. For example, in the case of a threefold X, $c_3(T_X)$ does not appear at all. A more efficient algorithm would only compute the monomials that are needed.

Procedure 4 Euler Characteristics

Input: Generators for homogeneous ideal $I \subseteq \mathbb{C}[x_0, \dots, x_r]$ defining a smooth variety $X \subset \mathbb{P}^r$ of dimension n and ideals $J_1 \supset I, \dots, J_s \supset I$ defining smooth divisors $f_i : D_i \to X$. The divisors are assumed to meet properly in the sense that any $d \leq n$ of them meet in a subscheme every component of which has codimension d.

Output: An array V containing the degrees of all products

$$\prod_{i=1}^{s} f_{i*}([P_{k_i-1}(D_i)])^{a_i} \prod_{j=1}^{n} [P_j(X)]^{m_j},$$

with $0 \leq m_j \leq n$, $1 \leq k_i \leq n$, $a_i \in \{0,1\}$ and $\sum_{i=1}^{s} a_i k_i + \sum_{j=1}^{n} j m_j \leq n$.

Let $M \subset \mathbb{N}^{n+2s}$,

$$M = \{(m_1, \dots, m_n, k_1, \dots, k_s, a_1, \dots, a_s) : 0 \leq m_j \leq n, 1 \leq k_i \leq n, 0 \leq a_i \leq 1, \sum_{i=1}^{s} a_i k_i + \sum_{j=1}^{n} j m_j \leq n\}.$$

Let V be an empty array.
for $(m_1, \dots, m_n, k_1, \dots, k_s, a_1, \dots, a_s) \in M$ **do**
  Let $A = \{i : a_i = 1\}$.
  Let $K = \sum_{j=1}^{n} \mathrm{ppc}(j, m_j, n, I) + \sum_{i \in A} \mathrm{ppc}(k_i - 1, a_i, n-1, J_i)$.
  Let $Y \subseteq \mathbb{P}^r$ be the subscheme defined by K. Adjoin $\deg(Y)$ to V.
end for

5. A gallery of examples

Example 5.1. Let M be a 2×4-matrix of general linear forms in 5 variables. Consider the ideal J generated by the 2×2-minors of M. The ideal J defines a smooth rational quartic curve $D \subset \mathbb{P}^4$. If we let I be an ideal generated by two general degree 2 elements of J, then I defines a smooth quartic surface X containing D.

We applied Procedure 4 in Section 4, implemented in Macaulay2 [11], to the ideals I and J. The result is:

$$\begin{aligned}
&\deg([P_0(X)]) = 4,\\
&\deg([P_1(X)]) = 8,\\
&\deg([P_2(X)]) = 12,\\
&\deg([P_1(X)] \cdot [P_1(X)]) = 16,\\
&\deg(f_*[P_0(D)]) = 4,\\
&\deg([P_1(X)] \cdot f_*[P_0(D)]) = 8,\\
&\deg(f_*[P_1(D)]) = 6,\\
&\dim(X^*) = 3, \deg(X^*) = 12,\\
&\mathrm{ED}(X) = 24,
\end{aligned}$$

where $f : D \to X$ is the inclusion. Let H be the hyperplane class of X. For the Chern classes of X, the table above and the discussion in Section 4.1 give $\deg(c_1) = \deg(3H - [P_1(X)]) = 12 - 8 = 4$ and $\deg(c_2) = \deg([P_2(X)] - 3H^2 + 2Hc_1) = 12 - 12 + 2 \cdot 4 = 8$. Moreover, $\deg(c_1^2) = \deg([P_1(X)]^2 - 9H^2 + 6Hc_1) = 16 - 9 \cdot 4 + 6 \cdot 4 = 4$. Let $d_j = f_*(c_{j-1}(T_D))$ for $j \in \{1,2\}$. Then $\deg(d_1) = 4$ and $\deg(c_1 d_1) = \deg(3Hd_1 - [P_1(X)] \cdot f_*[P_0(D)]) = 12 - 8 = 4$. Also, $\deg(d_2) = \deg(2Hd_1 - f_*[P_1(D)]) = 8 - 6 = 2$. Finally, we have that $\deg(c_1 D) = \deg(c_1 d_1) = 4$ and by the adjunction formula $\deg(D^2) = \deg(c_1 d_1 - d_2) = 4 - 2 = 2$. Putting this together we get that for $a \in \mathbb{Z}$:

$$\chi(X, aD) = \frac{1}{12}(4+8) + \frac{1}{2}4a + \frac{1}{2}2a^2 = 1 + 2a + a^2.$$

Example 5.2. For this example we consider the Veronese embedding

$$\nu : \mathbb{P}^2 \to \mathbb{P}^5 : (x,y,z) \mapsto (x^2, y^2, z^2, xy, xz, yz).$$

The image $X = \pi(\nu(\mathbb{P}^2))$ of the projection

$$\pi : \mathbb{P}^5 \to \mathbb{P}^4 : (a,b,c,d,e,f) \mapsto (a+c, b+c, d, e, f)$$

is smooth. Consider a divisor $D \subset X$ corresponding to a general cubic curve in $\mathbb{P}^2$. The equations for X and D embedded in $\mathbb{P}^4$ can be found via elimination, let the corresponding ideals be denoted I and J. We will consider the Euler characteristic of the line bundle corresponding to $aD + bH$, where $H \in A_1(X)$ is the hyperplane class and $a, b \in \mathbb{Z}$.

Running Procedure 4 in Section 4 on I and J, results in the following:

$$\begin{aligned}
&\deg([P_0(X)]) = 4,\\
&\deg([P_1(X)]) = 6,\\
&\deg([P_2(X)]) = 3,\\
&\deg([P_1(X)] \cdot [P_1(X)]) = 9,\\
&\deg(f_*[P_0(D)]) = 6,\\
&\deg([P_1(X)] \cdot f_*[P_0(D)]) = 9,\\
&\deg(f_*[P_1(D)]) = 12,\\
&\dim(X^*) = 3, \deg(X^*) = 3,\\
&\mathrm{ED}(X) = 13.
\end{aligned}$$

Solving for the degrees of the needed monomials in the Chern classes and D, we get the result in Table 1.

TABLE 1

	H^2	c_1	c_2	c_1^2	d_1	$c_1 d_1$	d_2	$c_1 D$	D^2
degree	4	6	3	9	6	9	0	9	9

For the Euler characteristic of $aD + bH$ we get:

$$\chi(X, aD + bH) = \frac{1}{12}(c_1^2 + c_2) + \frac{1}{2}c_1(aD + bH) + \frac{1}{2}(aD + bH)^2 = 1 + \frac{9}{2}a + 3b + \frac{9}{2}a^2 + 6ab + 2b^2.$$

Example 5.3. Let $J \subseteq \mathbb{C}[x_0, \ldots, x_5]$ be generated by three general forms of degree 2 and let D be the corresponding complete intersection surface. If we let I be an ideal generated by two general degree 2 elements of J, then I defines a smooth quartic threefold X containing D.

Running Procedure 4 on I and J gives the following result:

$$\begin{aligned}
&\deg([P_0(X)]) = 4,\\
&\deg([P_1(X)]) = 8,\\
&\deg([P_2(X)]) = 12,\\
&\deg([P_3(X)]) = 16,\\
&\deg([P_1(X)]\cdot[P_1(X)]) = 16,\\
&\deg([P_1(X)]\cdot[P_2(X)]) = 24,\\
&\deg(f_*[P_0(D)]) = 8,\\
&\deg([P_1(X)]\cdot f_*[P_0(D)]) = 16,\\
&\deg([P_2(X)]\cdot f_*[P_0(D)]) = 24,\\
&\deg([P_1(X)]\cdot[P_1(X)]\cdot f_*[P_0(D)]) = 32,\\
&\deg(f_*[P_1(D)]) = 24,\\
&\deg([P_1(X)]\cdot f_*[P_1(D)]) = 48,\\
&\deg(f_*[P_2(D)]) = 48.
\end{aligned}$$

Solving for the degrees of the needed monomials in the Chern classes and D as above we get the result in Table 2.

TABLE 2

	c_1	c_2	c_1^2	c_1c_2	d_1	c_1d_1	c_2d_1	$c_1^2d_1$	d_2	c_1d_2	d_3	c_1^2D	c_2D	c_1D^2	D^3
degree	8	12	16	24	8	16	24	32	0	0	24	32	24	32	32

This results in the Euler characteristic

$$\chi(X,aD) = 1 + \frac{14}{3}a + 8a^2 + \frac{16}{3}a^3.$$

Notice that the algorithm gives $\dim(X^*) = 4, \deg(X^*) = 16$ and $\mathrm{ED}(X) = 40$.

Example 5.4. Let M be a 2×3-matrix of general linear forms in $\mathbb{C}[x_0,\dots,x_5]$ and consider the variety X defined by the 2×2-minors of M. The threefold X is isomorphic to $\mathbb{P}^1\times\mathbb{P}^2$ and we may consider a general divisor D of type $(1,2)$. To get equations for D one can pick a random multihomogeneous polynomial p in $\mathbb{C}[a,b,x,y,z]$ of type $(1,2)$, consider the graph of the Segre embedding $\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^5$ intersected with $\{p=0\}$, and eliminate the variables a,b,x,y,z.

Running Procedure 4 on the ideals defining X and D gives the following result:

$$\begin{aligned}
&\deg([P_0(X)]) = 3,\\
&\deg([P_1(X)]) = 4,\\
&\deg([P_2(X)]) = 3,\\
&\deg([P_3(X)]) = 0\\
&\deg([P_1(X)]\cdot[P_1(X)]) = 5,\\
&\deg([P_1(X)]\cdot[P_2(X)]) = 3,\\
&\deg(f_*[P_0(D)]) = 5,\\
&\deg([P_1(X)]\cdot f_*[P_0(D)]) = 7,\\
&\deg([P_2(X)]\cdot f_*[P_0(D)]) = 6,\\
&\deg([P_1(X)]\cdot[P_1(X)]\cdot f_*[P_0(D)]) = 9,\\
&\deg(f_*[P_1(D)]) = 10,\\
&\deg([P_1(X)]\cdot f_*[P_1(D)]) = 14,\\
&\deg(f_*[P_2(D)]) = 12.
\end{aligned}$$

Solving for the degrees of the needed monomials in the Chern classes and D as above we get the result in Table 3.

TABLE 3

	c_1	c_2	c_1^2	c_1c_2	d_1	c_1d_1	c_2d_1	$c_1^2d_1$	d_2	c_1d_2	d_3	c_1^2D	c_2D	c_1D^2	D^3
degree	8	9	21	24	5	13	15	33	5	13	7	33	15	20	12

This results in the Euler characteristic

$$\chi(X, aD) = 1 + 4a + 5a^2 + 2a^3.$$

Notice that $\mathrm{ED}(X) = 10$ and $\dim(X^*) = 3$ since $\deg([P_3(X)]) = 0$ and $\deg([P_2(X)]) = 3$.

REFERENCES

[1] P. Aluffi, *Computing characteristic classes of projective schemes*, Journal of Symbolic Computation 35, 3-19 (2003).

[2] D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler, *Bertini: Software for Numerical Algebraic Geometry*. Available at bertini.nd.edu with permanent doi: dx.doi.org/10.7274/R0H41PB5.

[3] P. Baum, W. Fulton, R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. 45, 101-145 (1975).

[4] P. Bürgisser, M. Lotz, *The complexity of computing the Hilbert polynomial of smooth equidimensional complex projective varieties*, Foundations of Computational Mathematics, 51-86 (2007).

[5] S. Di Rocco, D. Eklund, C. Peterson, A. J. Sommese, *Chern numbers of smooth varieties via homotopy continuation and intersection theory*, Journal of Symbolic Computation 46 (1), 23-33 (2011).

[6] D. J. Bates, D. Eklund, C. Peterson, *Computing intersection numbers of Chern classes*, Journal of Symbolic Computation 50, 493-507 (2013).

[7] C. Jost, *A Macaulay2 package for characteristic classes and the topological Euler characteristic of complex projective schemes*, Journal of Software for Algebra and Geometry 7, 31-39 (2015).

[8] J. Draisma, E. Horobet, G. Ottaviani, B. Sturmfels, R. Thomas, *The Euclidean distance degree of an algebraic variety*, Foundations of Computational Mathematics 16, 99-149 (2016).

[9] W. Fulton, *Intersection Theory*, Springer-Verlag (1998).

[10] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston 1994.

[11] D. R. Grayson, M. E. Stillman, *Macaulay2: a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2`

[12] C. Harris, *Computing Segre classes in arbitrary projective varieties*, Journal of Symbolic Computation 82, 26-37 (2017).
[13] K. Kohn, *Coisotropic Hypersurfaces in the Grassmannian*, arXiv.org:1607.05932
[14] M. Helmer, B. Sturmfels, *Nearest Points on Toric Varieties*, To Appear in Math. Scandinavica, Software available at: `https://math.berkeley.edu/~mhelmer/Software/toricED`
[15] R. Piene, *Polar classes of singular varieties*, Annales Scientifiques de l'École Normale Supérieure (4) 11, 247-276 (1978).
[16] R. Piene, *Cycles polaires et classes de Chern pour les variétés projectives singulières*, Introduction à la théorie des singularités, II, Travaux en cours, Vol. 37, Hermann, Paris, 7-34 (1988).
[17] R. Piene, *Polar varieties revisited*, Lecture Notes in Comput. Sci 8942: "Computer algebra and polynomials", 139-150 (2015).

DEPARTMENT OF MATHEMATICS, KTH, 100 44 STOCKHOLM, SWEDEN
E-mail address: dirocco@math.kth.se
URL: http://www.math.kth.se/∼sandra

DEPARTMENT OF MATHEMATICS, KTH, 100 44 STOCKHOLM, SWEDEN
E-mail address: daek@math.kth.se
URL: http://www.math.kth.se/∼daek

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523
E-mail address: peterson@math.colostate.edu
URL: http://www.math.colostate.edu/∼peterson