A Scenario Based Approach to Robust Experiment Design

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Abstract: Robust optimal experiment design is an infinite dimensional optimisation problem. Typically it is solved by discretisation of the design space resulting in a discrete semi-infinite convex programming problem which is computationally expensive. In this paper we propose a new computational approach to solve robust optimal experiment design problems based on a recently developed method for robust convex optimisation known as the 'scenario approach'.

1. INTRODUCTION

Essentially, experiment design involves the adjustment (design) of the experimental conditions such that maximal information is gained from the experiment. It is well known that the experimental conditions have a strong influence on the accuracy of models obtained from system identification and hence requires a consolidated design. This has motivated substantial research on experiment design during the last century. Early research in the statistics literature includes Cox [1958], Fedorov [1972], Wald [1943], Whittle [1973], Wynn [1972], and, in the engineering literature, Gagliardi [1967], Goodwin et al. [1973], Goodwin and Payne [1977], Hildebrand and Gevers [2003b], Levadi [1966], Mehra [1974], Zarrop [1979]. More recent surveys are contained in Gevers [2005], Hjalmarsson [2005], Pronzato [2008] where many additional references can be found. In general, the focus in the engineering literature has been on experiment design for dynamic system identification.

In dynamical systems, a critical issue for experiment design is that the model is, typically, nonlinearly parameterised. This means that the Fisher information matrix [Goodwin and Payne, 1977], which is typically used as the basis for experiment design, depends, inter alia, on the true system parameters, i.e. the very thing that the experiment is aimed at finding.

Preliminary work in the engineering literature on robust experiment design includes substantial work on iterative design [Gevers, 2005, Hjalmarsson, 2005], and an insightful sub-optimal min-max solution for a one parameter problem in Walter and Pronzato [1997]. Also, a number of recent engineering publications refer to the idea of min-max optimal experiment design [Gevers and Bombois, 2006, Mårtensson and Hjalmarsson, 2006, Rojas et al., 2007].

A min-max robust design criterion is the basis of the approach described in the current paper. Specifically, we assume that we have available a-priori information that the parameters can take any value in a compact set Θ. We also constrain the allowable set of input signals. A typical constraint [Goodwin and Payne, 1977, Zarrop, 1979, Walter and Pronzato, 1997] used in experiment design is one placed on the input energy. The purpose of min-max robust experiment design is to optimise the input spectrum for the worst case performance of the identification procedure (typically measured as a scalar function of the information matrix of the model parameters).

The min-max optimisation problem can be considered as a special case of a robust convex program [Ben-Tal and Nemirovski, 1998]. In this case a linear objective function is minimised subject to a number of convex constraints, one for each instance of the uncertainty.

In robust experiment design it is usual to describe the uncertainty as a continuous set. Presenting this as a robust convex optimisation problem would give rise to an infinite number of constraints. This leads to a semi-infinite optimisation problem that is known to be difficult to solve and possibly NP-Hard [Ben-Tal and Nemirovski, 1998].

An approach that has been recently developed in Calafiore and Campi [2005, 2006] to deal with semi-infinite convex programming at a general level is known as the 'scenario approach'. The advantage of this method is that solvability can be obtained through random sampling of constraints provided that a probabilistic relaxation of the worst case robust paradigm is accepted. The probabilistic relaxation consists in being content with robustness against the large majority of situations rather than against all situations. In the scenario approach the number of situations is under the control of the designer and can be made arbitrarily close to the set of 'all' situations.

Utilising this approach we will show that the min-max experiment design problem can be approximated quite closely with considerable gains made in the reduction of computation time.

It is important to note that alternative computational approaches to solve robust experiment design problems have appeared in the literature. For example, in [Pronzato and Walter, 1988], the relaxation algorithm of Shimizu and Aiyoshi [Shimizu and Aiyoshi, 1980] has been proposed to solve these problems. However, the performance of this approach depends on how the cost function and the constraints relate to the true parameter. Also note that this relaxation algorithm can only achieve local optimality in the general case. The scenario approach, on the other hand, does not impose any conditions upon the dependence on the true parameter, providing the nominal problem is convex.

The layout of the remainder of the paper is as follows: Section 2 describes the basic setup of the robust experiment design problem. The scenario approach for solving robust convex programs is explained in Section 3. Section 4 describes a probabilistic bound on the minimum number of scenarios required to obtain a given level of accuracy. Some numerical examples of the methodology developed in this paper are presented in Section 5. Finally, Section 6 provides the conclusions.

2. ROBUST EXPERIMENT DESIGN

2.1 The Information Matrix

An intuitive way to compare different experiments is to choose a measure related to the expected accuracy of the parameter estimator of the model to be obtained from the experimental data. However, the accuracy of the parameter estimator is a function of both the experimental conditions and the form of the estimator. Since we would prefer to have an 'estimator-independent' measure, we may assume that the estimator used is statistically efficient in the sense that the parameter covariance matrix achieves the Cramér-Rao lower bound [Goodwin and Payne, 1977], i.e.

$$\text{cov}\,\hat{\theta} = M^{-1},$$

where M is the Fisher's information matrix [Casella and Berger, 2002, Silvey, 1970]. Note that estimators are denoted by a superscript '$\widehat{\ }$' and implicitly depend on the data length, N. Therefore, the first step is to determine an expression for M.

To be specific, consider a single-input single-output (SISO) linear continuous time system, with input $u(t)$ and output $y(t)$, of the form

$$y(t) = G(p)u(t) + H(p)w(t)$$

where G and H are stable rational transfer functions, p is the time derivative operator, H is minimum phase with $H(\infty) = 1$, and $w(t)$ is zero mean Gaussian white noise of intensity σ^2. We assume that the system is operating in open loop, hence $u(t)$ and $w(t)$ are independent. We let $\theta := [\rho^T\ \eta^T\ \sigma^2]^T$ where ρ denotes the parameters in G and η denotes the parameters in H. Therefore, we assume that G, H and σ^2 are independently parameterised.

Assume that the input $u(t)$ has a zero order hold mechanism, with sampling period h, and that we sample the output $y(t)$ with the same sampling period h. Then for estimation purposes we will have N samples $\{u(kh), y(kh)\}_{k=1}^N$. Fisher's information matrix M is given by [Goodwin and Payne, 1977]

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where M_1 is the part of the information matrix related to ρ, and M_2 is independent of the input. Assuming N is large, it is more convenient to work with the scaled average information matrix for the parameters ρ [Goodwin and Payne, 1977, Walter and Pronzato, 1997],

$$\begin{aligned}\overline{M}(\theta, \Phi_u) &:= \lim_{N\to\infty} \frac{1}{Nh} M_1 \sigma^2 \\ &= \int_0^\infty \widetilde{M}(\theta, \omega)\Phi_u(\omega)d\omega. \end{aligned} \qquad (1)$$

where

$$\widetilde{M}(\theta, \omega) := \text{Re}\left\{ \frac{\partial G(j\omega)}{\partial \rho} |H(j\omega)|^{-2} \left[\frac{\partial G(j\omega)}{\partial \rho}\right]^H \right\}, \qquad (2)$$

G and H are continuous time transfer functions (assumed independently parameterised) and Φ_u is the continuous time input spectral density.

2.2 Criteria for Nominal Experiment Design

Since $\overline{M}$ is a matrix, we need a scalar measure of $\overline{M}$ for the purpose of experiment design. In the nominal case, typically treated in the engineering literature (i.e. when a fixed prior estimate of θ is used), several measures of the 'size' of $\overline{M}$ have been proposed which measure the 'goodness' of the experiment. Some examples include,

(i) D - optimality [Goodwin and Payne, 1977]

$$J_d(\theta, \Phi_u) := [\det \overline{M}(\theta, \Phi_u)]^{-1}. \qquad (3)$$

(ii) Experiment design for robust control [Hildebrand and Gevers, 2003a,b, Hjalmarsson, 2005].

$$J_{rc}(\theta, \Phi_u) := \sup_\omega g(\theta, \omega)^H \overline{M}^{-1} g(\theta, \omega) \qquad (4)$$

where g is a frequency dependent vector related to the ν-gap [Hildebrand and Gevers, 2003a,b].

Many other criteria have been described in the statistics literature, such as A-optimality ($\text{tr}\,\overline{M}(\theta, \Phi_u)^{-1}$), L-optimality ($\text{tr}\,W\overline{M}(\theta, \Phi_u)^{-1}$, for some $W \geq 0$) and E-optimality $\big(\lambda_{\max}(\overline{M}(\theta, \Phi_u)^{-1})\big)$; see Kiefer [1974]. On the other hand, in the engineering literature, Bombois et al. [2006] proposed a criterion that specifies the required accuracy to achieve a given level of robust control performance.

A common feature of all these nominal experiment design approaches is that they are aimed at choosing Φ_u to minimise a function of the type such as in (3) and (4). Most criteria are convex in Φ_u, so in the sequel, we will consider that the chosen criterion has this property.

2.3 Min-Max Robust Design

A min-max robust design criterion is the basis of our experiment design technique. Specifically, we assume that a-priori information is available indicating that the parameters can take any value in a compact set Θ. We also constrain the allowable set of input signals. Typically in experiment design, a constraint is imposed on input energy [Goodwin and Payne, 1977, Walter and Pronzato, 1997, Zarrop, 1979]. Here we define the constraint as [1]

$$\mathscr{S}(\mathbb{R}_0^+) := \left\{ \Phi_u : \mathbb{R} \to \mathbb{R}_0^+ : \quad \Phi_u \text{ is even and } \int_{-\infty}^{\infty} \Phi_u(\omega)d\omega = 1 \right\}.$$

The min-max robust optimal input spectral density, Φ_u^{opt}, is then chosen as

$$\Phi_u^{opt} = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \sup_{\theta\in\Theta} J(\theta, \overline{M}(\theta, \Phi_u)) \qquad (5)$$

where J is an appropriate scalar measure of $\overline{M}$. We assume that Φ_u^{opt} exists and is unique; see Rojas et al. [2007]. Notice also that we allow J to depend explicitly on θ.

[1] In general, given a set $X \subseteq \mathbb{R}_0$, we will denote by $\mathscr{S}(X)$ the set of all even generalised functions Φ_u on $\mathbb{R}$ [Rudin, 1973] such that Φ_u is the derivative of some probability distribution function on $\mathbb{R}$, and $\text{supp}\,\Phi_u \subseteq X \cup (-X)$, where $\text{supp}\,\Phi_u$ is the support of Φ_u (i.e. roughly speaking, $\mathscr{S}(X)$ is the set of all even (generalised) probability density functions on $X \cup (-X)$).

2.4 Discrete Approximation to the Optimal Input

Note that (5) is an infinite dimensional optimisation problem. In order to solve this problem we must approximate (1) by discretisation of the design space. To this end, we first restrict the positive support of Φ_u to a compact interval, say $K := [\underline{\omega}, \overline{\omega}] \subset \mathbb{R}_0^+$, hence $\Phi_u \in \mathscr{S}(K)$. Next we approximate the integral in equation (1) by a Riemann sum. Specifically, we choose a grid of $d+1$ points $\omega_m \in [\underline{\omega}, \overline{\omega}]$ for $m = 0, \ldots, d$ such that $\omega_0 = \underline{\omega}$, $\omega_d = \overline{\omega}$. Then

$$\begin{aligned} \overline{M}(\theta, \Phi_u) &:= \int_{\underline{\omega}}^{\overline{\omega}} \widetilde{M}(\theta, \omega)\Phi_u(\omega) d\omega \\ &\approx \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n)\Phi_u(\omega_n)(\omega_{n+1} - \omega_n) \qquad (6) \\ &= \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n) E_n \end{aligned}$$

where $E_n := \Phi_u(\omega_n)(\omega_{n+1} - \omega_n)$. We can now state the following discrete semi-infinite convex programming approximation to (5):

$$\begin{aligned} &\min_{t\in\mathbb{R},\, E\in\mathbb{R}^d} \ t \\ &\text{s.t.} \quad J\left(\theta, \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n) E_n\right) \le t, \qquad \theta \in \Theta \\ &\qquad\quad \sum_{n=0}^{d-1} E_n = 1 \qquad (7) \\ &\qquad\quad E_n \ge 0, \qquad n = 0, \ldots, d-1. \end{aligned}$$

where ‘s.t.’ denotes ‘subject to’.

3. THE SCENARIO APPROACH

In this section we briefly review the Scenario Approach for solving a general robust convex problem. The scenario approach presumes a probabilistic description of uncertainty, that is, the uncertainty is characterised through a set Δ describing the set of admissible situations, and a probability distribution P_r over Δ.

As shown in Section 2.4, the min-max optimisation problem, when converted to a robust convex optimisation program yields an unwieldy number of constraints, c.f. (7). The scenario approach involves selecting a small number of these constraints to include in the optimisation problem. Therefore by extracting, at random, N instances or ‘scenarios’ of the uncertainty parameter δ according to some probability P_r we consider only the corresponding constraints in the scenario optimisation problem.

Consider the following general Robust Convex Program:

$$RCP: \begin{array}{ll} \min\limits_{\gamma\in\mathbb{R}^d} & c^T\gamma \\ \text{s.t.} & f_\delta(\gamma) \le 0, \quad \delta \in \Delta. \end{array} \qquad (8)$$

where $f_\delta : \mathbb{R}^d \to \mathbb{R}$ is convex for every $\delta \in \Delta$. The scenario-based approximation is described next.

Scenario-Based Optimisation [Calafiore and Campi, 2006]

Extract N independent identically distributed samples $\delta^{(1)}, \ldots, \delta^{(N)}$, according to the probability P_r and solve the scenario convex program:

$$SCP_N: \begin{array}{ll} \min\limits_{\gamma\in\mathbb{R}^d} & c^T\gamma \\ \text{s.t.} & f_{\delta^{(i)}}(\gamma) \le 0, \quad i = 1, \ldots, N. \end{array} \qquad (9)$$

It can be seen from (9) that it is a standard finite dimensional convex optimisation problem with a finite number of constraints. Therefore the computational cost, provided N is not large, will be significantly smaller than the cost associated with the min-max optimisation problem.

4. BOUNDS ON THE NUMBER OF SCENARIOS

By considering only a finite subset of constraints, which are chosen in a random manner, we would like the scenario-based optimisation program SCP_N to provide a solution γ^{opt} which, with high probability, say $1-\beta$, satisfies all the constraints in Δ, except for a fraction with a small probability, say ϵ (with respect to the probability measure P_r). Here β is denoted as the ‘confidence parameter’ and ϵ is the ‘violation parameter’. These variables are user choices which determine the minimum number of scenarios N to be randomly selected.

Several bounds on the minimum number of scenarios required have been derived in the literature, see Alamo et al. [2007, 2008], Calafiore and Campi [2006], Campi and Garatti [2007]. To date, the tightest bound has been established in Campi and Garatti [2007], according to which N has to satisfy

$$\sum_{i=0}^{d-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \le \beta. \qquad (10)$$

This relationship establishes an implicit dependence of N on d, and comes from the following proposition, first established in Calafiore and Campi [2005]:

Theorem 1. Consider the convex program:

$$\mathcal{P}: \begin{array}{ll} \min\limits_{x\in\mathbb{R}^d} & c^T x \\ \text{s.t.} & x \in \mathcal{X}_i, \quad i = 1, \ldots, m, \end{array}$$

where $c \in \mathbb{R}^d$, and $\mathcal{X}_i$, $i = 1, \ldots, m$, are closed convex sets in $\mathbb{R}^d$. Also define for every $k = 1, \ldots, m$,

$$\mathcal{P}_k: \begin{array}{ll} \min\limits_{x\in\mathbb{R}^d} & c^T x \\ \text{s.t.} & x \in \mathcal{X}_i, \quad i = 1, \ldots, k-1, k+1, \ldots, m. \end{array}$$

Let x^{opt}, x_k^{opt} be any optimal solutions of $\mathcal{P}$ and $\mathcal{P}_k$, $k = 1, \ldots, m$, respectively. We say that $\mathcal{X}_k$ is a *support constraint* for $\mathcal{P}$ if $c^T x_k^{opt} < c^T x^{opt}$. Then, the number of support constraints for $\mathcal{P}$ is at most d.

Proof. See Calafiore and Campi [2005].

According to (10), the bound on N is an increasing function of d. On the other hand, for robust experiment design, the size of d is related to the discretisation described in Section 2.4. This appears to give rise to a huge curse of dimensionality, since in order to obtain a reasonable degree of approximation, the required number of scenarios might be too large for a practical implementation. However, in most practical cases, it is possible to replace d in (10) by a much smaller number, asymptotically independent of the degree of approximation made in Section 2.4. To show this, we require the following result:

Theorem 2. Assume that $\widetilde{M}(\theta, \omega)$ in (2) is uniformly bounded with respect to θ, strictly positive in $\Theta \times \mathbb{R}$, analytic with respect to ω for every $\theta \in \Theta$, and such that $\lim_{|\omega|\to\infty} \widetilde{M}(\theta, \omega) = 0$

for every $\theta \in \Theta$. Assume also that the criterion J is an analytic convex function of $\overline{M}$ for every $\theta \in \Theta$, such that $\lim_{\overline{M}\to\infty} J(\theta, \overline{M}) = 0$ and $\lim_{\overline{M}\to 0} J(\theta, \overline{M}) = \infty$ uniformly in $\theta \in \Theta$. Then, any solution Φ_u^{opt} of (5) with compact support has finite support.

Proof. This theorem is a variation of Theorems 7.1.1 and 7.1.3 of [Karlin, 1959, Volume II]. $\square$

Under the conditions of this result, it follows that if d is large enough, most of the variables, E_n, in (7) will be zero. Therefore, by Theorem 1, the number of support constraints of (7) can be bounded by a number independent of d. In fact, the bound can be made equal to twice the cardinality of the support of Φ_u^{opt} in (5), since every support point of Φ_u^{opt} will give rise to at most two nonzero variables [2], E_n. This means that the bound (10) can be replaced by

$$\sum_{i=0}^{M-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta, \tag{11}$$

where M is twice the number of support points of Φ_u^{opt}, which is a priori unknown but usually small and independent of d.

Notice that the number of terms in the sum of (11) could be further reduced, since the input power constraint and the nonnegativity of Φ_u are necessarily support constraints.

The previous argument provides some theoretical support for the practical applicability of the scenario approach to the robust experiment design problem. This will be confirmed by examples given in the following section.

Remark 1. For the bound (11) to hold, it is not necessary to know, a priori, exactly which variables E_n are nonzero. This is because the proof of (10) in Campi and Garatti [2007] only relies on a bound for the number of support constraints, which, by Theorem 1, depends on the number M of nonzero decision variables of problem $\mathcal{P}$. Unfortunately, M is not known a priori, but it can roughly estimated in a preliminary stage by using the scenario approach, as explained in the examples of Section 5.

Remark 2. Notice that the scenario approach involves two levels of randomization, since it provides a (highly) probably correct solution to a convex program with chance constraints (see e.g. [Nemirovski and Shapiro, 2006]). In fact, for $\beta = 0$, the problem corresponds to a convex program with a design criterion of the "quantile type", as defined in Pázman and Pronzato [2007], where a steepest descent algorithm (which converges to a local optimum) is developed.

5. NUMERICAL EXAMPLES

In this section, two examples are presented. The first example utilises a one parameter first order system, which can be solved in principle using linear programming. The second example involves three parameters, which shows the real potential of the scenario approach.

5.1 First Order Problem [Rojas et al., 2007]

Consider a model given by $H(s) = 1$ and

$$G(s) = \frac{1}{s/\theta + 1},$$

where it is assumed that $\theta \in [0.1, 10]$. For this model structure, the 'single frequency' normalised information matrix is given by

$$\widetilde{M}(\theta, \omega) = \frac{\omega^2/\theta^4}{(\omega^2/\theta^2 + 1)^2}.$$

Consider a criterion of the form

$$J(\theta, \overline{M}(\theta, \Phi_u)) = \frac{1}{\theta^2 \overline{M}(\theta, \Phi_u)}.$$

The reason for multiplying $\overline{M}$ by θ^2 is that $\overline{M}^{-1}$ is a variance measure and thus $[\theta^2 \overline{M}]^{-1}$ gives relative (mean square) errors.

As shown in Rojas et al. [2007], this robust experiment design problem can be solved by discretising the interval for θ, and rewriting the problem as a linear program. This approach is similar to the one described in Section 3, except for the fact that in Rojas et al. [2007] a deterministic (in fact, uniform) sampling of the constraints has been used.

Considering an interval $[0.1, 10]$ for the support of Φ_u (which, according to Rojas et al. [2007], actually contains the optimal spectrum), $d = 30$, $\epsilon = 0.01$, and $\beta = 10^{-10}$, the bound (10) shows that N should be at least 7864. For this value of N, and a distribution P_r which is uniform on $\ln\theta$, the scenario approach gives the spectrum presented in Figure 1.

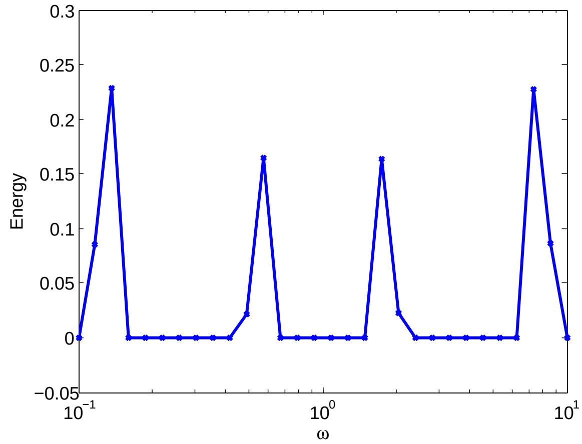

Fig. 1. Values of E for the discretised robust optimal input of Example 1.

Note, from Figure 1, that the optimal input seems to have only 4 spectral lines, hence by bound (11) (with $M = 2 \times 4 = 8$) N should be at least 4044, independently of d. This means that we can increase the resolution of the spectrum (by choosing a larger d) without having to increase the number of scenarios.

Now, we interpret the meanings of β and ϵ. Consider the optimal cost $J(\theta, \overline{M}(\theta, \Phi_u^{opt}))$, as shown in Figure 2. According to the definition of ϵ, with probability $1 - \beta$, the fraction of the constraints being violated is at most ϵ (in terms of the probability measure P_r, which corresponds to a uniform distribution in $\ln\theta$ for this example). Figure 3 in fact shows that a small fraction of the constraints are violated at the boundaries of the interval $[0.1, 10]$.

[2] This statement is asymptotically true, as the optimal solution of the discretized problem can be made as close as possible to the optimal solution, in a weak convergence sense, by increasing d [Owen, 1968, page 78].

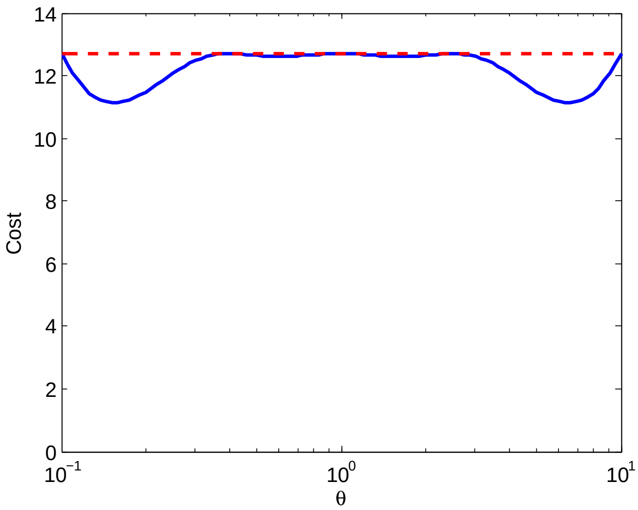

Fig. 2. Cost $J(\theta, \overline{M}(\theta, \Phi_u^{opt}))$ as a function of θ for Example 1 (solid), and the optimal cost (dashed).

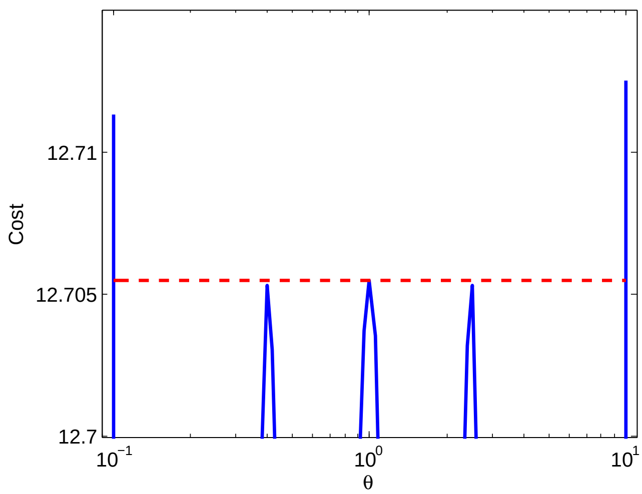

Fig. 3. Zoom in of Figure 2.

5.2 Multiparameter D-Optimal Design

Consider now a model given by $H(s) = 1$ and

$$G(s) = \frac{K}{s^2 + a_1 s + a_0},$$

where it is assumed that $a_1 \in [1, 2]$, $a_0 \in [1, 9]$ and $K \in [1, 2]$. Consider a criterion of the form

$$J(\theta, \overline{M}(\theta, \Phi_u)) = (\det\{S_\theta \overline{M}(\theta, \Phi_u) S_\theta\})^{-1}, \qquad (12)$$

where $\det$ denotes the determinant and S_θ is a parameter dependent scaling matrix. One possible choice for S_θ is $\mathrm{diag}(a_0, a_1, K)$. The motivation for this choice is that $\overline{M}(\theta, \Phi_u)^{-1}$ is a measure of the parameter covariance matrix. Hence $S_\theta^{-1} \overline{M}(\theta, \Phi_u)^{-1} S_\theta^{-1}$ is the covariance normalised by the nominal values of each parameter. Therefore it is a measure of the relative error. This seems to be an important property in the robust design context (where we maximise over $\theta \in \Theta$) since it ensures that one is maximising (over Θ) the relative errors. These errors are normalised and thus better scaled for comparison purposes.

Equation (12) corresponds to a modified D-criterion, and is log-convex in Φ_u (furthermore, it can be optimised for a fixed θ

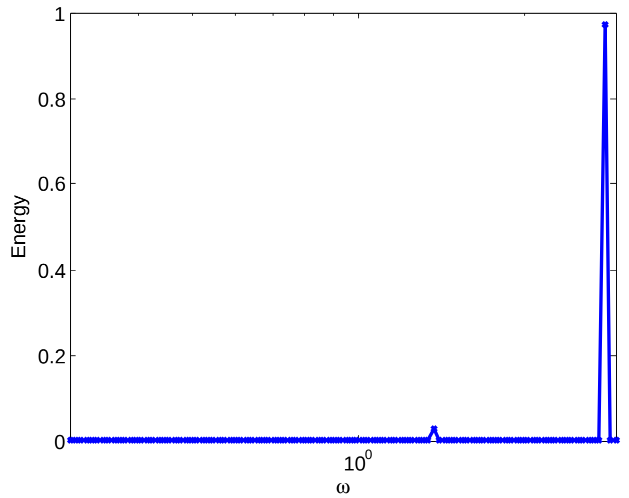

Fig. 4. Values of E for the discretised robust optimal input of Example 2.

via semidefinite programming, see Nesterov and Nemirovskii [1994]). Considering $d = 100$, $N = 4000$, a frequency range $[\underline{\omega}, \overline{\omega}] = [0.3, 3]$, and a uniform probability distribution P_r on $[1, 2] \times [1, 9] \times [1, 2]$, the scenario approach gives the solution presented in Figure 4. This figure shows that the optimal spectrum has 2 spectral lines, hence by (11), with probability $1 - \beta = 1 - 10^{-10}$, the fraction of constraints being violated is less than 0.006.

The resulting scenario program has been solved using semidefinite programming with the LMI parser YALMIP [Löfberg, 2004] and the solver SeDuMi. In a PC with Intel Core 2 Duo CPU of 2.53GHz and 3.48 Gb of RAM the program is solved in 606.7 seconds.

6. CONCLUSION

This paper has proposed a robust optimal experiment design procedure based on a scenario approach. The technique appears quite promising when compared to approaches based on a finely discretised optimisation problem. The scenario approach gives a performance that approaches that of the optimal cost. It also allows one to trade off cost versus computation by selecting the number of scenarios to be used based on a probability that some constraints may be violated. Two examples are provided which highlight the efficacy of this approach.

REFERENCES

T. Alamo, R. Tempo, and E. F. Camacho. Improved sample size bounds for probabilistic robust control design: A packet-based strategy. In *Proceedings of the 46th IEEE Conference on Decision and Control (CDC)*, pages 6178–7183, 2007.

T. Alamo, R. Tempo, and E. F. Camacho. Statistical learning theory: A packet-based strategy for uncertain feasibility and optimization problems. In V. D. Blondel, S. P. Boyd, and H. Kimura, editors, *Recent Advances in Learning and Control*. Springer-Verlag, London, 2008.

A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

X. Bombois, G. Scorletti, M. Gevers, P. Van den Hof, and R. Hildebrand. Least costly identification experiment for control. *Automatica*, 42(10):1651–1662, October 2006.

G. Calafiore and M. C. Campi. Uncertain convex programs: Randomized solutions and confidence levels. *Mathematical Programming, Series A*, 102:25–46, 2005.

G. C. Calafiore and M. C. Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51(5):742–753, 2006.

M. C. Campi and S. Garatti. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM Journal on Optimization (to appear)*, 2007. Available at Optimization Online, http://www.optimization-online.org/.

G. Casella and R. L. Berger. *Statistical Inference, 2nd Edition*. Duxbury, California, 2002.

D. R. Cox. *Planning of Experiments*. Wiley, New York, 1958.

V. V. Fedorov. *Theory of Optimal Experiments*. Academic Press, New York, 1972.

R. M. Gagliardi. Input selection for parameter idenfication in discrete systems. *IEEE Transactions on Automatic Control*, AC-12(5):133–147, 1967.

M. Gevers. Identification for control: From the early achievements to the revival of experiment design. *European Journal of Control*, 11:1–18, 2005.

M. Gevers and X. Bombois. Input design: From open-loop to control-oriented design. In *14th IFAC Symposium on System Identification (SYSID'06)*, pages 1329–1334, Newcastle (Australia), 2006.

G. C. Goodwin and R. L. Payne. *Dynamic System Identification: Experiment Design and Data Analysis*. Academic Press, New York, 1977.

G. C. Goodwin, J. C. Murdoch, and R. L. Payne. Optimal test signal design for linear single input – single output system identification. *International Journal of Control*, 17(1):45–55, 1973.

R. Hildebrand and M. Gevers. Minimizing the worst-case ν-gap by optimal input design. In *Proceedings of 13th IFAC Symposium on System Identification*, pages 665–670, 2003a.

R. Hildebrand and M. Gevers. Identification for control: Optimal input design with respect to a worst-case ν-gap cost function. *SIAM Journal on Control and Optimization*, 41(5): 1586–1608, 2003b.

H. Hjalmarsson. From experiment design to closed-loop control. *Automatica*, 41(3):393–438, March 2005.

S. Karlin. *Mathematical Methods and Theory in Games, Programming, and Economics*. Addison-Wesley, Reading, Massachussets, 1959.

J. Kiefer. General equivalence theory for optimum designs (aproximate theory). *Annals of Statistics*, 2(5):849–879, 1974.

V. S. Levadi. Design of input signals for parameter estimation. *IEEE Transactions on Automatic Control*, 11(2):205–211, 1966.

J. Löfberg. YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. Available from http://control.ee.ethz.ch/˜joloef/yalmip.php.

J. Mårtensson and H. Hjalmarsson. Robust input design using sum of squares constraints. In *14th IFAC Symposium on System Identification (SYSID'06)*, pages 1352–1357, Newcastle, Australia, 2006.

R. K. Mehra. Optimal input signals for parameter estimation in dynamic systems – survey and new results. *IEEE Transactions on Automatic Control*, 19(6):753–768, December 1974.

A. Nemirovski and A. Shapiro. Scenario approximations of chance constraints. In *Probabilistic and Randomized Methods for Design under Uncertainty*, chapter 1, pages 3–47. Springer, London, 2006.

Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM, Philadelphia, 1994.

G. Owen. *Game Theory*. W. B. Saunders, Philadelphia, 1968.

A. Pázman and L. Pronzato. Quantile and probability-level criteria for nonlinear experimental design. In J. Lopez-Fidalgo, J.M. Rodriguez-Diaz, and B. Torsney, editors, *mODa'8 - Advances in Model-Oriented Design and Analysis, Proceedings of the 8th International Workshop*, pages 157–164, Almagro, Spain, 2007. Physica Verlag.

L. Pronzato. Optimal experimental design and some related control problems. *Automatica*, 44(2):303–325, February 2008.

L. Pronzato and E. Walter. Robust experiment design via maxmin optimisation. *Mathematical Biosciences*, 89:161–176, 1988.

C. R. Rojas, J. S. Welsh, G. C. Goodwin, and A. Feuer. Robust optimal experiment design for system identification. *Automatica*, 43(6):993–1008, 2007.

W. Rudin. *Functional Analysis*. McGraw-Hill, New York, 1973.

K. Shimizu and E. Aiyoshi. Necessary conditions for min-max problems algorithms by a relaxation procedure. *IEEE Transactions on Automatic Control*, 25(1):62–66, 1980.

S. D. Silvey. *Statistical Inference*. Chapman and Hall, London, 1970.

A. Wald. On the efficient design of statistical investigations. *Annals of Mathematical Statistics*, 14:134–140, 1943.

E. Walter and L. Pronzato. *Identification of Parametric Models from Experimental Data*. Springer-Verlag, Berlin, Heidelberg, New York, 1997.

P. Whittle. Some general points in the theory of optimal experimental design. *Journal of the Royal Statistical Society*, 35(1):123–130, 1973.

H. P. Wynn. Results in the theory and construction of D-optimum experimental designs. *Journal of the Royal Statistical Society*, 34(2):133–147, 1972.

M. Zarrop. *Optimal Experiment Design for Dynamic System Identification*, volume 21 of *Lecture Notes in Control and Information*. Springer, Berlin, New York, 1979.