Robust Experiment Design

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I hereby certify that the work embodied in this thesis is the result of original research, the greater part of it which was completed subsequent to admission to candidature for the degree. Note that, during the course of the candidature, several papers have been coauthored with my academic supervisors (together with Dr. Juan C. Agüero and Professor Arie Feuer) based on normal candidate-supervisor practice. Chapters 7 and 8 are based on collaborative work with Märta Barenthin and Professor Håkan Hjalmarsson, from KTH, Sweden. The contributions of these chapters are considered to be equally distributed between Märta Barenthin and myself, Cristian R. Rojas.

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To my parents,
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CONTENTS

ABSTRACT

This Thesis addresses the problem of robust experiment design, i.e., how to design an input signal to maximise the amount of information obtained from an experiment given limited prior knowledge of the true system. The majority of existing literature on experiment design specifically considers optimal experiment design, which, typically depends on the true system parameters, that is, the very thing that the experiment is intended to find. This obviously gives rise to a paradox. The results presented in this Thesis, on robust experiment design, are aimed at resolving this paradox.

In the robust experiment design problem, we assume that the parameter vector is a-priori known to belong to a given compact set, and study the design of an input spectrum which maximises the worst case scenario over this set. We also analyse the problem from a different perspective where, given the same assumption on the parameter vector, we examine cost functions that give rise to an optimal input spectrum independent of the true system features. As a first approach to this problem we utilise an asymptotic (in model order) expression for the variance of the system transfer function estimator. To enable the extension of these results to finite order models, we digress from the main topic and develop several fundamental integral limitations on the variance of estimated parametric models. Based on these results, we then return to robust experiment design, where the input design problems are reformulated using the fundamental limitations as constraints. In this manner we establish that our previous results, obtained from asymptotic variance formulas, are valid also for finite order models.

Robustness issues in experiment design also arise in the area of 'identification for (robust) control'. In particular, a new paradigm has recently been developed to deal with experiment design for control, namely 'least costly experiment design'. In the Thesis we analyse least costly experiment design and establish its equivalence with the standard formulation of experiment design problems.

Next we examine a problem involving the cost of complexity in system identification. This problem consists of determining the minimum amount of input power required to estimate a given system with a prescribed degree of accuracy, measured as the maximum variance of its frequency response estimator over a given bandwidth. In particular, we study the dependence of this cost on the model order, the required accuracy, the noise variance and the size of the bandwidth of interest.

Finally, we consider the practical problem of how to optimally generate an input signal given its spectrum. Our solution is centered around a Model Predictive Control (MPC) based algorithm, which is straightforward to implement and exhibits fast convergence that is empirically verified.

CHAPTER 1

INTRODUCTION

In engineering, to design a given device or system, it is usually crucial to have a mathematical model of a portion of the physical reality that is directly related to the design problem we need to solve. This is particularly true in automatic control, where the goal is to design a feedback mechanism to force a physical dynamic system to behave in a prescribed way.

Techniques for developing mathematical models can be categorised into two main groups:

1. *Physical (White-Box or First-Principles) Modelling*, where a model is developed based on well known physical laws, and

2. *System Identification (or Black-Box Modelling)*, where we assume that the physical system can be described in terms of a particular mathematical structure (e.g. a linear differential equation), and use data-based statistical techniques to find suitable parameters for this structure.

A third group, which is essentially a hybrid combination of the two above groups, is known as Grey-Box (or Semi-Physical) Modelling. Basically it consists of using physical modelling principles to derive a suitable model structure to describe the system, and then to employ statistical techniques to estimate the parameters of this model from input-output data.

In System Identification, we are generally not interested in obtaining a deep understanding of the true system. What we require is a reasonable mathematical approximation of the true system suitable for use in design, simulation or prediction. This task involves a tradeoff between precision and complexity which can, in general, be solved by invoking the Parsimony Principle: 'Do not use extra parameters for describing a dynamic phenomenon if they are not needed' (Stoica and Söderström 1982). Furthermore, the intended use of the model should also be taken into consideration. For example, in order to regulate an industrial process, it is often enough to use a simple low order model. However, to gain physical insight into the system we need more complex and sophisticated models.

1.1 System Identification

According to (Linz 1979), many problems of applied mathematics can be formulated in mathematical terms as that of solving the equation

$$Tx = y,$$

where $x \in X$, $y \in Y$, X and Y are linear spaces, and $T : X \to Y$. This equation involves three quantities, hence it is possible to distinguish three different types of problems:

1. *The direct problem.* Given T and x, find y.
2. *The inverse problem.* Given T and y, find x.
3. *The identification problem.* Given x and y, find T.

In the jargon of systems engineering, x, y and T correspond to the input, the output, and the system respectively.

In this Thesis we focus on the last problem, namely the system identification problem, where one attempts to establish a mathematical model representative of the physical laws governing a system from a finite number of input-output data points. In more precise terms, system identification deals with the construction of models representing static or dynamic systems from discrete-time measurement data, based on some prior information, described in terms of a suitable set of possible models and assumptions on the data. For more information on the topic of system identification, the reader is referred to (Eykhoff 1974, Goodwin and Payne 1977, Ljung 1999, Norton 1986, Pintelon and Schoukens 2001, Söderström and Stoica 1989, Solo 1986).

Most existing work in this area has been focused on linear, time invariant, finite dimensional dynamic systems (Kailath 1980), in continuous- or discrete-time, which can be described in either the time or frequency domain. These are the type of systems we consider in this Thesis.

The construction of a model, using system identification, involves several stages:

- Experiment design.
- Data collection.
- Model structure selection.
- Model parameter estimation.

- Model validation.

We focus here on the first stage, experiment design. Specifically, we address the problem of robust experiment design, i.e., how to design an input signal which maximises the amount of information obtained from an experiment, given that the true system is a-priori unknown.

1.2 Experiment Design

As discussed in (Söderström and Stoica 1989), the result of an identification experiment may be affected by four factors:

1. The system, which corresponds to the physical reality
2. The model structure, i.e., the set of possible models from which the identification procedure selects one based on the data.
3. The identification method, that is, the statistical procedure which, based on the data, chooses a particular model from the model structure.
4. The experimental condition. Roughly speaking, this corresponds to a description of how the identification experiment is performed. This involves issues such as the input signal (with or without feedback from the system's output), the sampling mechanism (i.e., sampling interval, holding mechanism, presampling filtering, etc.), data prefiltering, etc.

Of these factors, only the system is to be considered fixed, since, in general, it cannot be modified by the identification experiment.

The design of the experiment is highly important. It is performed before, and specifies the conditions of, the actual experiment. The data collected from the experiment is then used by the identification method to discriminate among the different models in the model structure, and to determine the one which best represents the input-output behaviour of the true system. At this point it is useful to remember the GIGO (Garbage In Garbage Out) principle: If we start with bad data, it is very difficult to obtain a reliable model.

Background to the problem of experiment design can be found in the statistics literature (Cox 1958, Fedorov 1972, Fedorov and Hackl 1997, Karlin and Studden 1966, Kempthorne 1952, Kiefer and Wolfowitz 1960, Pukelsheim 1993, Silvey 1980, Titterington 1980, Wald 1943, Whittle 1973, Wynn

1972) as well as in the engineering literature (Arimoto and Kimura 1973, Gagliardi 1967, Goodwin et al. 1973a, Goodwin and Payne 1973, Goodwin et al. 1973b, Goodwin and Payne 1977, Hildebrand and Gevers 2003b, Levadi 1966, Mehra 1974a, Zarrop 1979). Recent surveys are contained in (Gevers 2005, Pronzato 2008) where many additional references can be found. The focus in the engineering literature has been predominately on experiment design for dynamic system identification.

Among the several design issues which constitute the experimental condition, we will focus on the design of the input signal. The ultimate goal is to design an input which maximises the amount of information in the data, within the constraints imposed on the experimental condition. These constraints can include amplitude and power restrictions on the input, output or internal variables of the system, and the total duration of the experiment.

In order to design an input signal, it is necessary to know some properties of the true system. If nothing is known about the system, it is in general impossible to design an optimal/efficient experiment. This gives rise to a paradox, and can be posed as a question in the following manner: How can we design an experiment to estimate a mathematical model of the true system, if we do not know what the true system is before performing the experiment?

1.3 Thesis Overview

1.3.1 Min-Max Experiment Design

A key issue with experiment design for dynamic systems is that the model is usually nonlinearly parameterised. This means, amongst other things, that the Fisher information matrix (Goodwin and Payne 1977), which is typically used as the basis for experiment design, depends, inter alia, on the true system parameters (Ford et al. 1989) (i.e. the nominal optimal experiment depends on the very thing that the experiment is aimed at finding).

One may imagine that, provided a good a-priori estimate of the parameter vector is available, then an experiment designed on the basis of this parameter vector, called a *locally optimal design* (Chernoff 1953, Fedorov 1972, Ford et al. 1989, Pronzato 2008), would still be useful in practice. However, one does need to be careful that the experiment does not simply reinforce what one already knows (or believes to know) instead of making a genuine exploration of 'unchartered territories'.

The issue of robustness to nominal parameter values has been analysed in both the statistics and engineering literature. Suggested strategies for dealing with robustness include:

- Sequential design, where one iterates between parameter estimation, on the one hand, and experiment design using the current parameter estimates, on the other – see (Chernoff 1972; 1975, Ford and Silvey 1980, Ford et al. 1985, Hjalmarsson et al. 1996, Müller and Pötscher 1992, Walter and Pronzato 1997, Wu 1985).

- Bayesian design (Atkinson et al. 1993, Atkinson and Doner 1992, Chaloner and Larntz 1989, Chaloner and Verdinelli 1995, El-Gamal and Palfrey 1996, Sebastiani and Wynn 2000). The Bayesian approach is characterised by the minimisation of the expected value (over the prior parameter distribution) of a local optimality criterion related to the information matrix.

- Min-max design (Biedermann and Dette 2003, D'Argenio and Van Guilder 1988, Dette et al. 2003, Fedorov 1980, Landaw 1984, Melas 1978, Pronzato and Walter 1988).

However, there has been little work on robust experiment design for engineering problems. This has been highlighted in the recent survey paper (Hjalmarsson 2005), where it is stated that '... as usual in experiment design, in order to compute the optimal design the true system has to be known. Methods that are robust with respect to uncertainty about the system is a wide open research field.'

Preliminary work in the engineering literature on robust experiment design includes substantial work on iterative design (Gevers 2005, Hjalmarsson 2005) and an insightful sub-optimal min-max solution for a one parameter problem (Walter and Pronzato 1997, page 339). This latter problem will be discussed, in detail, in Chapter 2. Also, a number of very recent engineering papers refer to the idea of min-max optimal experiment design – see for example papers presented at SYSID'06, e.g., (Gevers and Bombois 2006, Mårtensson and Hjalmarsson 2006).

In Chapter 2 we develop the idea of min-max optimal experiment design for dynamic system identification. To gain insight into this approach, we explore two illustrative examples in depth. For these examples we determine several theoretical properties of the min-max optimal input, namely its existence and uniqueness, and the fact that its spectrum has finite support (even though the cardinality of its support is usually not bounded). The uniqueness of the min-max optimal input is quite unexpected, since in nominal experiment design the optimal experiment is in general non unique (Goodwin and Payne 1977, page 137). Also, the fact that the robust optimal input has finite support is very surprising, as the nominal optimal input for these examples is a single sinusoid, hence one would expect that the robust optimal signal should have a continuous spectrum to account for all the possible values of the parameter. However, this property also highlights that it is very easy to implement such signals, since they are multisines.

We assume prior information in the form that the system parameters, θ, are contained in a given compact set Θ. We then choose a design criterion $f(M(\theta),\theta)$ where $M(\theta)$ is the Fisher information matrix, evaluated at θ, and design the experiment to optimise the worst case of $f(M(\theta),\theta)$ over Θ. Note that this differs from the usual approaches to experiment design in the engineering literature which typically optimise $f(M(\theta_0),\theta_0)$ for some given nominal value θ_0.

Our approach is more akin to the usual formulation of robust optimal control which typically considers the worst case (Zhou et al. 1996). Indeed, there are substantial links between the work presented here and continuous game theory (Başar and Bernhard 1995, Başar and Olsder 1995, Fudenberg and Tirole 1991, Owen 1995, Szép and Forgó 1985). These links are discussed in Chapter 2.

The merits of our approach are illustrated by two examples which show, for a first order system and a second order resonant system, that an order of magnitude improvement in the worst case performance in experiment design can be achieved at the expense of only a few percent degradation in the nominal performance.

It is also shown that a bandlimited '$1/f$' noise input performs surprisingly well as a 'robust' input signal. This suggests that we should not use (near) white inputs. Furthermore it suggests that it may be worthwhile to investigate binary inputs whose energy distribution approximates bandlimited '$1/f$' noise. This will be examined in Chapter 9.

1.3.2 Optimal Experiment Design with Diffuse Prior Information

A surprising observation from our work on min-max optimal experiment design is that bandlimited '$1/f$' noise is actually quite close to optimal for particular problems. Indeed, in Chapter 2, bandlimited '$1/f$' noise is shown to have a performance which is within a factor of 2 from the performance of robust optimal designs for first-order and resonant systems. It is important to note, however, that the proof of near optimality depends on a particular property of these systems which allows the parameters to be scaled with respect to frequency.

In Chapter 3 we ask a more general question; if we are just beginning to experiment on a system and thus have very little (i.e. diffuse) prior information, what would be a 'good' initial experiment to use to estimate the system?

We consider, as diffuse prior information, that the interesting part of the frequency response of the system lies in an interval $[a,b]$. This implies that we are seeking an experiment which is 'good' over a very broad class of possible systems. In Chapter 3, we propose a possible solution to this problem,

namely that the experiment should consist of bandlimited '$1/f$' noise.

In Chapter 3 we consider a broad class of systems, so we first need to study the problem of measuring the 'goodness' of an experiment that uses a system independent criterion. We then propose some desirable properties that one would expect this measure of 'goodness' to possess. Next, a preliminary result is developed for selecting a cost function which satisfies the desirable properties. With this result we then design a suitable cost function to account for the assumption that only diffuse prior information is available. The final form of the cost function that satisfies the desired properties is then specified. Finally we show that bandlimited '$1/f$' noise is an optimal input signal according to the cost function developed for diffuse prior information.

1.3.3 Fundamental Limitations on the Variance of Estimated Parametric Models

Fundamental Limitations are of importance since they quantify the possible and the impossible. In feedback control, the development of fundamental limitations has given insight and understanding into the achievable performance of a feedback control system (Bode 1945, Goodwin et al. 2001, Seron et al. 1997). Knowledge of these limitations also allows informed decisions to be made regarding the tradeoffs between conflicting performance criteria, e.g., the Bode integral shows that increasing performance in a particular frequency region will reduce performance in another. This is known as the *water-bed effect* (Seron et al. 1997).

The original motivation for the study of fundamental limitations was in feedback design for control systems. However there have also been a number of limitations developed in other areas. For example, the Cramér-Rao Bound is an important relationship in estimation theory (Cramér 1946, Lehmann and Casella 1998). In information theory there is the Shannon Theorem (Shannon 1948), which is sometimes known as the fundamental theorem of information theory. Again the limitations described by these two results provide inescapable performance bounds.

To date, there has been relatively few publications dealing with fundamental limitations in system identification. Previous work in this area has examined integral constraints on systematic errors (bias) for least-squares estimators (de Moor et al. 1991; 1994, Gevers 1990, Ninness 1996, Salgado et al. 1990). In spectral estimation, a fundamental limitation has been developed in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004). Specifically, an integral constraint on the relative variance was established for the parametric estimation of a signal spectrum. This result was used to demonstrate the 'water-bed' effect in spectral estimation (Stoica et al. 2004).

Chapter 4 establishes fundamental limitations on the variance of estimated parametric models by generalising the result in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004). Specifically we show the relationship between the results presented in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004), which obtain a lower bound on the variance of parametric spectral estimators, with the new results obtained in this chapter. Furthermore the subtle differences between both sets of results are shown to be due to the use of different covariance expressions. Limitations are obtained for both open and closed loop identification. With respect to closed loop identification, both direct and indirect methods are considered. For the case of direct identification, bounds are established in lieu of an exact expression.

As an application of the novel fundamental limitation results, it is shown that for multisine inputs, a well known asymptotic (in model order) variance expression (Ljung 1985) corresponds to an upper bound on the actual variance of the estimated models of finite order.

1.3.4 Experiment Design Considering Models of Finite Order

Based on the fundamental limitations derived in Chapter 4, we then return to the problem of robust experiment design. In particular, a closed form expression is derived for the input spectrum which minimises the maximum value of a weighted integral on the variance of the frequency response estimator, over all model structures with a given number of parameters. The idea utilises fundamental limitations on the variance as constraints to reformulate the optimisation as a simple constrained variational problem.

Alternative approaches to the problem of designing robust input signals appear in (Hjalmarsson et al. 2006, Mårtensson 2007, Mårtensson and Hjalmarsson 2006; 2007).

The problem studied in Chapter 3 involved finding a class of cost functions, dependant on the relative variance of the frequency response estimator, that give an optimal input independent of the true system and noise dynamics. In Chapter 5 this is reconsidered in the light of the new approach to robust experiment design detailed above. Specifically, the results obtained in Chapter 3, based on an asymptotic (in model order) variance expression (Ljung 1985), are shown to be valid also for finite order models.

We also revise Yuan & Ljung's unprejudiced optimal input design (Yuan and Ljung 1985), where the effect of both bias and variance errors in experiment design is considered. Utilising the fundamental limitations of Chapter 4 we study their approach, based on an asymptotic (in model order) variance

expression, and show how to derive an unprejudiced optimal input for finite order models.

1.3.5 Least Costly and Traditional Experiment Design for Control

A primary motivating factor for system identification and hence experiment design, is to obtain accurate system models for use in one of the many model based control system design methodologies. To this end there has been a substantial amount of recent work aimed specifically at investigating experiment design for control (Gevers and Bombois 2006, Hildebrand and Gevers 2003b, Hjalmarsson 2005, Jansson 2004).

In the more recent research on experiment design for control, a new paradigm has been proposed, namely Least Costly Identification Experiments for Control (Bombois et al. 2004a;b; 2006). The objective of this paradigm is to design an experiment based on a robust control performance specification with respect to quality constraints on the estimated model and which also incurs the smallest possible cost (Bombois et al. 2006). The cost considered is generally associated with the input and/or output power, or the duration of the experiment. This least costly experiment design formulation can be contrasted with the traditional approach where one determines the optimal experiment that minimises, for example, a control oriented measure of the model accuracy, subject to input and/or output power constraints.

Least costly open loop experiment design for control (Bombois et al. 2004a) seeks to minimise the input power under the constraint that the controller, where the design is based on the identified model, is guaranteed to stabilise and achieve a desired $\mathscr{H}_\infty$ performance on the true system. This is accomplished in a two step procedure. The first step determines the size of the identified model uncertainty that can be tolerated which, when used for the design of a controller, will satisfy $\mathscr{H}_\infty$ performance specifications. The least powerful input signal is then designed in a second step such that the identified model uncertainty is less than that determined in the first step.

In closed loop (Bombois et al. 2006), the cheapest experiment is considered to be that which minimises the impact of the perturbation as observed on either the input or output of the system. A methodology, analogous to that used in the least costly open loop case, is then used for the experiment design.

The results of Bombois et al. (2004a;b; 2006) are based on variance errors only (although the least costly paradigm has been recently extended to also include bias errors (Bombois and Gilson 2006)). Early results (Bombois et al. 2004b) use variance expressions which are asymptotic in the model order

and data length, whereas the recent results (Bombois et al. 2004a;b; 2006) are based on more accurate parameter covariance expressions which are non-asymptotic in the model order (Ljung 1999).

The identification procedure considered in the least costly approach to experiment design is the Prediction Error Method (PEM). In the closed loop case the results have been specifically developed for the direct identification method.

In Chapter 6 we explore the relationship between two approaches to experiment design which differ in the way the input/output power is considered as part of the optimisation. We will say that a given experiment design formulation is in the *traditional* framework if the input/output power is included as a constraint in the optimisation problem, such that the purpose of the optimal experiment is to maximise a given quantity related to the model quality, i.e. a function of the parameter covariance matrix under an input/output power constraint. On the other hand, a *least costly* experiment design formulation is defined as an optimisation problem where the input/output power is minimised subject to a model quality constraint, given as a function of the parameter covariance matrix.

In Bombois et al. (2006) the least costly paradigm is stated to be a 'dual approach' to the traditional optimal experiment design problem. The results presented in Chapter 6 establish the equivalence between the two paradigms and hence show that they are indeed dual problems. Specifically, we show equivalence between the traditional optimal experiment design problem and the results for the least costly approach for both open and closed loop systems. Since there is no unification between the open and closed loop cases with respect to a measure of the model quality, we establish equivalence for each of the cases previously analysed in the least costly framework (Bombois et al. 2004a;b; 2006).

The results in Chapter 6 also show that solutions of several experiment design problems in the least costly framework are equivalent to scaled versions of solutions to corresponding traditional experiment design problems. This implies that it is possible to make use of computationally efficient algorithms developed for one framework in the other framework. In particular, during the last few years, very efficient LMI formulations have been developed independently for both frameworks (Bombois et al. 2006, Jansson 2004, Jansson and Hjalmarsson 2005a;b). The results we obtain can then be used to translate these formulations between the two different approaches.

Additionally, the equivalence results allow the incorporation of additional constraints into the least costly framework. They also allow the results obtained in one framework to be interpreted in the other. For example, there are usually hard constraints on the input power or amplitude, due to actuator

limitations. Thus, if the power of the optimal least costly input (where the cost has been measured in terms of the input power) exceeds the maximum allowed value, the equivalence results show that this is due to an excessively tight constraint on the model quality. Thus, by translating the problem into the traditional framework and then reverting back to the least costly framework, it is possible to modify this constraint in an appropriate way, in order to satisfy the hard input power constraint.

1.3.6 The Cost of Complexity

A key factor allowing system identification to work in practice is the nature of the input signal, and hence experiment design. It has been noted that experiment design can emphasise system properties of interest, while properties of little or no interest can be 'hidden' (Hjalmarsson 2005, Hjalmarsson et al. 2006). As remarked in Hjalmarsson et al. (2006), some properties can be more easily estimated than others, in the sense that the amount of input power required to estimate them with a given level of accuracy does not depend on the complexity of the system. For example, it has been shown that the cost of estimating the transfer function at a particular frequency, or one non-minimum phase zero, is independent of the model order (Hjalmarsson et al. 2006). However, some properties do depend on the model order.

Chapter 7 can be considered as an extension to the study of these observations. Here we investigate the minimum amount of input power required to estimate a given linear system with a prescribed degree of accuracy, as a function of the model complexity. This quantity is defined to be the 'cost of complexity'. The degree of accuracy considered is the maximum variance of the discrete-time transfer function estimator over a frequency range $[-\omega_B, \omega_B]$. For simplicity, we restrict the model class and consider only Finite Impulse Response (FIR) models. Furthermore, we assume that there is no undermodelling, i.e., the true system belongs to the model structure.

The contributions in Chapter 7 consist of establishing several key properties for the dependence of the cost on the model complexity. Some of these seem self-evident, but others are quite unexpected. For example, if ω_B is very close (but not necessarily equal) to π, we show that the optimal input actually satisfies the model quality constraint for all frequencies!

In Chapter 8 we extend several of the results derived in Chapter 7 to more general model structures such as Output Error (OE), fixed denominator and Laguerre models.

We believe that these results provide a better understanding of the relationship between the amount of information that we ask to be extracted from a system, and the sensitivity of the cost of the identi-

fication with respect to the model complexity. This appears to be a key point for understanding why system identification can work successfully for very complex systems.

In order to study the cost of complexity, we employ a semidefinite optimisation approach (Bombois et al. 2006, Hildebrand and Gevers 2003b, Jansson and Hjalmarsson 2005a). In particular, the input design problem is formulated in terms of *Linear Matrix Inequalities* (LMIs), which reduces to studying the positivity of a specific Toeplitz matrix.

It is worth noting that G. Zames and collaborators also studied the cost of identifying a linear system (Zames 1979, Zames and Owen 1993), however they pursued a different approach, using concepts such as ε-entropy and ε-dimension, from Kolmogorov's theory of complexity.

1.3.7 An Algorithm to Generate Binary Signals

In many fields, the problem of generating a signal with specified second order properties arises, see for example (Cule and Torquato 1999, Gujar and Kavanagh 1968, Koutsourelakis and Deodatis 2005, Liu and Munson 1982, Sheehan and Torquato 2001, Yeong and Torquato 1998a;b). For instance, in experiment design (Goodwin and Payne 1977, Ljung 1999) one typically obtains an optimal test signal specified in terms of its spectral properties. This leads to the problem of implementing a real signal with a specified spectrum. Moreover, it is usual that the input should also be constrained in its amplitude, i.e. the amplitude must lie in an interval $[a,b] \subset \mathbb{R}$. In general, frequency domain techniques do not work properly with this kind of constraint, and as such are translated into an 'equivalent' power constraint under which the input is designed to satisfy the conditions.

Implementing an input signal which, within the constraints of its amplitude, has maximum power is important in many applications. This is the case for example, in experiment design, where the quality of the estimation typically increases with the signal to noise ratio. The signal to noise ratio is obviously improved by choosing an input with high power. Binary signals have precisely this desirable property: their power is maximum for a given amplitude constraint (Tan and Godfrey 2001). This then motivates the question of how to design a binary signal with a given autocovariance satisfying input amplitude constraints whilst maximising power.

Several techniques have been proposed to solve this problem (see e.g. (Cule and Torquato 1999, Gujar and Kavanagh 1968, Koutsourelakis and Deodatis 2005, Liu and Munson 1982, Sheehan and Torquato 2001, Yeong and Torquato 1998a;b) and the references therein). For example, (Gujar and Kavanagh 1968) and (Liu and Munson 1982) consider a scheme consisting of a linear system followed by a static

nonlinearity. The nonlinear block, in this case, is used to force the output signal to be binary, and the linear system is tuned to produce an output signal with the desired autocovariance. However, it can be shown that this method has severe limitations, e.g. it cannot be used to generate binary signals with a bandlimited spectrum (Liu and Munson 1982, Wise et al. 1977). A similar procedure consisting of a linear system followed by a level crossing block is developed in (Koutsourelakis and Deodatis 2005). A simulated annealing method is proposed in (Yeong and Torquato 1998a) and (Yeong and Torquato 1998b). The methods outlined above generally involve complex calculations and are computationally intensive.

In Chapter 9 we develop a simple procedure to solve the above problem, based on the use of the Receding Horizon concept commonly employed in Model Predictive Control (Goodwin et al. 2001). Heuristically speaking, the idea is to solve, for each time instant, a finite horizon optimisation problem to find the optimal set of the next, say, T values of the sequence such that the sampled autocovariance sequence so obtained is as close as possible (in a prescribed sense) to the desired autocovariance. One then takes the first term of this optimal set for the sequence, advances time by one step and repeats the procedure. The idea behind this procedure is thus closely related to finite alphabet receding horizon control (Quevedo et al. 2003, Goodwin et al. 2005), where receding horizon concepts are employed to control a linear system whose input is restricted to belong to a finite set.

Notice that in order to find the true optimal binary sequence of length N, we would have to compute the sample autocovariance function of all sequences in $\{0,1\}^N$ and then choose the sequence whose autocovariance is closest to the desired one according to some prescribed norm. This procedure, however, would be computationally intractable as it involves 2^N comparisons, a truly large number in general.

Several kinds of measures can be used to compare the sampled autocovariance of the generated signal with the desired autocovariance, including the Euclidean or the infinity norm of their difference. However, we have verified via simulations that the Euclidean norm produces very good results when compared to other norms. Furthermore, the algorithm is shown to converge for a special case when the Euclidean norm is used.

The proposed algorithm is fast and easy to implement when compared with the existing methods, and can also be run in realtime. This allows the possibility of implementing adaptive input generation schemes, which can be useful when the signal properties must change with time, as in sequential experiment design procedures (Walter and Pronzato 1997).

To demonstrate the application of the algorithm, two examples, motivated by experiment design, are provided. A typical input signal used in system identification is bandlimited white noise (Ljung 1999). We show how the proposed algorithm can be used to generate this type of signal. We also provide the obtained spectrum to highlight how close it approximates the desired spectrum. The second example is inspired by Chapters 2 and 3 where it is shown that a more robust input is in fact one with a bandlimited '$1/f$' spectrum. We again provide the spectrum generated by the receding horizon algorithm as well as that of the prescribed signal, for the purpose of comparison.

1.4 Key Contributions

The work presented in Chapter 2 of this Thesis appears across several publications of the author's. Part of it has been published in Automatica (Rojas et al. 2007c) and as a chapter of the book 'Forever Ljung in System Identification' (Goodwin et al. 2006). Components have also been published in the proceedings of the IFAC Symposium on Dynamics and Control of Process Systems (Goodwin et al. 2007a), and submitted for publication to the Journal of Process Control (Goodwin et al. 2007b). The key contributions in this chapter are:

- Detailed derivation of the theoretical properties of the robust optimal experiment for two one-parameter examples (i.e., a first order system and a second order resonant system).
- Algorithmic aspects on the numerical determination of the robust optimal experiment for these examples.
- Study of the robust optimal experiment for multi-parameter cases.
- Study of the use of bandlimited '$1/f$' noise as an input signal and a comparison to the robust optimal experiments.

The results in Chapter 3 have been published by the author in the proceedings of the European Control Conference ECC'07 (Rojas et al. 2007a). The key contributions of this chapter are:

- Development of a general class of cost functions in experiment design for which the optimal solution is independent of the knowledge of the system and noise.
- The use of bandlimited '$1/f$' noise as a good input signal for experiment design when there is diffuse prior information.

The results from Chapters 2 and 3 have also been presented at the 2007 SIAM Conference on Computational Science and Engineering (CSE07), in Costa Mesa, California, on February 2007.

The results from Chapter 4 have been submitted for publication to the IEEE Transactions on Automatic Control (Rojas et al. 2008f). The key contributions of this chapter are:

- The derivation of several fundamental limitations on the variance of parametric models for open and closed loop situations.
- A proof showing that a well known asymptotic (in model order) variance expression provides an upper bound on the variance of parametric models of finite order.

The work developed in Chapter 5 will be submitted as a journal paper to Automatica (Rojas et al. 2008a). The main contributions of this chapter are:

- A closed form solution for a class of robust experiment design problems based on non-asymptotic variance expressions.
- Extension of the results of Chapter 3, on experiment design with diffuse prior information, to models of finite order.
- Derivation of an unprejudiced optimal input (in Yuan and Ljung's sense) for models of finite order.

The work presented in Chapter 6 have been accepted for publication in Automatica (Rojas et al. 2008b). The main contribution of this chapter is to establish the equivalence between a set of least costly experiment design formulations and traditional experiment design problems.

The research presented in Chapters 7 and 8 resulted from collaboration with Märta Barenthin and Professor Håkan Hjalmarsson, from the School of Electrical Engineering, KTH, in Stockholm, Sweden. Parts of it have been accepted for publication at the 17th IFAC World Congress (Rojas et al. 2008e). A submission has also been made to the IEEE Transactions on Automatic Control (Rojas et al. 2008c) for the FIR model case. The Output Error case will be submitted to Automatica (Rojas et al. 2008d). The key contributions in these chapters are:

- Determination of the theoretical properties of the cost, as a function of the model complexity, bandwidth of interest, noise variance and required precision.

- An LMI formulation of the problem, suitable for the numerical computation of the cost.

The work presented in Chapter 9 has been published in the Proceedings of the American Control Conference (Rojas et al. 2007b). The main contribution of this chapter is an algorithm based on the receding horizon concept to generate binary signals with a prescribed autocovariance and a proof of its convergence for the special case of generating pseudo white noise.

1.5 Publications

The following publications are a direct result of the research embodied in this Thesis:

- G. C. Goodwin, C. R. Rojas, and J. S. Welsh. Good, bad and optimal experiments for identification. In T. Glad, editor, *Forever Ljung in System Identification – Workshop on the ocassion of Lennart Ljung's 60th birthday*. September 2006.

- C. R. Rojas, J. S. Welsh, G. C. Goodwin, and A. Feuer. Robust optimal experiment design for system identification. *Automatica*, 43(6): 993–1008, 2007.

- C. R. Rojas, G. C. Goodwin, J. S. Welsh, and A. Feuer. Optimal experiment design with diffuse prior information. In *Proceedings of the European Control Conference (ECC)*, pages 935–940, Kos, Greece, July 2007.

- C. R. Rojas, J. S. Welsh, and G. C. Goodwin. A receding horizon algorithm to generate binary signals with a prescribed autocovariance. In *Proceedings of the 2007 American Control Conference (ACC)*, pages 122–127, New York, July 2007.

- G. C. Goodwin, J. C. Agüero, J. S. Welsh, G. J. Adams, J. I. Yuz, and C. R. Rojas. Robust identification of process models from plant data. In *Proceedings of the 8th IFAC Symposium on Dynamics and Control of Process Systems (DYCOPS)*, pages 1–18, Cancún, Mexico, 2007.

- C. R. Rojas, M. Barenthin, J. S. Welsh, and H. Hjalmarsson. The cost of complexity in identification of FIR systems. In *Proceedings of the 17th IFAC World Congress*, Seoul, South Korea, July 2008.

- C. R. Rojas, J. C. Agüero, J. S. Welsh, and G. C. Goodwin. On the equivalence of least costly and traditional experiment design for control. *Automatica (accepted for publication)*, 2008.

- G. C. Goodwin, J. C. Agüero, J. S. Welsh, G. J. Adams, J. I. Yuz, and C. R. Rojas. Robust identification of process models from plant data. *Journal of Process Control (accepted for publication*, 2008.

The following papers have been submitted based on the work in the Thesis:

- C. R. Rojas, M. Barenthin, J. S. Welsh, and H. Hjalmarsson. The cost of complexity in system identification: The FIR case. *IEEE Transactions on Automatic Control (submitted for publication)*, 2008.
- C. R. Rojas, J. S. Welsh and J. C. Agüero. Fundamental Limitations on the Variance of Estimated Parametric Models. *IEEE Transactions on Automatic Control (submitted for publication)*, 2008.

The following papers are currently in draft form:

- C. R. Rojas, M. Barenthin, J. S. Welsh, and H. Hjalmarsson. The cost of complexity in system identification: The OE case. *Automatica (in preparation)*, 2008.
- C. R. Rojas, J. C. Agüero and J. S. Welsh. Robustness in Experiment Design. *Automatica (in preparation)*, 2008.

Other publications written during the PhD candidature, which are not directly related to the main research theme described in the Thesis include:

- C. R. Rojas, G. C. Goodwin, M. M. Seron, and M. Zhang. Open-cut mine planning *via* closed loop receding-horizon optimal control. In R. S. Sánchez-Peña, J. Quevedo and V. Puig, editors, *Identification and Control: The Gap between Theory and Practice*, pages 43–62. Springer, London, 2007.
- J. S. Welsh and C. R. Rojas. Frequency localising basis functions for wide-band system identification: A condition number bound for output error systems. In *Proceedings of the European Control Conference (ECC)*, pages 4618–4624, Kos, Greece, July 2007.
- J. S. Welsh, C. R. Rojas, and S. D. Mitchell. Wideband parametric identification of a power transformer. In *Proceedings of the Australian Universities Power Engineering Conference (AUPEC)*, Perth, Australia, December 2007.

Notation

To aid the readability of the Thesis, a table of notation is provided in Appendix A.

CHAPTER 2

MIN-MAX EXPERIMENT DESIGN

2.1 Introduction

In this chapter we develop a min-max approach to robust experiment design for dynamic system identification. Although min-max experiment design has been explored in the statistics literature, the technique is virtually unknown to the engineering community, hence there has been little prior work on examining its properties when applied to dynamic system identification. Connections between our approach and continuous game theory (Başar and Bernhard 1995, Başar and Olsder 1995, Fudenberg and Tirole 1991, Owen 1995, Szép and Forgó 1985) are explored, providing a useful framework for dealing with min-max problems. Specifically we consider linear systems with energy (or power) bounded inputs, and assume that the parameters lie in a given compact set. The min-max approach to robust experiment design is then formulated so as to obtain an optimal input for the worst case over this set.

To study this robust experiment design approach we provide a detailed analysis of the solution for an illustrative one parameter example as well as a second order resonant system. Furthermore we propose a convex optimisation algorithm that can be applied more generally to a discretised approximation to the design problems. Several properties of the robust optimal input are established for these examples, namely existence and uniqueness, and the fact that the spectra have finite support. We also describe the extension to multi-parameter systems. The effect of different design criteria is examined, and some simulation examples are presented to illustrate the merits of the proposed approach.

2.2 Experiment Design Criteria

2.2.1 The Information Matrix

An intuitive way to compare different experiments is to choose a measure related to the expected accuracy of the parameter estimator of the model to be obtained from the data collected. However, the accuracy of the parameter estimator is a function of both the experimental conditions and the form of the estimator. Since we would prefer to have an 'estimator-independent' measure, we may assume that the estimator used is statistically efficient in the sense that the parameter covariance

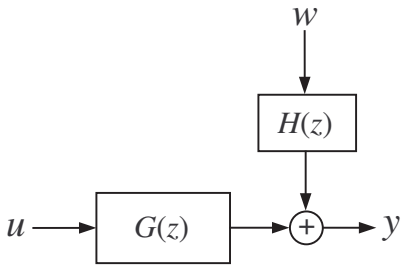

Figure 2.1. Block diagram describing the relationship between the input u, the noise w and the output y of the system G to be identified.

matrix achieves the Cramér-Rao lower bound (Goodwin and Payne 1977), i.e.

$$\operatorname{cov} \hat{\theta} = M^{-1},$$

where M is the Fisher's information matrix (Casella and Berger 2002, Silvey 1970). Note that estimators are denoted by a superscript '$\hat{}$' and implicitly depend on the data length, N. Therefore, our first step is to determine an expression for M.

To be specific, consider a single-input single-output (SISO) linear discrete time system, with input $\{u_t\}$ and output $\{y_t\}$, of the form

$$y_t = G(z)u_t + H(z)w_t$$

where G and H are stable rational transfer functions, z is the forward shift operator, H is minimum phase with $H(\infty) = 1$, and $\{w_t\}$ is zero mean Gaussian white noise of variance σ^2. We assume that the system is working in open loop, hence $\{u_t\}$ and $\{w_t\}$ are independent. We let $\theta := [\rho^T \ \eta^T \ \sigma^2]^T$ where ρ denotes the parameters in G and η denotes the parameters in H. Therefore, we assume that G, H and σ^2 are independently parameterised. Figure 2.1 shows the relationship between the input u, the noise w and the output y of the system.

The log likelihood function (Goodwin and Payne 1977) for data $Y := \{y_t\}_{t=1}^N$ given parameters θ, is

$$\ln p(Y|\theta) = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{N}\varepsilon_t^2 \tag{2.1}$$

where

$$\varepsilon_t := H(z)^{-1}[y_t - G(z)u_t]. \tag{2.2}$$

Fisher's information matrix is obtained by taking the following expectation (Goodwin and Payne

1977)

$$M := \mathrm{E}_{Y|\theta}\left[\left(\frac{\partial \ln p(Y|\theta)}{\partial \theta}\right)\left(\frac{\partial \ln p(Y|\theta)}{\partial \theta}\right)^T\right] \tag{2.3}$$

where, from (2.1)

$$\frac{\partial \ln p(Y|\theta)}{\partial \theta} = -\frac{1}{\sigma^2}\sum_{t=1}^{N}\varepsilon_t\frac{\partial \varepsilon_t}{\partial \theta} - \frac{1}{2\sigma^2}\frac{\partial \sigma^2}{\partial \theta}\left[N - \frac{1}{\sigma^2}\sum_{t=1}^{N}\varepsilon_t^2\right],$$

from (2.2)

$$\frac{\partial \varepsilon_t}{\partial \theta} = -H(z)^{-1}\left\{\frac{\partial H(z)}{\partial \theta}\varepsilon_t + \frac{\partial G(z)}{\partial \theta}u_t\right\}$$

and $\mathrm{E}_{Y|\theta}$ denotes the expectation over the distribution of the data given θ.

Taking expectations, as in (2.3), M can be partitioned as

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where M_1 is the part of the information matrix related to ρ, and M_2 is independent of the input. Thus,

$$M_1 := \frac{1}{\sigma^2}\sum_{t=1}^{N}\left(\frac{\partial \varepsilon_t}{\partial \rho}\right)\left(\frac{\partial \varepsilon_t}{\partial \rho}\right)^T \tag{2.4}$$

where $\partial \varepsilon_t/\partial \rho$ satisfies

$$\frac{\partial \varepsilon_t}{\partial \rho} = -H(z)^{-1}\frac{\partial G(z)}{\partial \rho}u_t\,.$$

Notice that M_1 depends on the full parameter vector θ. Assuming N is large, it is more convenient to work with the scaled average information matrix for the parameters ρ, (Goodwin and Payne 1977, Walter and Pronzato 1997),

$$\overline{M}(\theta, \Phi_u) := \lim_{N\to\infty}\frac{1}{N}M_1\sigma^2\,.$$

Utilising Parseval's Theorem,

$$\overline{M}(\theta, \Phi_u) = \frac{1}{\pi}\int_0^{\pi}\widetilde{M}(\theta, \omega)\Phi_u(e^{j\omega})d\omega \tag{2.5}$$

where

$$\widetilde{M}(\theta, \omega) := \mathrm{Re}\left\{\frac{\partial G(e^{j\omega})}{\partial \rho}\left|H(e^{j\omega})\right|^{-2}\left[\frac{\partial G(e^{j\omega})}{\partial \rho}\right]^H\right\}$$

and Φ_u is the discrete time input spectral density (considered as a generalised function). Here, $(\,\cdot\,)^H$ is the conjugate transpose operator.

It is also possible to do a parallel development (Goodwin and Payne 1977) for continuous time models. In the latter case, (2.5) is replaced by

$$\overline{M}(\theta, \Phi_u) = \int_0^{\infty}\widetilde{M}(\theta, \omega)\Phi_u(\omega)d\omega \tag{2.6}$$

where

$$\widetilde{M}(\theta,\omega) := \mathrm{Re}\left\{\frac{\partial G(j\omega)}{\partial \rho}\,|H(j\omega)|^{-2}\left[\frac{\partial G(j\omega)}{\partial \rho}\right]^{H}\right\},$$

G and H are continuous time transfer functions (assumed independently parameterised) and Φ_u is the continuous time input spectral density. In the remainder of the chapter we will consider only continuous time models.

Notice that the results do not depend on σ^2 since it appears as a scaling factor in (2.4). Also, we see from (2.6) that, in $\overline{M}(\theta,\Phi_u)$, H is essentially a frequency dependent weighting, which can be easily included in the analysis if it were known. However, for simplicity we assume white noise. Hence in the sequel we refer to θ as containing only ρ.

2.2.2 Brief Review of Design Criteria for Nominal Experiment Design

Since $\overline{M}$ is a matrix, we need a scalar measure of $\overline{M}$ for the purpose of experiment design. In the nominal case, typically treated in the engineering literature (i.e. when a fixed prior estimate of θ is used), several measures of the 'size' of $\overline{M}$ have been proposed which measure the 'goodness' of the experiment. Some examples are:

(i) D - optimality (Goodwin and Payne 1977)

$$J_d(\theta,\Phi_u) := [\det \overline{M}(\theta,\Phi_u)]^{-1}. \tag{2.7}$$

(ii) Experiment design for robust control (Hildebrand and Gevers 2003a;b, Hjalmarsson 2005).

$$J_{rc}(\theta,\Phi_u) := \sup_{\omega} g(\theta,\omega)^H \overline{M}^{-1} g(\theta,\omega) \tag{2.8}$$

where g is a frequency dependent vector related to the ν-gap (Hildebrand and Gevers 2003a;b).

Many other criteria have been described in the statistics literature, such as A-optimality ($\mathrm{tr}\,\overline{M}(\theta,\Phi_u)^{-1}$), L-optimality ($\mathrm{tr}\,W\overline{M}(\theta,\Phi_u)^{-1}$, for some $W \geq 0$) and E-optimality $\left(\lambda_{\max}(\overline{M}(\theta,\Phi_u)^{-1})\right)$; see (Kiefer 1974). On the other hand, in the engineering literature, (Bombois et al. 2005a) proposed a criterion that leads to the required accuracy to achieve a given level of robust control performance. Other criteria will be discussed in Section 2.5.

A common feature of all these nominal experiment design approaches is that they are aimed at choosing Φ_u to minimise a function of the type shown in (2.7) and (2.8). Notice, however, that the optimal input spectrum depends, inter-alia, on the unknown parameter vector θ.

2.2.3 Min-Max Robust Design

A min-max robust design criterion is the basis of our experiment design technique. Specifically, we assume that a-priori information is available indicating that the parameters can take any value in a compact set Θ. We also constrain the allowable set of input signals. Typically in experiment design a constraint is imposed on input energy (Goodwin and Payne 1977, Walter and Pronzato 1997, Zarrop 1979). Here we define the constraint as[1]

$$\mathscr{S}(\mathbb{R}_0^+) := \left\{ \Phi_u : \mathbb{R} \to \mathbb{R}_0^+ : \ \Phi_u \text{ is even and } \int_{-\infty}^{\infty} \Phi_u(\omega) d\omega = 1 \right\}.$$

The min-max robust optimal input spectral density, Φ_u^{opt}, is then chosen as

$$\Phi_u^{opt} = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \sup_{\theta \in \Theta} J(\overline{M}(\theta, \Phi_u), \theta) \tag{2.9}$$

where J is an appropriate scalar measure of $\overline{M}$. We are assuming for the moment that Φ_u^{opt} exists and is unique; these properties are studied in the following section. Notice also that we allow J to depend explicitly on θ, this point will be of practical importance and is discussed below.

2.2.4 A Mixed Policy Game Approach

An alternative approach to that described above would be to extend the space to include 'mixed policies' (Başar and Bernhard 1995) by introducing a (generalised) probability density ξ on Θ, i.e. $\xi \in \mathscr{S}(\Theta)$. The counterpart of (2.9) is then:

$$\Phi_u^{opt} = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \sup_{\xi \in \mathscr{S}(\Theta)} J'(\xi, \Phi_u), \tag{2.10}$$

where J' is an appropriate scalar measure of the form:

$$J'(\xi, \Phi_u) := \int f \left(\int S_\theta \widetilde{M}(\theta, \omega) S_\theta^T \Phi_u(\omega) d\omega \right) \xi(\theta) d\theta,$$

and f is a scalar-valued function, e.g. $f(L) = \operatorname{tr} L^{-1}$ or $f(L) = \lambda_{\max}(L^{-1})$, $\widetilde{M}$ is the single frequency information matrix and S_θ is a parameter dependent scaling matrix (see Section 2.5.1 for further discussion of S_θ).

Note that if f were linear, it could be introduced into the inner integral, in which case it can be shown that (2.10) is equivalent to (2.9) (see the proof of Theorem 2.3.1 for how such equivalence can be established).

[1] In general, given a set $X \subseteq \mathbb{R}_0$, we will denote by $\mathscr{S}(X)$ the set of all even generalised functions Φ_u on $\mathbb{R}$ (Rudin 1973) such that Φ_u is the derivative of some probability distribution function on $\mathbb{R}$, and $\operatorname{supp} \Phi_u \subseteq X \cup (-X)$, where $\operatorname{supp} \Phi_u$ is the support of Φ_u (i.e. roughly speaking, $\mathscr{S}(X)$ is the set of all even (generalised) probability density functions on $X \cup (-X)$).

2.3 An Illustrative First Order Example

Before considering the general multi-parameter case we first study, in detail, an illustrative continuous time one parameter problem to gain insight. We take $H(s) = 1$ and let

$$G(s) = \frac{1}{s/\theta + 1}. \tag{2.11}$$

Note that this problem has been discussed in (Goodwin and Payne 1977, page 142) for the case of nominal experiment design, and also in (Walter and Pronzato 1997, page 339) in the context of min-max robust experiment design. Note however that the latter analysis uses the restrictive assumption that the input is a single sinusoid. Actually we will see below that the later restriction unduely limits the solution space and does not lead to the optimal strategy when $\theta \in [\underline{\theta},\, \overline{\theta}]$, with $\overline{\theta}/\underline{\theta} > 2+\sqrt{3}$ (see Appendix 2.7). This is heuristically reasonable since if θ lies in an interval, then it makes sense to spread the input energy to cover all possible values of θ.

For the example (2.11), the scaled average information matrix is

$$\overline{M}(\theta, \Phi_u) = \int_0^\infty \widetilde{M}(\theta, \omega)\Phi_u(\omega)d\omega$$

where $\widetilde{M}$ is the 'single frequency' normalised information matrix given by

$$\widetilde{M}(\theta, \omega) = \left|\frac{\partial G(\omega)}{\partial \theta}\right|^2 = \frac{\omega^2/\theta^4}{(\omega^2/\theta^2+1)^2}\,. \tag{2.12}$$

2.3.1 Nominal Optimal Experiment Design

To place the robust design problem in context, we briefly review the nominal (or *locally optimal* (Chernoff 1953, Fedorov 1972, Ford et al. 1989, Pronzato 2008)) experiment design problem for the first order example. Here one assumes that a prior estimate, $\widehat{\theta}$, of θ is available. Based on this information, the function Φ_u is chosen to optimise some scalar-valued function of $\overline{M}(\widehat{\theta}, \Phi_u)$ subject to a constraint on the input power. In the nominal case it can be shown that we only need to use a single frequency input for the first order example (Goodwin and Payne 1977, page 143), namely, $\Phi_u(\omega) = \delta(\omega - \omega^{opt})$. Moreover, by differentiation of the single frequency information matrix given in (2.12), it is readily seen that the optimal input frequency is

$$\omega^{opt} = \theta. \tag{2.13}$$

This is an intuitively pleasing result, i.e. one places the test signal at the (nominal) 3dB break point. However, equation (2.13) reinforces the fundamental difficulty in nominal experiment design, namely, the optimal experiment depends on the very thing that the experiment is aimed at estimating.

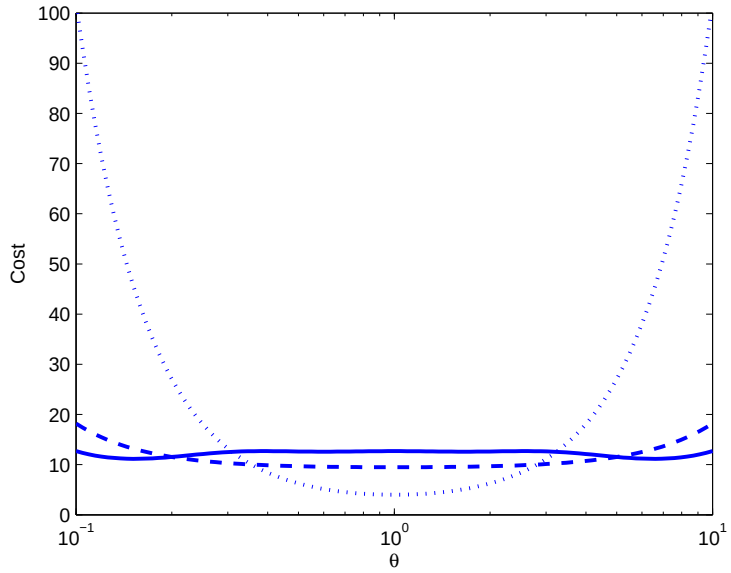

Figure 2.2. Cost $[\theta^2\overline{M}(\theta,\Phi_u)]^{-1}$ as a function of θ for nominal input (dotted), robust optimal input (solid) and bandlimited '$1/f$' noise (dashed).

To gauge the importance of the dependence on θ, we notice that $\widetilde{M}(\theta,\omega)$ in our example decays at the rate of 40dB per decade as a function of both θ and ω. Furthermore, given the prior estimate of the parameter, $\widehat{\theta}$, it makes sense to choose $\omega^{opt}=\widehat{\theta}$ for the input signal frequency. Also, assume that the true parameter lies in the range $(0.1\,\widehat{\theta}\leq\theta\leq 10\,\widehat{\theta})$, then $\min_{\theta\in\Theta}\widetilde{M}(\theta,\omega)$ is approximately $1/100^{th}$ of the nominal value! This suggests that nominal experiment design is limited to those cases where an extremely good prior estimate is available. This point is reinforced in Figure 2.2 which shows a plot of the cost $[\theta^2\overline{M}(\theta,\Phi_u)]^{-1}$, versus θ for the nominal optimal input.

Remark 2.3.1 *The reason for multiplying $\overline{M}$ by θ^2, as in Figure 2.2, then inverting is that $\overline{M}^{-1}$ is a variance measure and thus $[\theta^2\overline{M}]^{-1}$ gives relative (mean square) errors.*

An alternative way to normalise a nominal cost function in order to develop a min-max cost function is to consider an efficiency measure *$E(\theta,\Phi_u)$ of the form $E(\theta,\Phi_u)=J(\bar{M}(\theta,\Phi_u),\theta)/J(\bar{M}(\theta,\Phi_u^{opt}(\theta)),\theta)$, where $\Phi_u^{opt}(\theta)$ is the nominal optimal input spectrum for θ (i.e. for the cost function $J(\bar{M}(\theta,\Phi_u),\theta)$). This efficiency measure allows a comparison between an input spectrum Φ_u at a given θ and the optimal achievable experiment for that θ (Ford et al. 1989, Silvey 1980, Walter and Pronzato 1990; 1997). Interestingly, the cost function considered here, i.e. the relative variance of θ, turns out to be equivalent to the MMDE cost function used in (Walter and Pronzato 1997, page 339), which is an efficiency measure based on the D-optimality criterion.*

2.3.2 Robust Optimal Experiment Design

Next we examine robust experiment design as described in Section 2.2 and establish several properties for the input, namely its existence, uniqueness, and the fact that the spectrum has finite support. For the first order example, we use (see Remark 2.3.1 and Section 2.5)

$$J(\overline{M}(\theta,\Phi_u),\theta) := [\theta^2\,\overline{M}(\theta,\Phi_u)]^{-1}.$$

Thus, our min-max robust optimal experiment design can be stated as

$$\Phi_u^{opt} = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \overline{J}(\Phi_u) \tag{2.14}$$

where

$$\overline{J}(\Phi_u) := \sup_{\theta\in\Theta} \left[\int_0^\infty \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\,\Phi_u(\omega)\,d\omega \right]^{-1} \tag{2.15}$$

and $\Theta := [\underline{\theta},\ \overline{\theta}]$. In the sequel, we will give further insights into the above design problem.

We first observe that, since $\theta^2\widetilde{M}(\theta,\omega)$ in (2.15) is continuous in $\theta\in\Theta$ for every $\omega\in\mathbb{R}_0^+$ and it is bounded by an integrable function which is independent of θ (use e.g. C/ω^2, where C is large and independent of θ), the integral is continuous in θ (Bartle 1966, Corollary 5.8). This implies, with the compactness of Θ, that we can replace 'sup' in (2.15) with 'max'.

Furthermore, if we make the following change of variables

$$\begin{aligned} x &:= \frac{\ln\theta - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}} \\ y &:= \frac{\ln\omega - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}} \\ \Phi_u(\omega) &= \frac{2}{k\omega}\,\widetilde{\Phi}_u\left(\frac{\ln\omega - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}}\right) \\ k &:= 2(\ln\overline{\theta} - \ln\underline{\theta}) \end{aligned} \tag{2.16}$$

then the problem can be rewritten as

$$\widetilde{\Phi}_u^{opt} = \arg \max_{\widetilde{\Phi}_u\in\mathscr{S}(\mathbb{R})} \min_{x\in[0,1]} \int_{-\infty}^{\infty} \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2}\,\widetilde{\Phi}_u(y)dy. \tag{2.17}$$

To simplify the notation, let $F(x,y) := f(x-y)$, where $f(u) := e^{ku}/(e^{ku}+1)^2$.

In the following theorems we develop several properties of Φ_u^{opt} and $\widetilde{\Phi}_u^{opt}$.

First we note that some of the results below are based on the fact that if h is a continuous real-valued function on $[a,b]$, then

$$\min_{g\in\mathscr{S}([a,b])} \int_a^b h(x)g(x)dx = \min_{x\in[a,b]} h(x). \tag{2.18}$$

By choosing as g a Dirac delta at a point $x \in [a,b]$ for which $h(x)$ is minimum, we see that the right side of (2.18) is not less than its left side. The other inequality can be deduced from the Mean Value Theorem for integrals (Apostol 1974).

Lemma 2.3.1 (Compact support of the optimal input spectrum) *Considering the problem stated in* (2.17)*, the optimal input* $\widetilde{\Phi}_u^{opt}$*, if it exists, has all its energy inside* $[0,1]$*. Namely,*

$$\int_{\mathbb{R}\setminus[0,1]} \widetilde{\Phi}_u^{opt}(y)dy = 0.$$

Thus, the spectral density of the optimal input has compact support, i.e. $\widetilde{\Phi}_u^{opt} \in \mathscr{S}([0,1])$ *(or, equivalently,* $\Phi_u^{opt} \in \mathscr{S}(\Theta)$*), therefore we can replace* (2.17) *with*

$$\widetilde{\Phi}_u^{opt} = \arg \max_{\widetilde{\Phi}_u \in \mathscr{S}([0,1])} \min_{x\in[0,1]} \int_0^1 \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2}\, \widetilde{\Phi}_u(y)dy.$$

Proof. Notice that $\partial F(x,y)/\partial y > 0$ for $y < x$ and $\partial F(x,y)/\partial y < 0$ for $y > x$. It follows that for any $x \in [0,1]$ we have $\int_{-\infty}^{\infty} F(x,y)\widetilde{\Phi}_u^{opt}(y)dy \leq \int_{-\infty}^{\infty} F(x,y)\widetilde{\Phi}_u'(y)dy$, where $\widetilde{\Phi}_u'$ is given by

$$\widetilde{\Phi}_u'(y) := \widetilde{\Phi}_u^{opt}(y)\mathscr{X}_{[0,1]}(y) + \delta(y)\int_{-\infty}^{0_-} \widetilde{\Phi}_u^{opt}(\tau)d\tau + \delta(y-1)\int_{1_+}^{\infty} \widetilde{\Phi}_u^{opt}(\tau)d\tau$$

and $\mathscr{X}_{[0,1]}$ is the indicator function of $[0,1]$. The result follows. □

Theorem 2.3.1 (Existence of an optimal input) *For the problem stated in* (2.14) *or* (2.17)*, there exists at least one optimal input, that is, there exists a* $\Phi_u^{opt} \in \mathscr{S}(\mathbb{R}_0^+)$ *such that for every* $\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)$*,*

$$\overline{J}(\Phi_u^{opt}) \leq \overline{J}(\Phi_u)\,.$$

Proof. By Lemma 2.3.1, (2.17) can be related to a *two-person zero-sum game on the unit square* with *kernel* F, such that player x attempts to minimise F by using a *pure strategy*, and player y attempts to maximise this quantity by using a *mixed strategy* (Başar and Olsder 1995). Hence, in order to prove the existence of $\widetilde{\Phi}_u^{opt}$ (or of Φ_u^{opt}, which is the same), we can make use of a version of the Minimax Theorem, due to (Glicksberg 1950), which states that if F is an upper or lower semicontinuous function on $[0,1]\times[0,1]$, then

$$\begin{aligned}\inf_{\mu_x\in\mathscr{S}([0,1])}\sup_{\mu_y\in\mathscr{S}([0,1])}\int_0^1\int_0^1 F(x,y)\mu_x(x)\mu_y(y)\,dy\,dx& \\ = \sup_{\mu_y\in\mathscr{S}([0,1])}\inf_{\mu_x\in\mathscr{S}([0,1])}\int_0^1\int_0^1 F(x,y)\mu_x(x)\mu_y(y)\,dy\,dx& \quad (2.19)\\ =: V_m&\end{aligned}$$

where V_m is called the *average value* of the game. Furthermore, if F is continuous then, by a standard compactness argument (such as that given in the paragraph above (2.16)), there exist μ_x^{opt}, $\mu_y^{opt} \in \mathscr{S}([0,1])$ such that for every $\mu_x,\ \mu_y \in \mathscr{S}([0,1])$,

$$\begin{aligned}\int_0^1\int_0^1 F(x,y)\mu_x^{opt}(x)\mu_y(y)\,dy\,dx &\le \int_0^1\int_0^1 F(x,y)\mu_x^{opt}(x)\mu_y^{opt}(y)\,dy\,dx \\ &\le \int_0^1\int_0^1 F(x,y)\mu_x(x)\mu_y^{opt}(y)\,dy\,dx. \end{aligned} \tag{2.20}$$

It is evident from (2.20) that $(\mu_x^{opt},\mu_y^{opt})$ defines a *saddle point solution in mixed strategies* for the game (Başar and Olsder 1995). In our case F is continuous, therefore these results apply. Furthermore, by (2.20) and the compactness of $[0,1]$,

$$\begin{aligned}\int_0^1\int_0^1 F(x,y)\mu_x^{opt}(x)\mu_y^{opt}(y)\,dy\,dx &= \min_{\mu_x\in\mathscr{S}([0,1])}\int_0^1\int_0^1 F(x,y)\mu_x(x)\mu_y^{opt}(y)\,dy\,dx \\ &= \min_{x\in[0,1]}\int_0^1 F(x,y)\mu_y^{opt}(y)\,dy. \end{aligned} \tag{2.21}$$

From (2.19), (2.20) and (2.21), we have

$$\min_{x\in[0,1]}\int_0^1 F(x,y)\mu_y^{opt}(y)\,dy = \max_{\mu_y\in\mathscr{S}([0,1])}\min_{x\in[0,1]}\int_0^1 F(x,y)\mu_y(y)\,dy.$$

If we take $\widetilde{\Phi}_u^{opt} = \mu_y^{opt}$, we then have an optimal solution to (2.17). This proves the existence of an optimal input. □

Theorem 2.3.2 (Uniqueness of the optimal input) *For the problem stated in* (2.14) *or* (2.17)*, there is a unique optimal input. Moreover,* 0 *and* 1 *do not belong to the support of* $\widetilde{\Phi}_u^{opt}$ *(or, equivalently,* $\underline{\theta},\overline{\theta} \notin \operatorname{supp}\Phi_u^{opt}$*), and* $\widetilde{\Phi}_u^{opt}$ *is symmetric with respect to* $1/2$*, that is,* $\widetilde{\Phi}_u^{opt}(y) = \widetilde{\Phi}_u^{opt}(1-y)$*, hence* $\Phi_u^{opt}(\omega) = \Phi_u^{opt}(\overline{\theta}\underline{\theta}/\omega)$.

Proof. To establish the proof, we utilise results from (Karlin 1957). To begin we need to show that f is a *proper Pólya frequency function*. This means, in particular, that we need to show that for every $n \in \mathbb{N}$ and every set of values $\{x_i\}_{i=1,\dots,n}$ and $\{y_j\}_{j=1,\dots,n}$ such that $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, the determinant of the matrix $(f(x_i - y_j))_{i,j}$ is positive. Now,

$$f(x-y) = \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2} = \frac{e^{k(x+y)}}{(e^{kx}+e^{ky})^2}.$$

Now, if we let $z_i := e^{kx_i} > 0$ and $w_j := e^{ky_j} > 0$, then

$$\operatorname{sgn}\det(f(x_i-y_j))_{i,j} = \operatorname{sgn}\det\left(\frac{w_i z_j}{(w_i+z_j)^2}\right)_{i,j} = \operatorname{sgn}\det\left(\frac{1}{(w_i+z_j)^2}\right)_{i,j}.$$

The determinant in the last term is given by the following expression, known as *Borchardt's identity* (Krattenthaler 1998):

$$\det\left(\frac{1}{(w_i+z_j)^2}\right)_{i,j} = \frac{\prod_{1\le i<j\le n}(w_j-w_i)(z_j-z_i)}{\prod_{1\le i,j\le n}(w_i+z_j)}\operatorname{perm}\left(\frac{1}{w_i+z_j}\right)_{i,j}, \tag{2.22}$$

where perm(X) is the *permanent* (Horn and Johnson 1985) of a square matrix X, and is defined as

$$\operatorname{perm}\left(\frac{1}{w_i+z_j}\right)_{i,j} := \sum_{s\in S_n}\prod_{i=1}^{n}\frac{1}{w_i+z_{s(i)}} > 0, \tag{2.23}$$

where S_n denotes the symmetric group of order n (i.e. the set of all permutations on $\{1,\ldots,n\}$).

From (2.22), (2.23) and the ordering of $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$, we can see that the determinant of $(f(x_i-y_j))_{i,j}$ is indeed positive.

Now, since f is even, positive, analytic, and a proper Pólya frequency function such that $f'(0)=0$, we have by Theorems 1 and 2 of (Karlin 1957) that $\widetilde{\Phi}_u^{opt}$ is unique, $0,1\notin \operatorname{supp}\widetilde{\Phi}_u^{opt}$ and $\widetilde{\Phi}_u^{opt}$ is symmetric with respect to $1/2$. □

Theorem 2.3.3 (Finite support of the optimal input spectrum) *For the problem stated in* (2.14) *or* (2.17)*, the optimal input (Φ_u^{opt} or $\widetilde{\Phi}_u^{opt}$, respectively) has finite support. That is, if $\widetilde{\Phi}_u^{opt}$ is such that*

$$\min_{x\in[0,1]}\int_0^1 f(x-y)\widetilde{\Phi}_u^{opt}(y)dy = \max_{\widetilde{\Phi}_u\in\mathscr{S}([0,1])}\left[\min_{x\in[0,1]}\int_0^1 f(x-y)\widetilde{\Phi}_u(y)dy\right]$$

then $\operatorname{supp}\widetilde{\Phi}_u^{opt}$ *is finite.*

Proof. This proof is based on a result in (Karlin 1957), which is included here for the sake of completeness. For clarity we focus on the problem stated in (2.17).

We first show that if μ_x^{opt} is defined as in the proof of Theorem 2.3.1, and $y_0\in[0,1]$ is in the support of $\widetilde{\Phi}_u^{opt}$, then

$$\int_0^1 f(x-y_0)\mu_x^{opt}(x)dx = V_m\,. \tag{2.24}$$

From (2.20), we have that

$$\int_0^1 f(x-y)\mu_x^{opt}(x)dx \le V_m, \qquad y\in[0,1]\,. \tag{2.25}$$

If this inequality were strict for $y=y_0$, then by the continuity of f there would be an interval $[a,b]\subseteq[0,1]$ for which $a\le y_0\le b$ and

$$\int_0^1 f(x-y)\mu_x^{opt}(x)dx < V_m, \qquad y\in[a,b]\,. \tag{2.26}$$

Thus, integrating both sides of (2.25) weighted by $\widetilde{\Phi}_u^{opt}$, and taking (2.26) into account, we obtain

$$\int_0^1\int_0^1 f(x-y)\mu_x^{opt}(x)\widetilde{\Phi}_u^{opt}(y)dydx < V_m$$

which contradicts the definition of V_m. This proves (2.24).

Now, if $\operatorname{supp}\widetilde{\Phi}_u^{opt}$ is infinite, then (2.24) holds for an infinite number of points in a compact interval, such that these points have at least one limit point. On the other hand, the integral of the left side of this expression is an analytic function of y in some region Ω containing $\mathbb{R}$, and its right side is constant. Thus, we have two analytic functions which are equal in a set which has a limit point in Ω, hence by a well-known result of complex analysis (Rudin 1987, page 209) they must be equal in Ω. In particular it holds that

$$\int_0^1 f(x-y)\mu_x^{opt}(x)dx = V_m, \qquad y \in \mathbb{R}. \tag{2.27}$$

However, since f is bounded and $f(u) \to 0$ for $|u| \to \infty$,

$$\lim_{y\to\infty} \int_0^1 f(x-y)\mu_x^{opt}(x)dx = 0 \neq V_m$$

which contradicts (2.27). Thus, $\widetilde{\Phi}_u^{opt}$ has finite support. □

Remark 2.3.2 *Theorem 2.3.3 basically says that the robust optimal input is a finite linear combination of sinusoids. This is a rather surprising result, since the nominal optimal input is a single sinusoid of frequency equal to θ, hence one would expect that the robust optimal signal should have a continuous spectrum to account for all the possible values of this parameter. On the other hand, this property also says that it is very easy to implement such a signal; the only remaining problem is to determine the amplitudes and frequencies of the associated sinusoids. This is addressed in Appendix 2.7 (analytically) and Section 2.3.4 (numerically).*

Remark 2.3.3 *Note that Lemma 2.3.1 and Theorem 2.3.3 are different in that Lemma 2.3.1 states that the optimal input has compact support, which is a technical requirement for proving other results. Theorem 2.3.3, on the other hand, states that the optimal input has finite support, which is a stronger result than Lemma 2.3.1, but its proof relies on the previous theorems.*

2.3.3 Bandlimited '$1/f$' Noise Input

The results presented above are concerned with the optimal solution to the problem of interest. In this section we explore a sub optimal solution. In the latter context, the following result is useful, since it allows us to quantify the performance of a given input signal with respect to the robust optimal experiment.

Lemma 2.3.2 (Bounds on the min-max cost) *Let $\overline{\Phi}_u \in \mathscr{S}([0,1])$. Also let*

$$\alpha_{min}(\overline{\Phi}_u) := \min_{\theta\in\Theta} J(\overline{M}(\theta,\overline{\Phi}_u),\theta) \tag{2.28}$$

$$\alpha_{max}(\overline{\Phi}_u) := \max_{\theta\in\Theta} J(\overline{M}(\theta,\overline{\Phi}_u),\theta). \tag{2.29}$$

Then

$$\alpha_{min}(\overline{\Phi}_u) \leq \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \max_{\theta \in \Theta} J(\overline{M}(\theta, \Phi_u), \theta) \leq \alpha_{max}(\overline{\Phi}_u)\,. \tag{2.30}$$

Proof. The second inequality follows from the definition of the optimisation problem.

To establish the first inequality, we notice from (2.28) that

$$\frac{1}{\alpha_{min}(\overline{\Phi}_u)} = \frac{1}{\min\limits_{\theta \in \Theta} J(\overline{M}(\theta, \overline{\Phi}_u), \theta)} = \max_{\theta \in \Theta} [J(\overline{M}(\theta, \overline{\Phi}_u), \theta)]^{-1} = \max_{\theta \in \Theta} \int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\, \overline{\Phi}_u(\omega) d\omega.$$

Thus, for any feasible function Φ_u, we must have

$$\frac{1}{\alpha_{min}(\overline{\Phi}_u)} \geq \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \Phi_u(\theta) \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\, \overline{\Phi}_u(\omega) d\omega\, d\theta\,. \tag{2.31}$$

Now let us assume that the first inequality in (2.30) is false, i.e.

$$\alpha_{min}(\overline{\Phi}_u) > \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \max_{\theta \in \Theta} J(\overline{M}(\theta, \Phi_u), \theta) = \quad \max_{\theta \in \Theta} J(\overline{M}(\theta, \Phi_u^{opt}), \theta)$$

and therefore,

$$\frac{1}{\alpha_{min}(\overline{\Phi}_u)} < \min_{\theta \in \Theta} [J(\overline{M}(\theta, \Phi_u^{opt}), \theta)]^{-1} = \min_{\theta \in \Theta} \int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\, \Phi_u^{opt}(\omega) d\omega. \tag{2.32}$$

Hence, if we form a convex combination of the integrals on the right hand side of (2.32) using $\overline{\Phi}_u(\theta)$, we have

$$\frac{1}{\alpha_{min}(\overline{\Phi}_u)} < \int_{\underline{\theta}}^{\overline{\theta}} \overline{\Phi}_u(\theta) \int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\, \Phi_u^{opt}(\omega) d\omega\, d\theta\,. \tag{2.33}$$

However,

$$\frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2} = \frac{\theta^2/\omega^2}{(\theta^2/\omega^2+1)^2}. \tag{2.34}$$

Thus, changing the order of the variables of integration in (2.33) and using (2.34) gives

$$\frac{1}{\alpha_{min}(\overline{\Phi}_u)} < \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \Phi_u^{opt}(\theta) \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\, \overline{\Phi}_u(\omega) d\theta\, d\omega\,. \tag{2.35}$$

We see that (2.35) contradicts (2.31) if we choose Φ_u in (2.31) as Φ_u^{opt}. This contradiction establishes the result. □

Remark 2.3.4 *It is impossible to find an input Φ_u which brings $\alpha_{min}(\Phi_u)$ equal to $\alpha_{max}(\Phi_u)$. This is due to the fact that, for a fixed Φ_u, the cost function $J(\overline{M}(\theta, \Phi_u), \theta)$ is an analytic function of θ on $\mathbb{R}$, and vanishes as $|\theta| \to \infty$. Therefore, if we force $\alpha_{min}(\Phi_u) = \alpha_{max}(\Phi_u)$ then this cost function would be constant in the interval $[\underline{\theta}, \overline{\theta}]$, which implies by its analyticity, that it would be constant in $\mathbb{R}$, and hence equal to 0. This is impossible, since the integral of Φ_u over $[\underline{\theta}, \overline{\theta}]$ is equal to 1, and Φ_u can only take nonnegative values over that interval.*

Remark 2.3.5 *We see from Lemma 2.3.2 that, if a feasible design, Φ_u, is found such that $\alpha_{min}(\Phi_u)$ and $\alpha_{max}(\Phi_u)$ are 'close', then the corresponding cost function will be 'close' to optimal. In particular, if one could choose an input, Φ_u, such that $\alpha_{min}(\Phi_u) = \alpha_{max}(\Phi_u)$, then this input would be optimal. Alas, by Remark 2.3.4, there is no feasible input which brings $\alpha_{min}(\Phi_u)$ to $\alpha_{max}(\Phi_u)$. However, we will now examine a particular sub-optimal input such that $\alpha_{min}(\Phi_u)$ and $\alpha_{max}(\Phi_u)$ are within a factor of 2 of each other.*

With the above as background, we now consider the following feasible input, known as *bandlimited '$1/f$' noise*,

$$\Phi_u^{1/f}(\omega) := \begin{cases} \dfrac{1/\omega}{\ln\overline{\theta} - \ln\underline{\theta}}, & \omega \in [\underline{\theta}, \overline{\theta}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.36}$$

For this input, we have the following result.

Theorem 2.3.4 (Bounds on the min-max cost for bandlimited '$1/f$' noise) *Consider the bandlimited '$1/f$' noise input given in* (2.36)*. Let $\alpha_{min}(\Phi_u^{1/f})$ and $\alpha_{max}(\Phi_u^{1/f})$ be the corresponding limits as in* (2.28) *and* (2.29)*. Then*

$$\alpha_{min}(\Phi_u^{1/f}) = 2\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)\frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}$$
$$\alpha_{max}(\Phi_u^{1/f}) = 4\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)\frac{\overline{\theta}^2+\underline{\theta}^2}{\overline{\theta}^2-\underline{\theta}^2}.$$

Proof. Upon substitution of (2.36) into (2.15), we obtain

$$\frac{1}{J(\overline{M}(\theta,\Phi_u^{1/f}),\theta)} = \frac{1}{\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\frac{d\omega}{\omega} = \frac{\overline{\theta}^2-\underline{\theta}^2}{2\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\frac{\theta^2}{(\theta^2+\overline{\theta}^2)(\theta^2+\underline{\theta}^2)}.$$

Note that the function $f(\theta) = \theta^2/[(\theta^2+\overline{\theta}^2)(\theta^2+\underline{\theta}^2)]$, in Θ, increases to a maximum at $\theta = \sqrt{\underline{\theta}\,\overline{\theta}}$ and then decreases. Finally since $f(\underline{\theta}) = f(\overline{\theta})$ we conclude that

$$\frac{1}{\alpha_{max}(\Phi_u^{1/f})} = \frac{\overline{\theta}^2-\underline{\theta}^2}{2\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\frac{\theta^2}{(\theta^2+\overline{\theta}^2)(\theta^2+\underline{\theta}^2)}\bigg|_{\theta=\underline{\theta}} = \frac{\overline{\theta}^2-\underline{\theta}^2}{4\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\frac{1}{\overline{\theta}^2+\underline{\theta}^2}$$

and

$$\frac{1}{\alpha_{min}(\Phi_u^{1/f})} = \frac{\overline{\theta}^2-\underline{\theta}^2}{2\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\frac{\theta^2}{(\theta^2+\overline{\theta}^2)(\theta^2+\underline{\theta}^2)}\bigg|_{\theta=\sqrt{\underline{\theta}\overline{\theta}}} = \frac{1}{2\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)}\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}+\underline{\theta}}.$$

which completes the proof. □

Corollary 2.3.1 (Robust performance of bandlimited '$1/f$' noise) *For the bandlimited '$1/f$' noise input, the optimal cost, J^{opt}, must satisfy*

$$\alpha_{min}(\Phi_u^{1/f}) \leq J^{opt} \leq 2\alpha_{min}(\Phi_u^{1/f}) .$$

Proof. From Theorem 2.3.4,

$$\begin{aligned}
\alpha_{max}(\Phi_u^{1/f}) &= 4\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)\frac{\overline{\theta}^2+\underline{\theta}^2}{\overline{\theta}^2-\underline{\theta}^2} \\
&\leq 4\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)\frac{\overline{\theta}^2+\underline{\theta}^2+2\overline{\theta}\underline{\theta}}{\overline{\theta}^2-\underline{\theta}^2} \\
&= 4\ln\left(\frac{\overline{\theta}}{\underline{\theta}}\right)\frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}} \\
&= 2\alpha_{min}(\Phi_u^{1/f}) .
\end{aligned}$$

The result then follows from Lemma 2.3.2. □

Remark 2.3.6 *The above result is rather surprising since it shows that bandlimited '$1/f$' noise performs very well for this problem. This is an interesting result since 'conventional wisdom' suggests an input more akin to bandlimited white noise (e.g. a PRBS signal). However, one can easily verify that using $\underline{\theta} = 0.1$ and $\overline{\theta} = 10$, bandlimited '$1/f$' noise is almost an order of magnitude superior to bandlimited white noise – see Table 2.1 presented below.*

2.3.4 Discrete Approximation to the Optimal Input

As shown in Section 2.3.2, and is well known in the statistics literature (see e.g. (Walter and Pronzato 1997)), finding an exact solution to problems of the type (2.14), (2.15) is, in general, extremely difficult. Some algorithms have been proposed, e.g. the relaxation algorithm of (Shimizu and Aiyoshi 1980). Here we pursue an alternative idea of finding an approximate design by discretisation of the design space (see also (Walter and Pronzato 1997, page 341)). Since f is continuous, it is well known (Owen 1968) that this approach can approximate the optimal solution as closely as desired.

To develop this idea, we first approximate the integral in equation (2.15) by a Riemann sum. Specifically, utilising Lemma 2.3.1, we choose a grid of $N+1$ points $\underline{\theta} \leq \omega_m = \theta_m \leq \overline{\theta}$ for $0 \leq m \leq N$ such

that $\omega_0 = \theta_0 = \underline{\theta}$, $\omega_N = \theta_N = \overline{\theta}$. Then

$$\begin{aligned} J_m &:= \left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega^2/\theta_m^2}{(\omega^2/\theta_m^2+1)^2}\, \Phi_u(\omega) d\omega \right]^{-1} \\ &\approx \left[\sum_{n=0}^{N-1} \frac{\omega_n^2/\theta_m^2}{(\omega_n^2/\theta_m^2+1)^2}\, \Phi_u(\omega_n)(\omega_{n+1}-\omega_n) \right]^{-1} \\ &= \left[\sum_{n=0}^{N-1} A_{m,n} E_n \right]^{-1} \end{aligned} \tag{2.37}$$

where $A_{m,n} := (\omega_n^2/\theta_m^2)/(\omega_n^2/\theta_m^2+1)^2 > 0$ and $E_n := \Phi_u(\omega_n)(\omega_{n+1}-\omega_n)$. Notice that the matrix $A = \{A_{m,n}\}$ is symmetric and has positive entries.

We can now state the following discrete approximation to the optimisation problem in equation (2.14)

$$\mathbf{E}^{opt} = \arg \min_{\mathbf{E} \in \mathscr{S}_d} \max_{0 \le m < N} (\mathbf{e}_m^T A \mathbf{E})^{-1} \tag{2.38}$$

where $\mathscr{S}_d := \{\mathbf{E} \in \mathbb{R}^N : \mathbf{1}^T \mathbf{E} = 1, E_n \ge 0\}$, $\mathbf{E} := [E_0 \ \cdots \ E_{N-1}]^T$, $\mathbf{e}_m$ is the m^{th} column of the N dimensional identity matrix, and $\mathbf{1}$ is an N dimensional vector of ones.

It is well known that a finite dimensional min-max optimisation problem, such as (2.38), can be converted into a standard linear programming (LP) problem; see (Dantzig 1951, Gale et al. 1951, McKinsey 1952). To this end, let

$$\mathbf{F} := [x\, \mathbf{E}^T]^T \in \mathbb{R}^{N+1}.$$

Now we can readily show that (2.38) is equivalent to the following LP problem:

$$\begin{aligned} \max_{\mathbf{F}} \quad & \widetilde{C}\mathbf{F} \\ \text{subject to} \quad & \widetilde{A}\mathbf{F} \ge 0 \\ & \widetilde{B}\mathbf{F} = 1 \end{aligned}$$

where

$$\widetilde{A} := \begin{bmatrix} -\mathbf{1} & A \\ \mathbf{0} & I \end{bmatrix} \in \mathbb{R}^{2N \times (N+1)}$$

$$\widetilde{B} := [0\, \mathbf{1}^T] \in \mathbb{R}^{1 \times (N+1)}$$

$$\widetilde{C} := [1\, 0 \cdots 0] \in \mathbb{R}^{1 \times (N+1)}$$

and $\mathbf{1} := [1 \ \cdots \ 1]^T \in \mathbb{R}^{N \times 1}$.

It is also quite straightforward to compute a discrete approximation to the Bayesian optimal input for the first order example. For example, say that we use $J(\overline{M}(\theta, \Phi_u), \theta)$ as a Bayesian risk and, for the

sake of illustration assume that θ has a uniform distribution on Θ. Then, the Bayesian design problem becomes

$$\Phi_u^B = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \frac{1}{\overline{\theta} - \underline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_0^\infty \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2} \Phi_u(\omega) d\omega \right]^{-1} d\theta.$$

We can approximate this, as in (2.37), by

$$\mathbf{E}^B = \arg \min_{\mathbf{E} \in \mathscr{S}_d} \frac{1}{N} \sum_{k=0}^{N-1} \exp\left\{ \left[\frac{\ln \overline{\theta} - \ln \underline{\theta}}{N} \right] k + \ln \underline{\theta} \right\} (\mathbf{e}_k^T A \mathbf{E})^{-1}.$$

In the next section, we will also consider a Bayesian design for the case when $\ln \theta$ has a uniform distribution on $(\ln \underline{\theta}\ ,\ \ln \overline{\theta})$.

2.3.5 Numerical Results

We present below numerical results for the first order example. Here we take $\underline{\theta} = 0.1$, $\overline{\theta} = 10$, $N = 100$ and compare the following inputs using 3 different costs, namely $\max_{\theta \in \Theta} [\theta^2 \overline{M}(\theta, \Phi_u)]^{-1}$, and the Bayesian cost assuming uniform and logarithmic prior distributions for θ:

(i) A nominal input of frequency 1 [rad/s] (Notice that this is the optimal input if the initial estimate of the parameter is $\widehat{\theta} = 1$).

(ii) Bandlimited white noise input, limited to the frequency range $[0.1, 10]$ [rad/s].

(iii) Bandlimited '$1/f$' noise input, limited to the frequency range $[0.1, 10]$ [rad/s].

(iv) The approximate discretised robust optimal input generated by LP.

(v) The approximate discretised Bayesian optimal input for a uniform distribution on θ.

(vi) The approximate discretised Bayesian optimal input for a uniform distribution on $\ln \theta$.

Relative costs for the different experimental conditions are shown in Table 2.1. Note that the costs have been scaled such that the optimal value is 1.

Table 2.1 shows that bandlimited white noise gives poor performance under all criteria. Indeed, we see from the table that bandlimited '$1/f$' noise is almost an order of magnitude better than a bandlimited white noise input for all cost functions. Furthermore, the discretised min-max optimum gives an additional 40% improvement for the min-max cost function. The discretised min-max optimal input energy, Φ_u^{opt}, is shown in Figure 2.3. Notice that the above results are consistent with Theorem 2.3.3, which asserts that the unique optimal input has finite support. The corresponding values of $[\theta^2 \overline{M}(\theta, \Phi_u^{opt})]^{-1}$ as a function of θ were shown earlier in Figure 2.2, where they can be compared to

Table 2.1. Relative Values of Cost for the Different Input Signals

	$\max_{\theta\in\Theta}[\theta^2\overline{M}(\theta,\Phi_u)]^{-1}$	Bayesian cost on θ	Bayesian cost on $\ln\theta$
Single frequency at $\omega=1$	7.75	4.8	2.26
Bandlimited white noise	12.09	9.05	2.96
Bandlimited '$1/f$' noise	1.43	1.51	1.07
Robust min-max optimal input	1.00	1.45	1.12
Bayesian design (for uniform distribution on θ)	5.4	1.00	1.61
Bayesian design (for uniform distribution on $\ln\theta$)	1.53	1.46	1.00

the corresponding values for the nominal optimal input and bandlimited '$1/f$' noise. It is interesting to notice from Figure 2.2 that $[\theta^2\overline{M}(\theta,\Phi_u^{opt})]^{-1}$ is an almost constant function of θ. This should be compared with the comments in Remark 2.3.5. The comparative costs are given in Table 2.1.

2.4 A Resonant Second Order Example

In this section we consider a second order resonant system. This is in fact an ideal example to highlight the merits of robust experiment design, as will be shown below.

Let $H(s)=1$ and

$$G(s)=\frac{\theta^2}{s^2+2\xi\theta s+\theta^2},$$

where $\xi\in(0,1)$ is assumed known (to keep the example simple) and θ is assumed to belong to a given range $[\underline{\theta},\overline{\theta}]$.

The scaled single frequency information matrix is

$$\theta^2\widetilde{M}(\theta,\omega)=\frac{4[(\omega/\theta)^4+\xi^2(\omega/\theta)^2]}{\{[1-(\omega/\theta)^2]^2+4\xi^2(\omega/\theta)^2\}^2}.$$

The nominal optimal test can be realised by a single sinusoid at frequency (see Section 2.4.1 for the details)

$$\omega^{opt}=\sqrt{x}\,\theta,$$

where x is the unique positive root of

$$2x^3+3\xi^2x^2+2(2\xi^4-\xi^2-1)x-\xi^2=0.$$

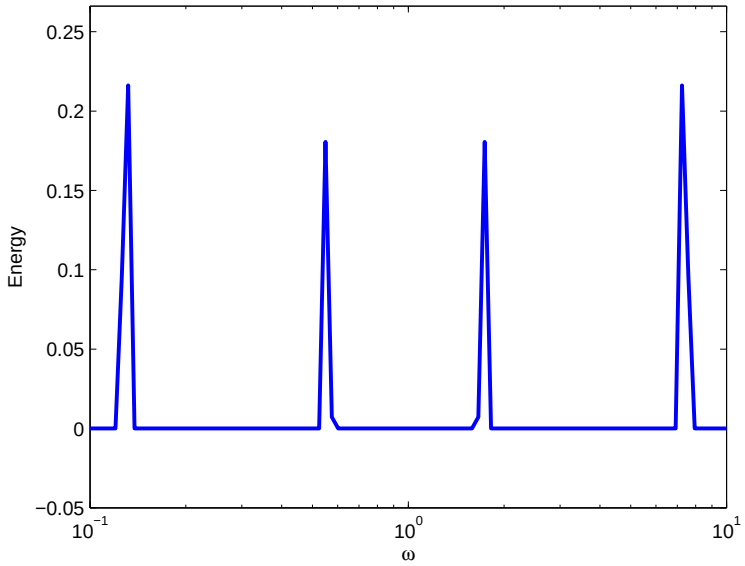

Figure 2.3. Values of E for the discretised robust optimal input.

Notice that $\omega^{opt} \to \theta$ as $\xi \to 0$. Again, this result is heuristically reasonable since one places the optimal test signal very near the resonance frequency.

In the results presented below, we assume that $\theta \in [0.5, 2]$ with a nominal value of 1. Also, we consider 3 values for ξ, namely, 0.1, 0.05 and 0.02.

Figure 2.4 shows a high sensitivity of the cost $[\theta^2 \widetilde{M}(\theta, \omega)]^{-1}$ with respect to θ when using the nominal optimal input designed for $\theta_0 = 1$. This figure provides a strong incentive to use robust experiment designs for this system. Whilst, as shown in the nominal optimal case, it makes sense to put the test signal energy near the resonance frequency, it can be reasonably assumed that this can totally 'miss the target' if the resonance frequency is not quite where it was believed to be.

In the next section we will establish important properties of the min-max optimal design for this system. However, to motivate the reader, we first present some numerical results. See Remark 2.4.1 in Section 2.4.2 for details regarding the frequency ranges considered for the optimal input spectrum, bandlimited white noise and bandlimited '$1/f$' noise.

Figure 2.5 shows the spectrum of the robust optimal input for $\xi = 0.1$.

The numerical results for this example have been obtained by discretising the experiment design problem, as with the first order example in Section 2.3.4. However, due to the large number of spikes

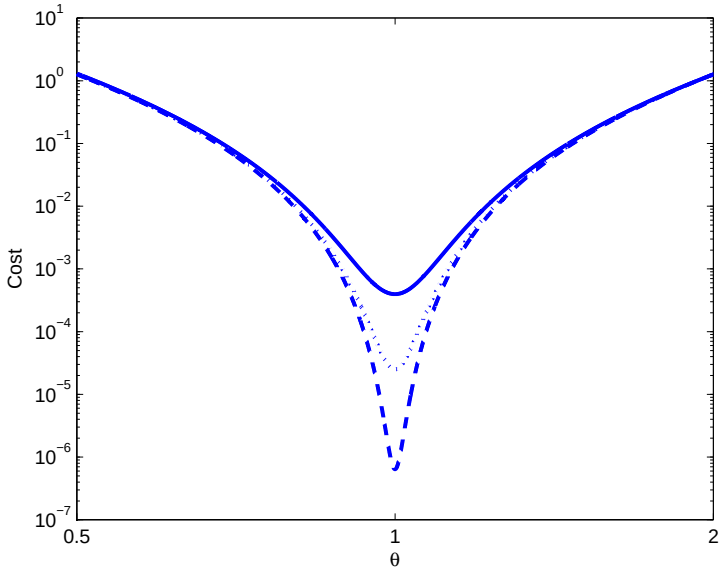

Figure 2.4. Variation of the cost function, $[\theta^2\widetilde{M}(\theta,\omega)]^{-1}$, with respect to θ, for $\xi = 0.1$ (solid), $\xi = 0.05$ (dotted) and $\xi = 0.02$ (dashed), when using the nominal optimal test signal designed for $\theta_0 = 1$.

in the spectrum of the optimal input (as shown in Figure 2.5), the discretised problem has not been solved using linear programming, but a game-theoretical technique known as Fictitious Play (Brown 1951), which consists in playing, repeatedly, the associated game. At each turn, each player (the maximiser and the minimiser) plays by assumming that the opponent will respond according to the empirical distribution of the previous moves. This method is very efficient for large games, and it has been established that the method converges to a saddle point for a finite, two-person, zero-sum game (Robinson 1951), such as the one associated with this example. See (Washburn 2001) for a fast version of this method.

The results are summarised in Table 2.2. (Note that the performance of the optimal input has again been normalised to 1 for ease of comparison.)

A startling observation from Table 2.2 is how well bandlimited '$1/f$' noise performs. Specifically, we see that it is within a factor of 1.82 for $\xi = 0.1$, a factor of 1.90 for $\xi = 0.05$ and a factor of 1.97 for $\xi = 0.02$, of the robust optimal input.

Figure 2.6 exhibits the sensitivity of the cost, $[\theta^2\widetilde{M}(\theta,\omega)]^{-1}$, with respect to θ for the test signals considered in Table 2.2 for $\xi = 0.1$.

From Figure 2.6 it can be seen that the min-max optimal input is indeed robust for the parameter range

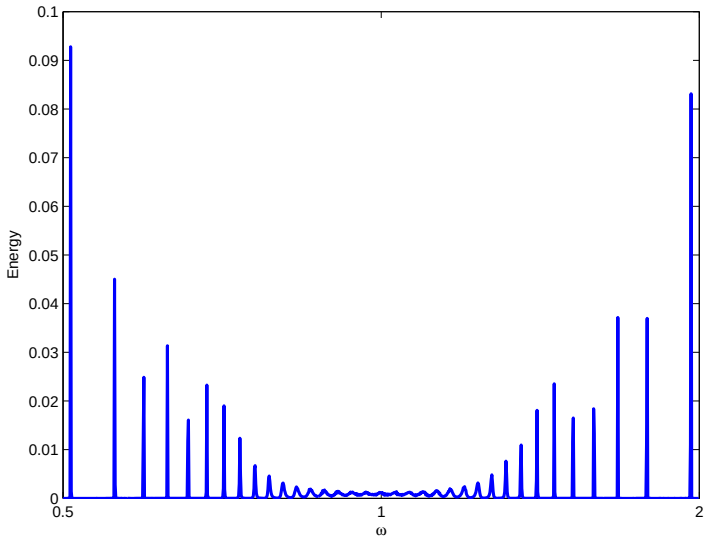

Figure 2.5. Discretised spectrum of the robust optimal input designed for the resonant second order system with $\xi = 0.1$.

Table 2.2. Relative Values of Cost ($\max_{\theta\in\Theta}\ [\theta^2\overline{M}(\theta,\Phi_u)]^{-1}$) for the Different Input Signals

	Relative cost for $\xi = 0.1$	Relative cost for $\xi = 0.05$	Relative cost for $\xi = 0.02$
Single frequency at $\omega = 1$	338.16	2,756.10	44,003.41
Bandlimited white noise	3.69	4.00	3.65
Bandlimited '$1/f$' noise	1.82	1.90	1.97
Robust min-max optimal input	1	1	1

$[0.5, 2]$ when contrasted against bandlimited '$1/f$' noise and bandlimited white noise. Furthermore, we observe that bandlimited '$1/f$' noise gives a performance that is 'nearly' as good as the robust min-max optimal input.

Recall that bandlimited '$1/f$' noise also performs extremely well in the case of the first order system studied in Section 2.3. This leads us to the following conjecture:

A signal such as bandlimited '$1/f$' noise is actually a *good* (as opposed to optimal) test signal in system identification. The reason is that it is not only robust with respect to parameter variations but is also robust with respect to model structure. This observation actually has motivated further work reported in Chapter 3.

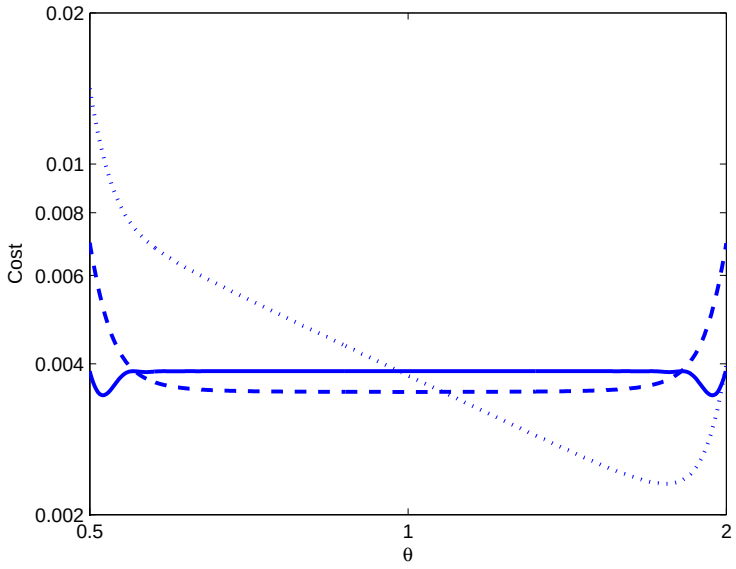

Figure 2.6. Variation of cost versus θ for the robust optimal input (solid), bandlimited white noise (dotted) and bandlimited '$1/f$' noise (dashed), for $\xi = 0.1$.

In the next subsections, we develop proofs of key properties of the robust optimal input for the second order resonant system, i.e. compact support, existence and finite support.

2.4.1 Nominal Experiment Design

To obtain some insight into the nature of the robust optimal input for the second order resonant system, we will first consider the nominal experiment design problem, which consists in finding a $\Phi_u^{opt} \in \mathscr{S}(\mathbb{R}_0^+)$, if it exists, such that

$$\Phi_u^{opt} = \arg \max_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \int_0^\infty \widetilde{M}(\theta_0, \omega)\, \Phi_u(\omega) d\omega \tag{2.39}$$

for a fixed $\theta_0 \in \mathbb{R}^+$, where $\widetilde{M}$ is given by

$$\widetilde{M}(\theta, \omega) := \frac{1}{\theta^2} \frac{4[(\omega/\theta)^4 + \xi^2(\omega/\theta)^2]}{\{[1-(\omega/\theta)^2]^2 + 4\xi^2(\omega/\theta)^2\}^2}. \tag{2.40}$$

The details are somewhat more involved than those of the first order system analysed in Section 2.3.

To solve problem (2.39), we need to understand how $\widetilde{M}(\theta_0, \omega)$ behaves as a function of ω. First notice that, from (2.40), $\widetilde{M}(\theta_0, \omega)$ is differentiable and nonnegative for every $\omega, \theta_0 \in \mathbb{R}^+$, while $\widetilde{M}(\theta_0, 0) = 0$ and $\lim_{\omega \to \infty} \widetilde{M}(\theta_0, \omega) = 0$, and $\widetilde{M}(\theta_0, \cdot)$ does not have zeros in $\mathbb{R}^+$, hence it must have at least one maximum in $\mathbb{R}^+$. Moreover,

$$\frac{\partial \widetilde{M}(\theta_0, \omega)}{\partial \omega} = -\frac{8(\omega/\theta_0)\{2(\omega/\theta_0)^6 + 3\xi^2(\omega/\theta_0)^4 + [4\xi^4 - 2\xi^2 - 2](\omega/\theta_0)^2 - \xi^2\}}{\theta_0^5((\omega/\theta_0)^4 + 2(2\xi^2 - 1)(\omega/\theta_0)^2 + 1)^3}. \tag{2.41}$$

The second order factor in the denominator of (2.41) is of the form $x^2 + 2(2\xi^2 - 1)x + 1$ (where $x := (\omega/\theta)^2$), and it has roots at

$$x_{1,2} = 1 - 2\xi^2 \pm 2\xi^2\sqrt{\xi^2 - 1} \notin \mathbb{R}.$$

Hence, the denominator of (2.41) does not change sign. We next evaluate the denominator at $\omega = 0$ and see that it is always positive.

The numerator of (2.41), on the other hand, vanishes at $\omega = 0$ and $\omega = \sqrt{x}\theta$, where x is a positive root of

$$2x^3 + 3\xi^2 x^2 + 2(2\xi^4 - \xi^2 - 1)x - \xi^2 = 0. \tag{2.42}$$

Now, the linear coefficient of (2.42) has roots at $\xi = \pm j/\sqrt{2}, \pm 1$, hence for $\xi \in (0,1)$ it is negative. Thus, the coefficients of (2.42) have only one change of sign for every $\xi \in (0,1)$, so by Descartes' rule of signs (Dickson 1914), (2.42) has exactly one positive root for every $\xi \in (0,1)$.

By the preceding analysis, $\widetilde{M}(\theta_0, \omega)$ has exactly one maximum at $\omega = \sqrt{x}\theta$, where x is the unique positive root of (2.42). Thus, the optimal nominal input should have all its energy at this particular frequency, i.e.

$$\Phi_u^{opt} = \arg \max_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \int_0^\infty \widetilde{M}(\theta_0, \omega)\, \Phi_u(\omega) d\omega = \delta_{\sqrt{x}\theta},$$

where δ_α is the Dirac distribution with support in α.

2.4.2 Robust Experiment Design

The robust optimal experiment design problem for the resonant second order system consists in finding a $\Phi_u^{opt} \in \mathscr{S}(\mathbb{R}_0^+)$, if it exists, such that

$$\Phi_u^{opt} = \arg \max_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \inf_{\theta \in \Theta} \int_0^\infty \frac{4(\omega/\theta)^2[\xi^2 + (\omega/\theta)^2]}{\{[1 - (\omega/\theta)^2]^2 + 4\xi^2(\omega/\theta)^2\}^2}\, \Phi_u(\omega) d\omega. \tag{2.43}$$

We first observe that, since the integrand in (2.43) is continuous in $\theta \in \Theta$ for every $\omega \in \mathbb{R}_0^+$ and it is bounded by an integrable function which is independent of θ (use e.g. C/ω^2, where C is large enough and independent of θ), the integral is continuous in θ; see (Bartle 1966). This implies, with the compactness of Θ, that we can replace the 'inf' in (2.43) with 'min'.

Furthermore, we make the following change of variables

$$
\begin{aligned}
x &:= \frac{\ln\theta - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}},\\
y &:= \frac{\ln\omega - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}},\\
\Phi_u(\omega) &= \frac{2}{k\omega}\,\widetilde{\Phi}_u\left(\frac{\ln\omega - \ln\underline{\theta}}{\ln\overline{\theta} - \ln\underline{\theta}}\right),\\
k &:= 2(\ln\overline{\theta} - \ln\underline{\theta}),
\end{aligned}
$$

such that the problem can be rewritten as

$$
\widetilde{\Phi}_u^{opt} = \arg\max_{\widetilde{\Phi}_u\in\mathscr{S}(\mathbb{R})}\min_{x\in[0,1]}\int_0^\infty \frac{4e^{k(y-x)}[\xi^2+e^{k(y-x)}]}{\{[1-e^{k(y-x)}]^2+4\xi^2e^{k(y-x)}\}^2}\,\widetilde{\Phi}_u(y)dy\,. \tag{2.44}
$$

To simplify the notation, let $F(x,y) := f(y-x)$, where

$$
f(u) := \frac{4e^{ku}[\xi^2+e^{ku}]}{\{[1-e^{ku}]^2+4\xi^2e^{ku}\}^2}\,.
$$

We next establish the properties of existence and finite support for Φ_u^{opt} and $\widetilde{\Phi}_u^{opt}$ in the following theorems.

Lemma 2.4.1 (Compact support of the optimal input spectrum) *For the problem stated in* (2.44)*, the optimal input $\widetilde{\Phi}_u^{opt}$, if it exists, has compact support. Namely,*

$$
\int_{\mathbb{R}\setminus[\alpha,1+\alpha]}\widetilde{\Phi}_u^{opt}(y)dy = 0\,,
$$

where $\alpha\in\mathbb{R}$ is the only real solution of the equation

$$
2e^{3ka}+3\xi^2e^{2ka}+2(2\xi^4-\xi^2-1)e^{ka}-\xi^2 = 0\,.
$$

Thus, $\widetilde{\Phi}_u^{opt}\in\mathscr{S}([\alpha,1+\alpha])$, hence we can replace (2.44) *with*

$$
\tilde{\widetilde{\Phi}}_u^{opt} = \arg\max_{\tilde{\widetilde{\Phi}}_u\in\mathscr{S}([0,1])}\min_{x\in[0,1]}\int_0^1 \frac{4e^{k(y-x+\alpha)}[\xi^2+e^{k(y-x+\alpha)}]}{\{[1-e^{k(y-x+\alpha)}]^2+4\xi^2e^{k(y-x+\alpha)}\}^2}\,\tilde{\widetilde{\Phi}}_u(y)dy\,, \tag{2.45}
$$

where $\tilde{\widetilde{\Phi}}_u(y) = \widetilde{\Phi}_u(y+\alpha)$ for every $y\in[0,1]$.

Proof. The derivative of f is

$$
\frac{df(u)}{du} = -\frac{4ke^{ku}[2e^{3ku}+3\xi^2e^{2ku}+(4\xi^4-2\xi^2-2)e^{ku}-\xi^2]}{[e^{2ku}+2(2\xi^2-1)e^{ku}+1]^3},
$$

which has essentially the same form as $\partial\widetilde{M}(\theta_0,\omega)/\partial\omega$ in (2.41), after replacing $(\omega/\theta_0)^2$ by e^{ku}. As in Section 2.4.1, by the application of Descartes' rule of signs we can show that $f(u)$ has a unique global and local maximum at $u=\alpha:=k^{-1}\ln x$, where x is the unique positive root of

$$2x^3+3\xi^2x^2+2(2\xi^4-\xi^2-1)x-\xi^2=0\,. \tag{2.46}$$

Moreover, since $\lim_{|u|\to\infty}f(u)=0$, we have that $\partial f/\partial u>0$ for $u<\alpha$ and $\partial f/\partial u<0$ for $u>\alpha$. Thus for any $x\in[0,1]$ we have

$$\int_{-\infty}^{\infty}F(x,y)\widetilde{\Phi}_u^{opt}(y)dy\le\int_{-\infty}^{\infty}F(x,y)\widetilde{\Phi}_u'(y)dy\,,$$

where $\widetilde{\Phi}_u'$ is given by

$$\widetilde{\Phi}_u'(y):=\widetilde{\Phi}_u^{opt}(y)\mathscr{X}_{[\alpha,1+\alpha]}(y)+\delta(y-\alpha)\int_{-\infty}^{\alpha_-}\widetilde{\Phi}_u^{opt}(\tau)d\tau+\delta(y-[1+\alpha])\int_{(1+\alpha)_+}^{\infty}\widetilde{\Phi}_u^{opt}(\tau)d\tau\,,$$

and $\mathscr{X}_{[\alpha,1+\alpha]}$ denotes the indicator function of $[\alpha,1+\alpha]$.

Finally, a simple change of variables gives (2.45). □

Remark 2.4.1 *Lemma 2.4.1 implies that*

$$\operatorname{supp}\Phi_u^{opt}\subseteq\left[\underline{\theta}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\alpha},\overline{\theta}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{\alpha}\right].$$

This means that for e.g. $\underline{\theta}=0.5$, $\overline{\theta}=2$ *and three values of* ξ, *namely* $\xi=0.1,0.05,0.02$, *we have that the support of* Φ_u^{opt} *is contained in* [0.499975, 1.99990], [0.499998, 1.99999] *and* [0.499999, 1.99999], *respectively.*

Remark 2.4.2 *Since* ξ *is usually unknown, it is convenient to find the smallest interval* $[\underline{\omega},\overline{\omega}]$ *which contains* $\operatorname{supp}\Phi_u^{opt}$ *for every* $\xi\in(0,1)$. *To this end, we determine the minimum and maximum values of the unique positive root of* (2.46) *for* $\xi\in(0,1)$. *Now,* $x=1$ *for* $\xi=0$ *and* $x=1/2$ *for* $\xi=1$, *therefore, by the continuity of the roots of a polynomial (Horn and Johnson 1985),*

$$\underline{\omega}\le\underline{\theta}\left(\frac{\overline{\theta}}{\underline{\theta}}\right)^{-\frac{1}{k}\ln 2}<\overline{\theta}\le\overline{\omega}\,.$$

Also, if we denote by $p_\xi(x)$ *the left side of* (2.46), *then we have that*

$$p_\xi(x+1/2)=2x^3+3(\xi^2+1)x^2+(4\xi^4+\xi^2-1/2)x+(2\xi^4-5/4\xi^2-3/4)$$
$$p_\xi(x+1)=2x^3+3(\xi^2+2)x^2+4(\xi^4+\xi^2+1)x+4\xi^4$$

The first two higher coefficients of $p_\xi(x+1/2)$ *are positive for* $\xi \in (0,1)$*, and its linear coefficient is negative for* $\xi \in (0,1)$ *(since it has roots at* $\xi = \pm 1, \pm j\sqrt{(}6)/4$*), therefore by Descartes' rule of signs, the positive root of* $p_\xi(x)$ *is not less than* $1/2$ *for* $\xi \in (0,1)$*. Similarly, all coefficients of* $p_\xi(x+1)$ *are positive, hence again by Descartes' rule of signs, the positive root of* $p_\xi(x)$ *is not greater than* 1 *for* $\xi \in (0,1)$*. This implies that*

$$[\underline{\omega}, \overline{\omega}] = \left[\underline{\theta} \left(\frac{\overline{\theta}}{\underline{\theta}} \right)^{-\frac{1}{k}\ln 2}, \overline{\theta} \right].$$

For our resonant system example where $\underline{\theta} = 0.5$ *and* $\overline{\theta} = 2$ *we have that* $\operatorname{supp} \Phi_u^{opt} \subseteq [2^{-3/2}, 2] \approx [0.3536, 2]$.

Theorem 2.4.1 (Existence of an optimal input) *For the problem stated in* (2.45)*, there exists at least one optimal input.*

Proof. This proof follows similarly to the proof of Theorem 2.3.1. □

Theorem 2.4.2 (Finite support of the optimal input spectrum) *For the problem stated in* (2.45)*, the optimal input has finite support, i.e.* $\operatorname{supp} \tilde{\tilde{\Phi}}_u^{opt}$ *is finite. This implies that the optimal solution of problem* (2.43) *has finite support as well.*

Proof. This proof follows similarly to the proof of Theorem 2.3.3. □

2.5 Generalisation to Multi-parameter Problems

For the multi-parameter case we return to the general expression for $\overline{M}(\theta, \Phi)$ given in (2.5) and (2.6). Again for simplicity, we assume white noise and hence restrict our attention to the parameters in the system transfer function, i.e. θ, although the extension to coloured noise offers no additional difficulties. We convert this problem into an approximate discrete form, as in Section 2.3, by discretising the input and parameter spaces. We write

$$Q_k(\mathbf{E}) := \sum_m A_{km} E_m, \qquad \theta_k \in \Theta$$

as an approximation to the integral in (2.6) i.e. Q_k is the information matrix corresponding to the k^{th} (discretised) element θ_k of the parameter set Θ, the index m denotes the frequency and E_m is the input energy at the m^{th} frequency.

There exist many possible choices for the inner design criterion $J(\overline{M}(\theta, \Phi_u), \theta)$ in the multi-parameter case (see Section 2.2.2). Three alternatives are discussed below.

2.5.1 Minimal Eigenvalue

The use of the minimum eigenvalue of the information matrix as a design criterion for *nominal* experiment design has previously been studied (Mareels et al. 1987). For the *robust* case, we propose to optimise the worst case of the following related criterion which uses the minimum eigenvalue of a scaled version of the information matrix,

$$J_1(\overline{M}(\theta,\Phi_u),\theta) := (\lambda_{\min}\{S_\theta \overline{M}(\theta,\Phi_u) S_\theta\})^{-1} \tag{2.47}$$

where $\lambda_{\min}$ denotes the minimum eigenvalue and S_θ is a parameter dependent scaling matrix. One possible choice for S_θ is $\mathrm{diag}[\theta_1 \ \cdots \ \theta_m]$. The motivation for this choice is that $\overline{M}(\theta,\Phi_u)^{-1}$ is a measure of the parameter covariance matrix. Hence $S_\theta^{-1}\overline{M}(\theta,\Phi_u)^{-1}S_\theta^{-1}$ is the covariance normalised by the nominal values of each parameter. Therefore it is a measure of the relative error. This seems to be an important property in the robust design context (where we maximise over $\theta \in \Theta$) since it ensures that one is maximising (over Θ) the relative errors. These errors are normalised and thus better scaled for comparison purposes.

Another useful property of $J_1(\overline{M}(\theta,\Phi_u),\theta)$ is that, due to the normalisation by S_θ, the scaled information matrix does not depend on the system gain. This simplifies the problem of discretisation of the set Θ by eliminating one degree of freedom (the gain). This also makes sense, heuristically speaking, since the system gain simply scales the output.

2.5.2 Relative Frequency Domain Errors

This criterion is motivated by robust control (Zhou et al. 1996). It is well known (Goodwin et al. 2001), that the achieved sensitivity, S, is related to the nominal sensitivity, S_0, via

$$S = \frac{S_0}{1 + T_0\,\Delta G/G}$$

where T_0 is the nominal complementary sensitivity and $\Delta G/G$ is the relative error in G. Indeed, this leads to the well known sufficient condition for robust stability, namely $\|T_o\,\Delta G/G\|_\infty < 1$.

Say we put an upper bound on $\|T_0\|_\infty$, then we see that what is important is the infinity norm of the relative error in G, $\Delta G/G$. Then, noting that the covariance of all unbiased estimates of θ are lower bounded by $(N\overline{M})^{-1}$ where N is the number of data points (Goodwin and Payne 1977), we can obtain a measure of the size of $\Delta G/G$ as

$$\left\| \frac{N\mathrm{E}\{|\Delta G|^2\}}{|G|^2} \right\|_\infty = \max_\omega \frac{\frac{\partial G(j\omega)^T}{\partial \theta}\,\overline{M}^{-1}\,\frac{\partial G(-j\omega)}{\partial \theta}}{|G(j\omega)|^2} =: J_2(\overline{M}(\theta,\Phi_u),\theta). \tag{2.48}$$

Note that here we use the per-sample information matrix $\overline{M}$.

It is readily seen that $J_2(\overline{M}(\theta,\Phi_u),\theta)$ is a dimensionless quantity. Thus the associated experiment design is independent of the system gain in the same way that this was true for J_1 (see (2.47)).

Remark 2.5.1 *We see that the criterion J_2 has the form*

$$J_2(\overline{M}(\theta,\Phi_u),\theta) = \max_{\omega} g(\omega)^* \overline{M}^{-1} g(\omega)$$

where

$$g(\omega) := \frac{\partial G(-j\omega)/\partial\theta}{|G(j\omega)|}. \tag{2.49}$$

Thus (2.48) *maximises $x^*\overline{M}^{-1}x$ where x is restricted to the particular set of vectors given in* (2.49). *This can be compared with $J_1(\overline{M}(\theta,\Phi_u),\theta)$ which is actually equivalent to maximising $y^*\overline{M}^{-1}y$ over the set of vectors y where $y = S_\theta Z$ and $Z^*Z = 1$.*

2.5.3 A Criterion Related to the ν Gap

Hildebrand and Gevers (2003a;b) have suggested the following criterion for nominal experiment design such that the worst case ν gap is minimised,

$$\begin{aligned} J_3(\overline{M}(\theta,\Phi_u),\theta) &:= \max_{\omega} \left. \frac{\lambda_{\max}\left\{ \begin{bmatrix} \mathrm{Re}\,\frac{\partial G}{\partial\theta} \\ \mathrm{Im}\,\frac{\partial G}{\partial\theta} \end{bmatrix} \overline{M}^{-1} \begin{bmatrix} \mathrm{Re}\,\frac{\partial G}{\partial\theta} & \mathrm{Im}\,\frac{\partial G}{\partial\theta} \end{bmatrix} \right\}}{[1+|G|^2]^2} \right|_{\omega} \\ &= \max_{\omega} \frac{\lambda_{\max}\begin{bmatrix} R_\omega^T\overline{M}^{-1}R_\omega & R_\omega^T\overline{M}^{-1}I_\omega \\ I_\omega^T\overline{M}^{-1}R_\omega & I_\omega^T\overline{M}^{-1}I_\omega \end{bmatrix}}{[1+|G_\omega|^2]^2} \end{aligned} \tag{2.50}$$

where the subscript ω denotes 'frequency ω', $R_\omega := \mathrm{Re}\{\partial G(\omega)/\partial\theta\}$ and $I_\omega := \mathrm{Im}\{\partial G(\omega)/\partial\theta\}$.

Remark 2.5.2 *Not surprisingly, there is a connection between $J_2(\overline{M}(\theta,\Phi_u),\theta)$ and $J_3(\overline{M}(\theta,\Phi_u),\theta)$ since both are motivated by robust control. Specifically, it is readily seen that*

$$\frac{\partial G}{\partial\theta}^T \overline{M}^{-1} \frac{\partial \bar{G}}{\partial\theta} = R_\omega^T\overline{M}^{-1}R_\omega + I_\omega^T\overline{M}^{-1}I_\omega = \mathrm{tr}\,L_\omega$$

where L_ω appears in (2.50)*, i.e.*

$$L_\omega := \begin{bmatrix} R_\omega^T\overline{M}^{-1}R_\omega & R_\omega^T\overline{M}^{-1}I_\omega \\ I_\omega^T\overline{M}^{-1}R_\omega & I_\omega^T\overline{M}^{-1}I_\omega \end{bmatrix}.$$

We notice that $\lambda_{\max}(L_\omega) \leq \operatorname{tr} L_\omega \leq 2\lambda_{\max}(L_\omega)$. *Hence, we see that the criteria* $J_2(\overline{M}(\theta,\Phi_u),\theta)$ *and* $J_3(\overline{M}(\theta,\Phi_u),\theta)$ *are loosely connected. Moreover, Remark 2.5.1 links both criteria to* $J_1(\overline{M}(\theta,\Phi_u),\theta)$.

A potential issue with the criterion $J_3(\overline{M}(\theta,\Phi_u),\theta)$ *is that, unlike* $J_1(\overline{M}(\theta,\Phi_u),\theta)$ *and* $J_2(\overline{M}(\theta,\Phi_u),\theta)$, *it is not dimensionless. This is not an issue in the case of nominal experiment design. However, it could be a problem with respect to robust design when one wishes to compare the criteria for different values of* $\theta \in \Theta$. *A possible normalisation for* $J_3(\overline{M}(\theta,\Phi_u),\theta)$ *is given in (Welsh et al. 2006).*

Remark 2.5.3 *Notice that the above criteria are convex in terms of* Φ_u. *This follows since the supremum of a family of convex functions is itself convex.*

2.5.4 Numerical Results

To illustrate the merits of robust optimal experiment design on a realistic example, we have evaluated a discretised approximation to each of the criteria $J_1(\overline{M},\theta)$, $J_2(\overline{M},\theta)$ and $J_3(\overline{M},\theta)$ on a multi-parameter example. The system is given by $H(s) = 1$ and

$$G(s) = \frac{K}{s^2 + a_1 s + a_0}\,.$$

We assume prior knowledge of the parameters as follows:

$$\theta_1 := a_1 \in [1,2], \quad \theta_2 := a_0 \in [1,9], \quad \theta_3 := K \in [1,2]\,.$$

The parameter and frequency ranges were all divided into logarithmically spaced grids for the optimisation. For our example we chose each range to contain 20 values. The Matlab® optimisation toolbox was utilised to carry out the min-max designs.

In all our simulations we approximated the integral in (2.6) by the following discretisation

$$\overline{M}(\theta,\Phi_u) = \int_0^\infty \operatorname{Re}\{Q(\omega)\}\,\Phi_u(\omega)d\omega \approx \sum_{n=1}^{20} \operatorname{Re}\{Q(\omega_n)\}\int_{\Delta_n}\Phi_u(\omega)d\omega = \sum_{n=1}^{20}\operatorname{Re}\{Q(\omega_n)\}E_n$$

where

$$Q(\omega) := \frac{\partial G(j\omega)}{\partial\theta}|H(j\omega)|^{-2}\left[\frac{\partial G(j\omega)}{\partial\theta}\right]^H$$

and $E_n := \int_{\Delta_n}\Phi_u(\omega)d\omega$ is the input energy in the frequency range Δ_n. We have chosen $\Delta_n := \omega_{n+1} - \omega_n$, where $\omega_n = 0.3(10)^{(n-1)/20}$.

The discrete approximation to the robust optimal input was found for each of the criteria $\max_\theta J_1$, $\max_\theta J_2$ and $\max_\theta J_3$. For those criteria depending on a maximum over a frequency range (i.e. $\max_\theta J_2$ and $\max_\theta J_3$), we limited ω to $[0.3, 3]$ [rad/s] (this choice was motivated by the region of possible pole locations).

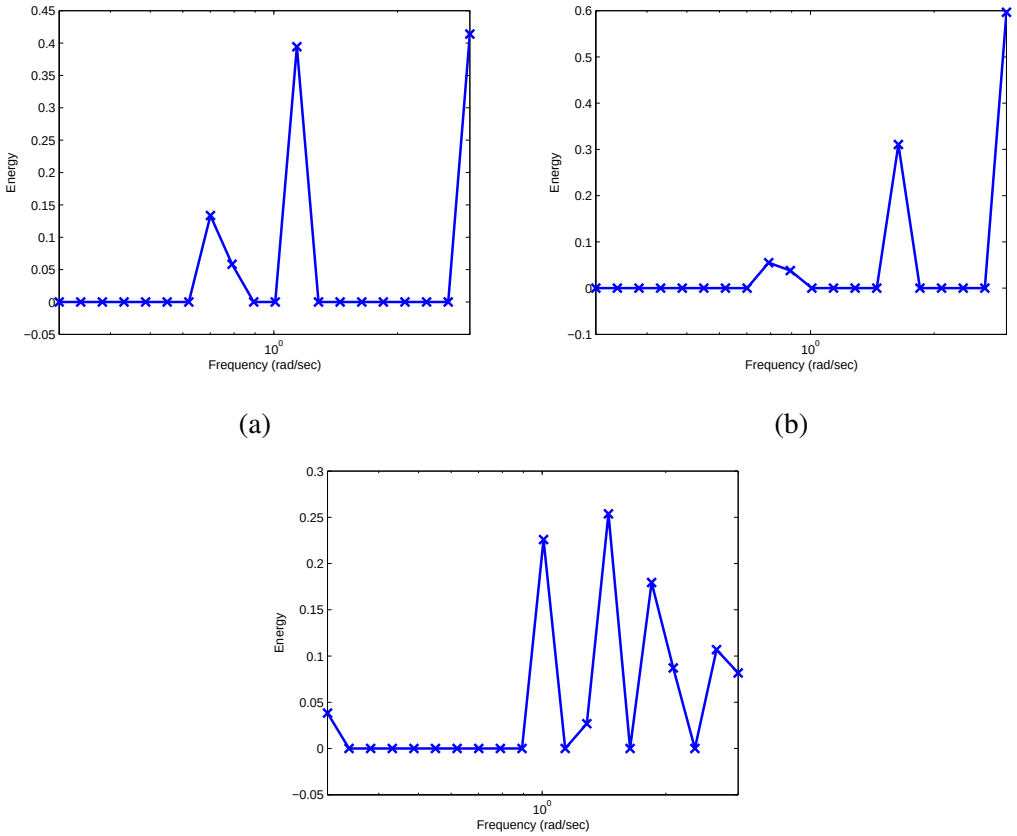

Figure 2.7. Values of the input energy E for the discretised robust optimal input obtained from criteria: (a) $\max_\theta J_1$, (b) $\max_\theta J_2$, and (c) $\max_\theta J_3$.

Results are shown in Figure 2.7. Figure 2.7(a) shows the discretised optimal input energy distribution for criterion $\max_\theta J_1$. Notice again that the input has finite support. Figure 2.7(b) shows the discretised optimal input energy for $\max_\theta J_2$. We see from Figures 2.7 (a) and (b) that the optimal input is approximately the same whether we use $\max_\theta J_1$ or $\max_\theta J_2$. Figure 2.7(c) shows the discretised optimal input energy distribution for $\max_\theta J_3$. Finally, Table 2.3 compares the cost functions obtained with different design criteria for different inputs. The table also shows the values of the corresponding cost functions for bandlimited '$1/f$' noise and bandlimited white noise.

The use of bandlimited '$1/f$' noise is motivated by earlier results in Section 2.3 where we showed that bandlimited '$1/f$' noise was near optimal for robust experiment design for the illustrative problem.

The results in Table 2.3 have been normalised so that the optimal design gives a value of 1. Observations from this table are:

1. Bandlimited '$1/f$' noise gives much better results in the multi-parameter case for all criteria

Table 2.3. Values of Cost for the Criteria

	$\max_\theta J_1$	$\max_\theta J_2$	$\max_\theta J_3$
Optimal input for $\max_\theta J_1$	1	1.73	1.55
Optimal input for $\max_\theta J_2$	1.14	1	1.32
Optimal input for $\max_\theta J_3$	1.78	2.43	1
Bandlimited '$1/f$' noise	2.08	3.13	1.22
Bandlimited white noise	5	8.8	1.93

than does bandlimited white noise. We believe this to be a surprising and interesting observation! Note that the observation is consistent with results obtained earlier for the first and second order examples.

2. For $\max_\theta J_1$, the discretised robust optimal input is approximately twice as good as bandlimited '$1/f$' noise and about 5 times as good as bandlimited white noise.

3. For $\max_\theta J_2$, the discretised optimal input is about 1.7 times better than the optimal input for $\max_\theta J_1$. The robust optimal input for $\max_\theta J_2$ is about 3 times as good as bandlimited '$1/f$' noise and almost 9 times as good as bandlimited white noise.

4. For $\max_\theta J_3$, the criterion seems to be less sensitive to the test signal.

5. We also notice that the discretised optimal inputs for $\max_\theta J_1$ and $\max_\theta J_2$ are quite similar whilst the discretised optimal result for $\max_\theta J_3$ is considerably different.

To further motivate our robust design approach, we also tried nominal experiment design for this example. Here we assumed nominal parameter values in the centre of the a-priori region, i.e. we chose $\hat{\theta}_1 = 1.5$, $\hat{\theta}_2 = 5$ and $\hat{\theta}_3 = 1.5$. We then found the corresponding exact nominal optimal input using J_1 as our design criterion. For this input, in the case where the true parameters take any value in the a-priori region, we found that the range of the cost is 30 to 2700. This lies in stark contrast to the range of cost for the discretised robust optimal input which turns out to be 26 to 400. Thus we see that the discretised robust optimal input gives almost 700% improvement in the worst case performance relative to the nominal optimal input. However, this is achieved with a negligible change (10%) in the best case performance, which provides a strong incentive to move to a robust design criterion.

2.6 Summary

We have described and analysed a min-max approach to robust optimal experiment design for dynamic system identification. Several properties were established for the robust optimal input of the one-parameter examples, namely existence and uniqueness, and the fact that the spectra have finite support. Also, we evaluated and compared several different design criteria. Three illustrative examples have been presented, two with a scalar parameter and one with multiple parameters, showing that substantial improvements in the worst case performance are achieved using a discretised robust design procedure relative to that achieved via a nominal experiment design procedure. It was also shown that bandlimited '$1/f$' noise performs well when compared to white inputs such as PRBS. This indicates that it may be valuable to investigate binary inputs whose energy distribution approximates bandlimited '$1/f$' noise, as examined in Chapter 9.

2.7 Appendix: Explicit Form of the Robust Optimal Input for the First Order Example

To obtain an explicit solution to the robust experiment design problem for the illustrative first order example, we use ideas from (Bohnenblust et al. 1950, Karlin 1957, McKinsey 1952).

From Section 2.3.2, the problem can be stated as

$$\mu_y^{opt} = \arg \max_{\mu_y \in \mathscr{S}([0,1])} \min_{\mu_x \in \mathscr{S}([0,1])} \int_0^1 \left[\int_0^1 \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2} \, \mu_x(x)\, dx \right] \mu_y(y)\, dy . \tag{2.51}$$

The kernel of the associated game is $F(x,y) := f(x-y)$, where $f(u) := e^{ku}/(e^{ku}+1)^2$.

Let us denote the optimal mixed strategies of players x and y of game (2.51) by μ_x^{opt} and μ_y^{opt} $(= \widetilde{\Phi}_u^{opt})$, respectively. Also, let

$$\Psi_x(x) := \int_0^1 \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2} \, \mu_y^{opt}(y)\, dy$$
$$\Psi_y(y) := \int_0^1 \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2} \, \mu_x^{opt}(x)\, dx .$$

When k is very small, $F(x,y)$ is strictly concave in $y \in [0,1]$ for every $x \in [0,1]$. Thus, $\Psi_y(y)$ (which appears in (2.51) by taking $\mu_x = \mu_x^{opt}$) is a linear combination of strictly concave functions in y, therefore it is strictly concave as well, and has a unique maximum at $y = 1/2$ (because of the symmetry of F). This means that

$$\widetilde{\Phi}_u^{opt}(y) = \mu_y^{opt}(y) = \delta(y - 1/2) . \tag{2.52}$$

Note that this coincides with the single sinusoid robust design given in (Walter and Pronzato 1997, page 339). On the other hand, by the Minimax Theorem (Glicksberg 1950), μ_x^{opt} must satisfy

$$\mu_x^{opt} = \arg \min_{\mu_x \in \mathscr{S}([0,1])} \int_0^1 \frac{e^{k(x-1/2)}}{(e^{k(x-1/2)}+1)^2}\, \mu_x(x)\, dx$$

that is,

$$\mu_x^{opt}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)\,. \tag{2.53}$$

If we increase the value of k, $F(x,y)$ eventually ceases to be strictly concave in y for every x. This implies that there is a number $k_1 \in \mathbb{R}^+$ such that, for $k > k_1$, $\Psi_y(y)$ has at least two maxima. This value can be computed by setting the second derivative of $\Psi_y(y)$ equal to zero for $y = 1/2$, which gives an equation whose only positive root is $k_1 = 2\ln(2+\sqrt{3}) \approx 2.6339$.

Thus, (2.52) and (2.53) hold for $0 < k \leq k_1$. However, for values of k slightly greater than k_1, (2.53) still holds, hence $\Psi_y(y)$, with μ_x^{opt} given by (2.53), has two maxima at, say, y' and $1-y'$, where

$$\frac{\partial}{\partial y}\left[\int_0^1 \frac{e^{k(x-y)}}{(e^{k(x-y)}+1)^2}\, \mu_x^{opt}(x)\, dx\right]\Bigg|_{y=y'} = 0\,.$$

This equation has only one real solution y' between 0 and $1/2$, from which we obtain

$$\widetilde{\Phi}_u^{opt}(y) = \mu_y^{opt}(y) = \frac{1}{2}\delta(y-y') + \frac{1}{2}\delta(y-[1-y'])\,. \tag{2.54}$$

Expressions (2.53) and (2.54) hold as long as $\Psi_y(y)$ has two maxima, which is true while (2.53) satisfies

$$\begin{aligned}\mu_x^{opt} &= \arg \min_{\mu_x \in \mathscr{S}([0,1])} \int_0^1 \Psi_x(x)\mu_x(x)\, dx \\ &= \arg \min_{\mu_x \in \mathscr{S}([0,1])} \int_0^1 \frac{1}{2}\left[\frac{e^{k(x-y')}}{(e^{k(x-y')}+1)^2} + \frac{e^{k(x-(1-y'))}}{(e^{k(x-(1-y'))}+1)^2}\right]\mu_x(x)\, dx\,.\end{aligned} \tag{2.55}$$

$\Psi_x(x)$ has local minima at $x = 0$, $x = 1$ and $x = 1/2$, so (2.53) and (2.54) hold for $k_1 < k \leq k_2$, where $k_2 \in \mathbb{R}^+$ is such that

$$\begin{aligned}&\frac{\partial}{\partial y}\left[\int_0^1 \frac{e^{k_2(x-y)}}{(e^{k_2(x-y)}+1)^2}\, \mu_x^{opt}(x)\, dx\right]\Bigg|_{y=y'} = 0 \\ &\frac{e^{k_2(1/2-y')}}{(e^{k_2(1/2-y')}+1)^2} + \frac{e^{k_2(y'-1/2)}}{(e^{k_2(y'-1/2)}+1)^2} = \frac{e^{-k_2 y'}}{(e^{-k_2 y'}+1)^2} + \frac{e^{-k_2(1-y')}}{(e^{-k_2(1-y')}+1)^2}.\end{aligned}$$

The first equation gives y' in terms of $k = k_2$, and the last equation gives the minimal value of k, say k_2, for which $x = 1/2$ is a global minimum of $\Psi_x(x)$. This system of equations gives $k_2 \approx 3.6855$.

For values of k slightly higher than k_2, (2.54) still holds, however (2.53) becomes

$$\mu_x^{opt}(x) = \alpha\delta(x) + (1-2\alpha)\delta(x-1/2) + \alpha\delta(x-1) \tag{2.56}$$

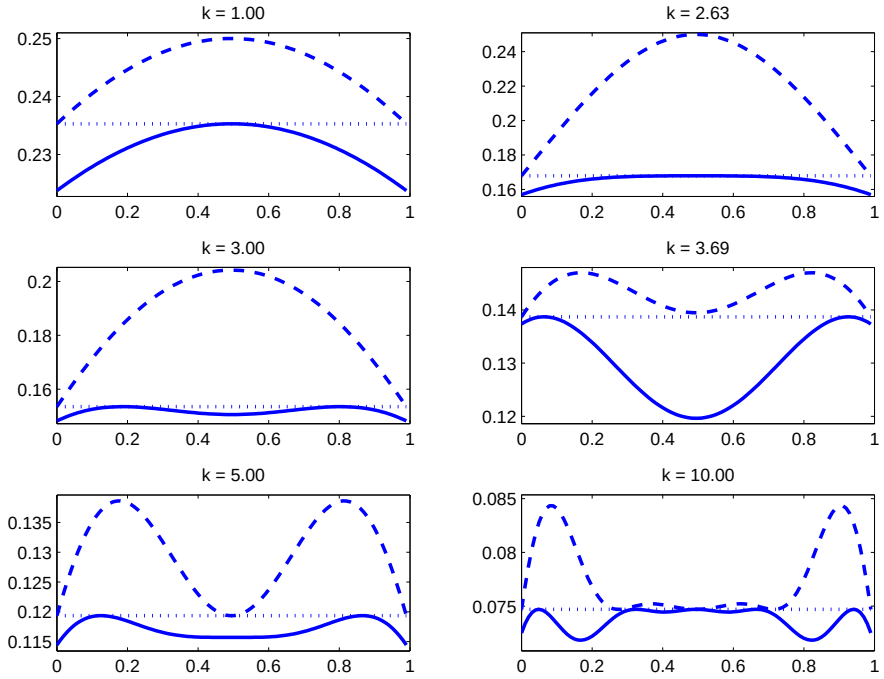

Figure 2.8. Ψ_x (dashed) and Ψ_y (solid) for various values of k. The dotted line corresponds to the average value, V_m.

where $\alpha \in [0, 0.5]$. This expression satisfies (2.55) for every α, however we must assure that (2.54) still satisfies (2.51) when using (2.56), which happens for a particular choice of α. One way to obtain the optimal value of α is to substitute (2.56) into (2.51) and to force the derivative of the integral in (2.51) with respect to y equal to zero for $y = y'$ (see the proof of Theorem 12.5 of (McKinsey 1952) for an example of how to use this idea in convex games).

Continuing in this way, we can see that it is possible, at least in principle, to obtain an 'explicit' solution to the robust experiment design problem.

In Figure 2.8 we can see the shapes of Ψ_x and Ψ_y for various values of k. These figures have been generated by an LP algorithm as explained in Section 2.3.4. We can also see from the figures that the minimum value of Ψ_x coincides with the maximum value of Ψ_y; this is a consequence of the Minimax Theorem, which states that both values are equal, and their common value is the so-called average value, V_m, of a game on the unit square with kernel F.

It can be further shown (Karlin 1957) that the number of support points of both μ_x^{opt} and μ_y^{opt} tend to infinity as $k \to \infty$, that they are lower semicontinuous in k and that they differ by at most 1. Namely,

the number of support points of μ_x^{opt} is not less than that of μ_y^{opt}.

CHAPTER 3

OPTIMAL EXPERIMENT DESIGN WITH DIFFUSE PRIOR INFORMATION

3.1 Introduction

In this chapter we address the general question: What would be a 'good' initial experiment in order to estimate the parameters of a system if we have very little (i.e. diffuse) prior information? Here we consider diffuse prior information to be, that, the 'interesting part' of the system frequency response lies in an interval $[a,b]$. This implies that we are seeking an experiment which is 'good' over a broad class of systems. A possible answer to this question was proposed in Chapter 2, i.e. the experiment should consist of bandlimited '$1/f$' noise. The results presented in this chapter provide theoretical support for a bandlimited '$1/f$' noise input.

Due to the fact that we are considering a broad class of systems, we first discuss the problem of measuring the 'goodness' of an experiment utilising a system independent criterion. Then, considering this measure of 'goodness', we propose some desirable properties that one would expect such a measure to possess. Next, a preliminary result is developed for selecting a cost function which satisfies the desirable properties. With this result we design a suitable cost function to take into account that only diffuse prior information is available. The final form of the cost function that satisfies these desired properties is then specified. Finally we show that bandlimited '$1/f$' noise is an optimal input signal according to the cost function developed for diffuse prior information. The advantages of bandlimited '$1/f$' noise are then illustrated by means of an example.

3.2 Optimal Experiment Design

3.2.1 A Measure of the 'Goodness' of an Experiment

With the aim of designing an experiment which is 'good' for a broad class of systems, we need a measure of 'goodness' for an experiment which is system independent. To construct such a measure, we make use of a result from (Ljung 1985) which shows that, for a broad class of linear systems, the

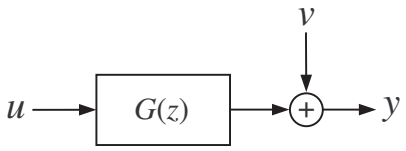

Figure 3.1. Block diagram describing the relationship between the input u, the noise v and the output y of the system G to be identified.

variance of the error in the estimated discrete time frequency response takes the asymptotic (in both system order and data points) form,

$$\mathrm{Var}(\hat{G}(e^{j\omega})) = K\frac{\Phi_v(\omega)}{\Phi_u(\omega)}, \qquad \omega \in [0, 2\pi], \tag{3.1}$$

where Φ_v is the noise spectral density and Φ_u is the input spectral density. Here K is a function of the number of system parameters and the number of observations. Figure 3.1 shows the relationship between the input u, the noise v and the output y of the system, i.e.

$$y_t = G(z)u_t + v_t,$$

where G is the transfer function of the system.

An interesting and highly desirable property of (3.1) is that it is essentially independent of the system parameters, as it depends only on Φ_v and Φ_u. Of course, Φ_v is somewhat problematic since it would be desirable to have (3.1) independent of the real characteristics of the noise. This will also be part of our consideration.

As argued in Chapter 2, absolute variances are not particularly useful when one wants to carry out an experiment design that applies to a broad class of systems. Specifically, an error of standard deviation equal to 10^{-2} in a variable of nominal size 1 would be considered insignificant, whereas the same error standard deviation of 10^{-2} in a variable of nominal size 10^{-3} would be considered catastrophic. Hence, it seems preferable to work with relative errors. Thus, if $|G(e^{j\omega})|$ is the magnitude of the frequency response of the system at frequency ω, then equation (3.1) suggests that the relative variance at frequency ω is given by

$$\text{Rel. Var}\{\hat{G}(e^{j\omega})\} = K\frac{\Phi_v(\omega)}{\Phi_u(\omega)|G(e^{j\omega})|^2}, \qquad \omega \in [0, 2\pi].$$

Finally, rather than consider just a single frequency ω, we take an 'average' measure over a range of

frequencies. This leads to a general measure of the 'goodness' of an experiment,

$$J(\Phi_u) = \int_a^b F(\mathrm{Var}(\hat{G}(e^{j\omega}))/|G(e^{j\omega})|^2)W(\omega)d\omega = \int_a^b F\left(\frac{K\Phi_v(\omega)}{\Phi_u(\omega)|G(e^{j\omega})|^2}\right)W(\omega)d\omega, \quad (3.2)$$

where F and W are functions to be defined later, and $0 < a < b < 2\pi$.

In the next section we propose some desirable properties of the functions F and W, such that they can provide us with a sensible answer to the problem of designing an experiment with very little prior information.

3.2.2 Desirable Properties of the Cost Function

We consider two sets of criteria in defining the desirable properties of the cost function. The first relates principally to the function F, the second to the function W in (3.2). In addition to these properties, we assume that $F \in C^1([a,b],\mathbb{R}_0^+)$ and $W \in C^1([a,b],\mathbb{R}^+)$, where $C^1(X,Y)$ is the space of all functions from $X \subseteq \mathbb{R}$ to $Y \subseteq \mathbb{R}$ having a continuous derivative.

Criteria A

It is reasonable to consider a cost function, such as (3.2), whose minimum is achieved by a function which does not depend on the actual system characteristics. The reason for this is that the system characteristics are typically unknown at the time the experiment is performed, and in fact, it is the purpose of the experiment to reveal this information.

On the other hand, the cost function (3.2) should be a measure of the 'size' of the variance in the estimation of the system frequency response. Hence, loosely speaking, the cost function should increase when there is an increase of the variance at any frequency.

The above argument implies that the function F in the measure (3.2) should be chosen to satisfy the following requirements:

A.1) The optimal experiment, Φ_u^{opt}, which minimises J in (3.2), should be independent of the system $|G(e^{j\omega})|^2$ and the noise dynamics Φ_v.

A.2) The integrand in (3.2) should increase if the variance $\mathrm{Var}(\hat{G}(e^{j\omega}))$ increases at any frequency. This implies that F should be a monotonically increasing function.

Criterion B

Many properties of linear systems depend on the *ratio* of poles and zeros rather than on their absolute locations in the frequency domain (Bode 1945, Goodwin et al. 2001, Seron et al. 1997). This implies that if we scale the frequency ω by a constant, the optimal input must keep its shape, as the poles and zeros of the new system will have the same ratios as before. This invariance property must be reflected in the weighting function W, which has to give equal weight to frequency intervals whose endpoints are in the same proportion. Thus, the weighting function W should be such that for every $0 < \alpha < \beta < 2\pi$ and every $k > 0$ such that $0 < k\alpha < k\beta < 2\pi$ we have that

$$\int_{\alpha}^{\beta} W(\omega)d\omega = \int_{k\alpha}^{k\beta} W(\omega)d\omega.$$

Note that W can be considered, from a practical point of view, as a weighting function that allows the control engineer to define at which frequencies it would be preferable to obtain a better model (depending on the control requirements, but not necessarily on the true system characteristics).

3.2.3 Constraints

Our goal is to optimise a cost function such as (3.2), where the input Φ_u is constrained in some fashion. Typically, in experiment design, a constraint is placed on the total input energy (Goodwin and Payne 1977). Thus, here we optimise $J(\Phi_u)$ subject to a constraint of the form

$$\int_{a}^{b} \Phi_u(\omega)d\omega = 1. \tag{3.3}$$

Specifically our goal is to adjust F and W such that the optimal experiment that minimises the cost (3.2) subject to the constraint (3.3) satisfies the criteria A.1, A.2 and B in Section 3.2.2.

3.3 Preliminary Technical Result

Motivated by the need for a measure to be independent of the system and such that criteria A.1, A.2 and B are met subject to a constraint on the input, we establish the following result:

Lemma 3.3.1 (Choice of F to satisfy Criterion A.1) *For $0 < a < b < 2\pi$, let $g, F \in C^1([a,b], \mathbb{R}_0^+)$ and $W \in C^1([a,b], \mathbb{R}^+)$. Define, if it exists,*

$$f^{opt}(g) := \arg \min_{\substack{f \in C^1([a,b],\mathbb{R}^+) \\ \int_a^b f(x)dx = 1}} \int_a^b F\left(\frac{g(x)}{f(x)}\right) W(x)dx. \tag{3.4}$$

If $f^{opt}(g)$ does not depend on g, then there are constants $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$F(y) = \alpha \ln y + \beta, \quad \inf_{x\in[a,b]} \frac{g(x)}{f(x)} \leq y \leq \sup_{x\in[a,b]} \frac{g(x)}{f(x)},$$

and $f^{opt} = \gamma W$.

Proof. Let $g, F \in C^1([a,b], \mathbb{R}_0^+)$ and $W \in C^1([a,b], \mathbb{R}^+)$ be fixed such that $f^{opt}(g)$, as defined in (3.4), exists. Then, by (Luenberger 1969, Section 7.7, Theorem 2), there is a constant $\lambda \in \mathbb{R}$ for which $f^{opt}(g)$ is a stationary point of

$$J_\lambda(f) := \int_a^b F\left(\frac{g(x)}{f(x)}\right) W(x)dx + \lambda \int_a^b f(x)dx.$$

Thus, for any $h \in C^1([a,b], \mathbb{R}_0^+)$ we have that the Gateaux differential of J_λ at f^{opt} with increment h vanishes (Luenberger 1969), i.e. $\delta J_\lambda(f^{opt}; h) = 0$, which means (Luenberger 1969, Section 7.5) that

$$\int_a^b \left[F'\left(\frac{g(x)}{f^{opt}(x)}\right) \left(-\frac{g(x)}{(f^{opt}(x))^2}\right) W(x) + \lambda \right] h'(x)dx = 0,$$

thus, by (Luenberger 1969, Section 7.5, Lemma 1),

$$F'\left(\frac{g(x)}{f^{opt}(x)}\right) W(x) \frac{g(x)}{(f^{opt}(x))^2} = \lambda, \quad x \in [a,b]. \tag{3.5}$$

Let $l(x) := g(x)/f^{opt}(x)$, then (3.5) can be written as

$$F'(l(x))l(x) = \lambda \frac{f^{opt}(x)}{W(x)}, \quad x \in [a,b]. \tag{3.6}$$

The left side of (3.6) depends on g, but the right does not (because of the assumption on the independence of f^{opt} upon g). Thus both sides are equal to a constant, say, $\alpha \in \mathbb{R}$, which implies that

$$F'(l(x)) = \frac{\alpha}{l(x)}, \quad x \in [a,b].$$

Now, by integrating both sides with respect to l between $\inf_{x\in[a,b]} l(x)$ and $\sup_{x\in[a,b]} l(x)$, we obtain

$$F(l(x)) = \alpha \ln l(x) + \beta, \quad x \in [a,b]$$

for some constant $\beta \in \mathbb{R}$.

On the other hand, we have that

$$\lambda \frac{f^{opt}(x)}{W(x)} = \alpha,$$

therefore if $\gamma := \alpha/\lambda$, we conclude that $f^{opt} = \gamma W$. □

3.4 Design of the Cost Function

3.4.1 Choice of the Frequency-Wise Model Quality Function

Here we use the result of the previous section to derive a suitable function F which satisfies Criteria A.1 and A.2, and determines the optimal input signal for the resulting cost function.

We first examine the choice of the function F in (3.2). Now we may take, without loss of generality, $\alpha = 1$ and $\beta = 0$ for the function F given by Lemma 3.3.1 (if we extend its range to $\mathbb{R}$). The reason for this, according to Lemma 3.3.1, is that every cost function (3.2) satisfying Criteria A.1 and A.2 is minimised by the same $f \in C^1([a,b],\mathbb{R}^+)$. Thus, the cost function can be written as

$$J(\Phi_u) = \int_a^b \ln\left(\frac{K\Phi_v(\omega)}{\Phi_u(\omega)|G(e^{j\omega})|^2}\right) W(\omega)d\omega. \tag{3.7}$$

It is then relatively straightforward to optimise (3.7) subject to the constraint (3.3). Indeed, by Lemma 3.3.1 the optimal experiment is essentially given by a scaled version of W, i.e.

$$\Phi_u^{opt}(\omega) = \frac{1}{\int_a^b W(x)dx} W, \qquad \omega \in [a,b]. \tag{3.8}$$

The following Lemma establishes that Φ_u^{opt} gives not only an extremum, but a global minimum for the cost function (3.7).

Lemma 3.4.1 (Global optimality of Φ_u^{opt}) *The function Φ_u^{opt} defined in* (3.8) *gives the global minimum of the cost function* (3.7)*. In other words, for $0 < a < b < 2\pi$, let $W \in C^1([a,b],\mathbb{R}^+)$, then,*

$$\Phi_u^{opt} = \arg \min_{\substack{\Phi_u \in C^1([a,b],\mathbb{R}^+) \\ \int_a^b \Phi_u(\omega)d\omega = 1}} \int_a^b \ln\left(\frac{K\Phi_v(\omega)}{\Phi_u(\omega)|G(e^{j\omega})|^2}\right) W(\omega)d\omega.$$

Proof. The cost function (3.7) can be written as

$$J(\Phi_u) = C - \int_a^b \ln(\Phi_u(\omega))W(\omega)d\omega,$$

where C is a constant, independent of Φ_u, such that

$$C := \int_a^b \ln\left(\frac{K\Phi_v(\omega)}{|G(e^{j\omega})|^2}\right) W(\omega)d\omega.$$

Now, if Φ_u is any function in $C^1([a,b],\mathbb{R}^+)$ such that $\int_a^b \Phi_u(\omega)d\omega = 1$, then by (3.8) we have that

$$\begin{aligned}
J(\Phi_u) &= C - \int_a^b \ln[\Phi_u^{opt}(\omega) + (\Phi_u(\omega) - \Phi_u^{opt}(\omega))]W(\omega)d\omega \\
&= C - \int_a^b \ln(\Phi_u^{opt}(\omega))W(\omega)d\omega - \int_a^b \frac{1}{\Phi_u^{opt}(\omega)}(\Phi_u(\omega) - \Phi_u^{opt}(\omega))W(\omega)d\omega \\
&\quad - \int_a^b h(\Phi_u(\omega), \Phi_u^{opt}(\omega))W(\omega)d\omega \\
&= J(\Phi_u^{opt}) - \int_a^b h(\Phi_u(\omega), \Phi_u^{opt}(\omega))W(\omega)d\omega - \left(\int_a^b W(\omega)d\omega\right)\left(\int_a^b (\Phi_u(\omega) - \Phi_u^{opt}(\omega))d\omega\right) \\
&= J(\Phi_u^{opt}) - \int_a^b h(\Phi_u(\omega), \Phi_u^{opt}(\omega))W(\omega)d\omega,
\end{aligned}$$

where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is given by

$$h(x,y) := \ln x - \ln y - \frac{1}{y}(x-y).$$

Thus, since $w > 0$, to prove that Φ_u^{opt} gives the global minimum for the cost function (3.7), it suffices to show that $h(x,y) < 0$ for every $x, y \in \mathbb{R}^+$ such that $x \neq y$. To this end, notice that

$$\frac{\partial h}{\partial x}(x,y) = \frac{1}{x} - \frac{1}{y},$$

thus if $x > y$, then

$$h(x,y) = h(y,y) + \int_y^x \frac{\partial h}{\partial x}(\tilde{x},y)d\tilde{x} < 0,$$

and similarly for $x < y$. This proves the Lemma. □

The relationship given in (3.8) highlights the importance of choosing the correct function W so as to reflect the desired relative frequency weighting. The choice of W will be explored in the next section.

3.4.2 Choice of the Weighting Function

A weighting function which is reasonable in the sense that it satisfies Criterion B is described below:

Lemma 3.4.2 (Choice of W to satisfy Criterion B) *For $0 < a < b < 2\pi$, let $W \in C^1([a,b],\mathbb{R}^+)$. If W satisfies*

$$\int_\alpha^\beta W(\omega)d\omega = \int_{k\alpha}^{k\beta} W(\omega)d\omega \tag{3.9}$$

for every $a \leq \alpha < \beta \leq b$ and every $k > 0$ such that $a \leq k\alpha < k\beta \leq b$, then there is a $\lambda > 0$ such that $W(x) = \lambda/x$ for every $x \in [a,b]$.

Proof. Since W is continuous, we have from (3.9) that

$$W(a) = \lim_{\varepsilon \to 0_+} \frac{\int_a^{a+\varepsilon} W(\omega) d\omega}{\varepsilon} = \lim_{\varepsilon \to 0_+} k \frac{\int_{ka}^{ka+k\varepsilon} W(\omega) d\omega}{k\varepsilon} = kW(ka)$$

for $1 \leq k < b/a$. Thus,

$$W(ka) = \frac{1}{k} W(a), \quad a \leq ka < b,$$

or, by defining $x = ka$ and $\lambda = aW(a)$,

$$W(x) = \frac{a}{x} W(a) = \frac{\lambda}{x}, \quad a \leq x < b.$$

By the continuity of W, we also have that $W(b) = \lambda/b$. This proves the Lemma. □

With this last result, and those of the previous sections, we can now proceed to establish the form of a suitable measure for the 'goodness' of an experiment, and, an optimal input signal according to this cost function.

3.5 Bandlimited '$1/f$' Noise

If we apply the results of the previous sections to the cost function (3.7), we immediately see that a reasonable cost function for measuring the 'goodness' of an experiment, when only diffuse prior information is available about the system, is

$$J(\Phi_u) = \int_a^b \ln \left(\frac{K\Phi_v(\omega)}{\Phi_u(\omega)|G(e^{j\omega})|^2} \right) \frac{1}{\omega} d\omega.$$

Therefore, according to (3.8) and Lemma 3.4.1, the optimal input spectrum is given by

$$\Phi_u^{opt}(\omega) = \frac{1/\omega}{\int_a^b \frac{d\omega}{\omega}} = \frac{1/\omega}{\ln b - \ln a}, \quad \omega \in [a,b].$$

Figure 3.2 shows the spectral density of this type of signal, known as *bandlimited '$1/f$' noise*, for $a = 1$ and $b = 2$.

Thus, subject to the assumptions introduced above, i.e. Criteria A.1, A.2 and B, bandlimited '$1/f$' noise is a robust input signal for identifying a system when one has only diffuse prior information.

Remark 3.5.1 *The fact that bandlimited '$1/f$' noise is the solution of a variational problem means that it is possible to consider additional prior information by imposing constraints in the optimisation*

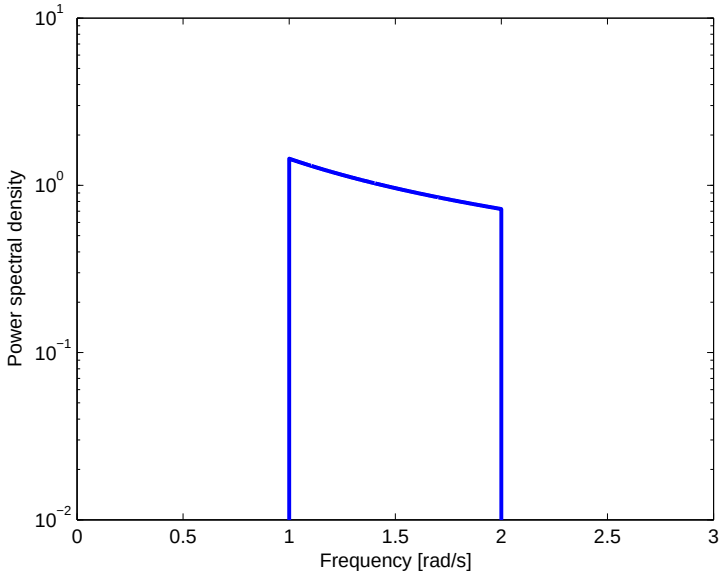

Figure 3.2. Power spectral density of bandlimited '$1/f$' noise signal for $a = 1$ and $b = 2$.

problem. In this sense, the problem of experiment design resembles the development of the Principle of Maximum Entropy as given in (Shore and Johnson 1980, Skilling 1988).

3.6 Numerical Example

We have shown above that bandlimited '$1/f$' noise can be regarded as a robust optimal test signal in the sense described in Section 3.5. This result is consistent with Chapter 2, where we showed that bandlimited '$1/f$' noise has near optimal properties for specific classes of systems. For example, it is known to yield a performance which is within a factor of 2 of the optimum for certain families of one-parameter problems (see Chapter 2), although similar results for multi-parameter problems are not yet available.

Table 3.1, reproduced from Chapter 2, shows some interesting results. This table shows the numerical results for the problem of designing an input signal to identify the parameter θ of the system

$$G(s) = \frac{1}{s/\theta + 1},$$

where it is assumed a-priori that θ lies in the range $\Theta := [0.1, 10]$, and $\Phi_v(\omega) = 1$. The cost function

Table 3.1. Relative Values of Cost for Different Input Signals

	$\max_{\theta\in\Theta}[\theta^2\overline{M}(\theta,\Phi_u)]^{-1}$
Single frequency at $\omega = 1$	7.75
Bandlimited white noise	12.09
Bandlimited '$1/f$' noise	1.43
Robust min-max optimal input (as developed in Chapter 2)	1.00

used for comparison is the worst case normalised variance of an efficient estimator of θ,

$$J'(\Phi_u) := \max_{\theta\in\Theta}\left[\int_0^\infty \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\Phi_u(\omega)d\omega\right]^{-1},$$

where the inputs being compared are

(i) A sine wave of frequency 1 (this is the optimal input if the true parameter is $\widehat{\theta} = 1$).

(ii) Bandlimited white noise input, limited to the frequency range $[0.1, 10]$.

(iii) Bandlimited '$1/f$' noise input, limited to the frequency range $[0.1, 10]$.

(iv) The approximate discretised robust optimal input generated by Linear Programming (see Chapter 2).

Notice that, for ease of comparison, the costs in Table 3.1 have been normalised such that the robust optimal input has cost 1.00. Figure 3.3 shows the performance of these signals according to the normalised variance obtained as a function of the true value of θ. Both Table 3.1 and Figure 3.3 demonstrate that bandlimited '$1/f$' noise does indeed yield good performance at least in terms of a specific example. The results presented in the current chapter give theoretical support to the earlier observations made in Chapter 2.

As a second example, we can recall Section 2.5.4, where bandlimited '$1/f$' noise was compared with the robust optimal input generated for three different cost functions in a multiparameter case. According to Table 2.3, the performance of bandlimited '$1/f$' noise is within a factor of 3 of the robust optimal signals. This gives further empirical support to the claim that bandlimited '$1/f$' noise is a good input signal when only diffuse prior information is available.

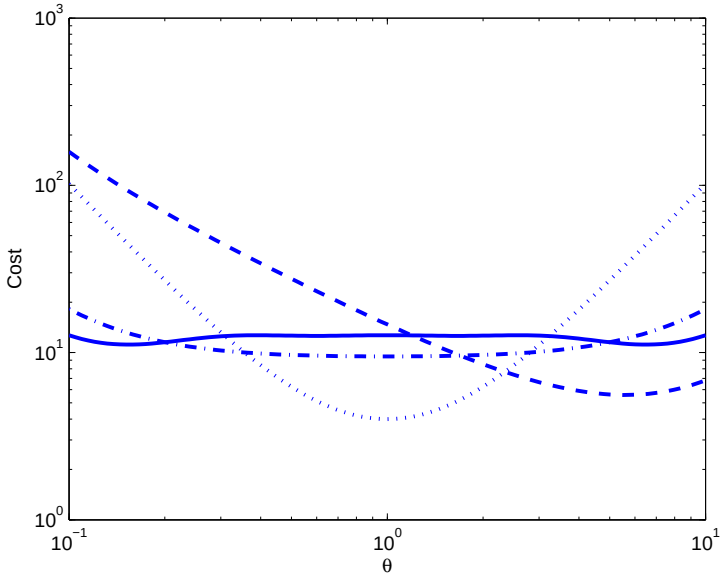

Figure 3.3. Variation of the normalised variance $\left[\int_0^\infty \frac{\omega^2/\theta^2}{(\omega^2/\theta^2+1)^2}\Phi_u(\omega)d\omega\right]^{-1}$ (cost) as a function of θ for various input signals: the robust optimal input (solid), a sine wave of frequency 1 (dotted), bandlimited white noise (dashed) and bandlimited '$1/f$' noise (dash-dotted).

3.7 Summary

In this chapter we have examined the problem of robust experiment design in the face of diffuse prior information. We have analysed a general class of criteria for measuring how good an experiment is, and found that there is a specific measure within this class that gives a system independent optimal experiment design. Furthermore this criterion is suitable for the case when one has only a vague idea about the system to be identified. We have also shown that bandlimited '$1/f$' noise is optimal according to this cost function.

CHAPTER 4

FUNDAMENTAL LIMITATIONS ON THE VARIANCE OF ESTIMATED PARAMETRIC MODELS

4.1 Introduction

As outlined in Chapter 1, Fundamental Limitations are of importance since they quantify the possible and the impossible. In feedback control, the development of fundamental limitations has given insight and understanding into the achievable performance of feedback control systems (Bode 1945, Goodwin et al. 2001, Seron et al. 1997). Knowledge of these limitations allows informed decisions to be made regarding the tradeoffs between conflicting performance criteria, e.g. the Bode integral shows that increasing performance in a particular frequency region will reduce performance in another. This is known as the *water-bed effect* (Seron et al. 1997).

In this chapter we establish fundamental integral limitations on the variance of estimated parametric models by generalising the result in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004) to parametric models possessing exogenous inputs. We establish the relationship between the results presented in the literature, which obtain a lower bound on the variance of parametric spectral estimators, with the new results obtained in this chapter, and show that the subtle differences between both sets of results are due to the use of different covariance expressions. Limitations are obtained for both open and closed loop identification. With respect to closed loop identification, both direct and indirect methods are considered. For the case of direct identification, bounds are established in lieu of an exact expression.

As an application of these results we show that, for multisine inputs, a well known and widely used asymptotic (in model order) variance expression (Ljung 1985) provides upper bounds on the actual variance of the estimated models of finite order. The derived fundamental limitations give rise to an estimation 'water-bed' effect, which is illustrated by an example.

4.2 Problem Description

Consider a SISO linear system given by

$$y_t = G_0(z)u_t + H_0(z)w_t,$$

where $\{u_t\}$ is a quasi-stationary signal (Ljung 1999) and $\{w_t\}$ is a zero mean Gaussian white noise sequence with variance σ^2. The noise transfer function, H_0, is assumed to be stable and minimum phase with $H_0(\infty) = 1$. In the sequel we denote $H_0(z)w_t$ by v_t.

Given the input-output data pairs $\{u_t, y_t\}_{t=1}^N$, a model,

$$y_t = G(z,\theta)u_t + H(z,\theta)\varepsilon_t,$$

is inferred. We assume no undermodeling, i.e. there exists a $\theta = \theta_0$ such that $G_0(z) = G(z,\theta_0)$ and $H_0(z) = H(z,\theta_0)$. Furthermore, we assume that the estimators of G_0 and H_0 are asymptotically efficient (e.g. Maximum Likelihood (ML), or Prediction Error Methods (PEM)).

We define the spectrum of a quasi-stationary signal $\{x_t\}$, according to (Ljung 1999), as

$$\Phi_x(\omega) := \sum_{\tau=-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}, \quad \omega \in [-\pi,\pi],$$

where $R_x(\tau) := \bar{E}\{x_t x_{t-\tau}\}$ is the autocovariance of $\{x_t\}$, and $\bar{E}\{f_t\} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^N E\{f_t\}$.

All integrals in the sequel are assumed to exist and be finite. Covariance expressions considered in this chapter are in general only valid as $N \to \infty$ (Ljung 1999) (i.e. they are correct up to order $1/N$). Note that all quantities involved in the fundamental limitations must be evaluated at their true values.

4.3 A Fundamental Limitation Result for Open Loop Identification

We first consider the open loop case, i.e. when $\{u_t\}$ and $\{w_t\}$ are independent, and develop an integral constraint on the variance of the estimated parametric model.

The following theorem is a generalisation of a result from (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004), where an integral limitation on parametric spectral estimation (i.e., for stochastic models without an exogenous signal) is established.

Theorem 4.3.1 (Limitations on open loop identification) *In open loop identification, where G and H are independently parameterised with n_G and n_H parameters respectively, and $(G(z,\theta_G), H(z,\theta_H))$*

is parameter identifiable *under* Φ_u *for the ML method (Söderström and Stoica 1989), then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_u(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega = \frac{n_G}{N}, \qquad (4.1)$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma^2}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{H}(e^{j\omega})]d\omega = \frac{n_H}{N}.$$

Proof. We have that (Ljung 1999)

$$\mathrm{Cov}[\hat{\theta}_G] = \left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma_G(e^{j\tau})\Gamma_G^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1},$$
$$\mathrm{Cov}[\hat{\theta}_H] = \left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma_H(e^{j\tau})\Gamma_H^H(e^{j\tau})\frac{\sigma^2}{\Phi_v(\tau)}d\tau\right]^{-1},$$

where θ_G and θ_H are the parameter vectors of G and H, respectively, and

$$\Gamma_G(z) := \frac{\partial G(z)}{\partial \theta_G}, \quad \Gamma_H(z) := \frac{\partial H(z)}{\partial \theta_H}.$$

Now, by the *Gauss' approximation formula* (Ljung 1999) (also called the *Delta method* (Casella and Berger 2002)),

$$\mathrm{Var}[\hat{G}(e^{j\omega})] = \Gamma_G^H(e^{j\omega})\,\mathrm{Cov}[\hat{\theta}_G]\Gamma_G(e^{j\omega}), \quad \mathrm{Var}[\hat{H}(e^{j\omega})] = \Gamma_H^H(e^{j\omega})\,\mathrm{Cov}[\hat{\theta}_H]\Gamma_H(e^{j\omega}).$$

Therefore,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_u(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega \\
&\qquad = \frac{1}{N}\,\mathrm{tr}\left\{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_G(e^{j\omega})\Gamma_G^H(e^{j\omega})\frac{\Phi_u(\omega)}{\Phi_v(\omega)}d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_G(e^{j\tau})\Gamma_G^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\right\} \\
&\qquad = \frac{n_G}{N}.
\end{aligned}$$

A similar argument applies to the integral of $\mathrm{Var}[\hat{H}(e^{j\omega})]$. □

Remark 4.3.1 *In general, if* Φ_u *is not persistently exciting of order* n_G*, the integral of* $\mathrm{Var}[\hat{G}]$ *will not be proportional to* n_G*. However, the integral will be proportional to the rank of the information matrix of* θ_G*, that is, to the number of spectral lines of* Φ_u*.*

It can be seen from (4.1) that, under the assumption of no undermodelling, a 'water-bed' effect exists on the variance of $\hat{G}$, since if $\mathrm{Var}[\hat{G}(e^{j\omega})]$ is small for some frequencies, it must necessarily be large at other frequencies, in order to satisfy (4.1).

4.4 Relationship to Previous Results

Theorem 4.3.1 establishes a fundamental limitation on the variance of estimators of the transfer functions G_0 and H_0. A result has been derived for the variance of spectral estimators in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004), which essentially establishes (in the notation of Theorem 4.3.1) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{|H_0(e^{j\omega})|^4}\,\mathrm{Var}[|\hat{H}(e^{j\omega})|^2]d\omega = \frac{2n_H}{N}. \tag{4.2}$$

In (Larimore 1982, Friedlander and Porat 1984) the term on the left side of (4.2) is considered as a measure of the accuracy of a spectral estimator, hence (4.2) provides a lower bound on the spectral accuracy. Note that, according to (Stoica et al. 2004), (4.2) imposes a water-bed effect on the variance of an asymptotically efficient spectral estimator.

The results presented in Theorem 4.3.1 differ from those established in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004). This difference is highlighted by the fact that Theorem 4.3.1 is based on Ljung's covariance expression (Ljung 1999):

$$\mathrm{Cov}[\hat{\theta}_H] = \left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\partial H(e^{j\tau})}{\partial\theta_H}\right)\left(\frac{\partial H(e^{j\tau})}{\partial\theta_H}\right)^H\frac{\sigma^2}{\Phi_v(\tau)}d\tau\right]^{-1},$$

whilst (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004) rely on '*Whittle's formula*' for the asymptotic covariance of an asymptotically efficient estimator $\hat{\theta}_H$ of θ_H (Whittle 1953, Porat 1994):

$$\mathrm{Cov}[\hat{\theta}_H] = \left[\frac{N}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\partial\Phi_v(\tau)}{\partial\theta_H}\right)\left(\frac{\partial\Phi_v(\tau)}{\partial\theta_H}\right)^T\frac{1}{\Phi_v^2(\tau)}d\tau\right]^{-1}.$$

Now by the Gauss' Approximation Formula,

$$\mathrm{Var}[|\hat{H}(e^{j\omega})|^2] = \left(\frac{\partial\Phi_v(\omega)}{\partial\theta_H}\right)^T\left[\frac{N}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\partial\Phi_v(\tau)}{\partial\theta_H}\right)\left(\frac{\partial\Phi_v(\tau)}{\partial\theta_H}\right)^T\frac{1}{\Phi_v^2(\tau)}d\tau\right]^{-1}\frac{\partial\Phi_v(\omega)}{\partial\theta_H}$$

and

$$\mathrm{Var}[\hat{H}(e^{j\omega})] = \left(\frac{\partial H(e^{j\omega})}{\partial\theta_H}\right)^H\left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\partial H(e^{j\tau})}{\partial\theta_H}\right)\left(\frac{\partial H(e^{j\tau})}{\partial\theta_H}\right)^H\frac{\sigma^2}{\Phi_v(\tau)}d\tau\right]^{-1}\frac{\partial H(e^{j\omega})}{\partial\theta_H}.$$

Since $\Phi_v = \sigma^2|H_0|^2$, it might seem straightforward to relate both expressions by establishing a connection between $\partial\Phi_v/\partial\theta_H$ and $\partial H/\partial\theta_H$. However, due to the complex-valued nature of H_0, this is not possible in general. Nonetheless, we can relate their integrals, as shown in the following theorem, which establishes the relationship between our result (Theorem 4.3.1) and the results in (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004) without using Whittle's or Ljung's expressions.

Theorem 4.4.1 (Relationship between the Noise Spectrum and its Transfer Function) *Let H_0 be a stable minimum phase transfer function such that $H_0(\infty) = 1$ and $H_0(\bar{z}) = \overline{H_0(z)}$ for all $z \in \mathbb{C}$. Also, let $\hat{H}_N = H(\hat{\theta}_N)$ be an asymptotically efficient estimator of H_0 (subject to the same constraints imposed on H_0), where N is the number of samples and $\hat{\theta}_N$ is an asymptotically efficient estimator of $\theta_0 \in \mathbb{R}^n$. Then,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{|H_0(e^{j\omega})|^2}\operatorname{Var}[\hat{H}_N(e^{j\omega})]d\omega = \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{1}{|H_0(e^{j\omega})|^4}\operatorname{Var}[|\hat{H}_N(e^{j\omega})|^2]d\omega.$$

Proof. See Appendix 4.9. □

Theorem 4.4.1 links the results of Theorem 4.3.1 for the noise transfer function with the results of (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004), thus showing that they can be considered, in some sense, equivalent. However, Theorem 4.3.1 also establishes a similar result for G, which has no resemblance with previous results in the literature, since (Larimore 1982, Friedlander and Porat 1984, Stoica et al. 2004) do not consider exogenous signals.

4.5 Fundamental Limitations in Closed Loop Identification

In closed loop identification, we consider the input $\{u_t\}$ to be generated as follows:

$$u_t = r_t - C(z)y_t,$$

where $\{r_t\}$ is a quasi-stationary reference signal, independent of $\{w_t\}$. Figure 4.1 shows the closed loop system. For general background on closed loop identification methods, we refer to (Forssell and Ljung 1999).

In order to derive fundamental limitations for closed loop identification, analogous to those developed for the open loop case, we define:

$$\begin{aligned}
S(z) &:= \frac{1}{1+G(z)C(z)},\\
G_{cl}(z) &:= \frac{G(z)}{1+G(z)C(z)},\\
H_{cl}(z) &:= \frac{H(z)}{1+G(z)C(z)},\\
\Phi_u^r(\omega) &:= \Phi_r(\omega)|S(e^{j\omega})|^2,\\
\Phi_v(\omega) &:= \sigma^2|H(e^{j\omega})|^2.
\end{aligned} \tag{4.3}$$

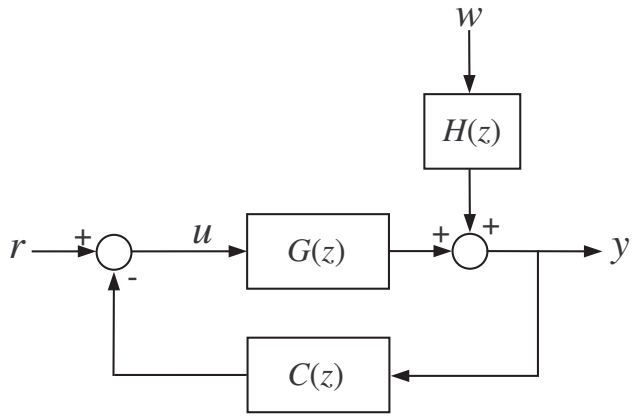

Figure 4.1. Block diagram describing the closed loop system.

Then

$$\frac{\partial G_{cl}(z)}{\partial \theta} = S^2(z)\frac{\partial G(z)}{\partial \theta},$$
$$\frac{\partial H_{cl}(z)}{\partial \theta} = -H(z)C(z)S^2(z)\frac{\partial G(z)}{\partial \theta} + S(z)\frac{\partial H(z)}{\partial \theta}.$$

Thus,

$$\Gamma_{cl}(z) = \Gamma_{ol}(z)\begin{bmatrix} S^2(z) & -H(z)C(z)S^2(z) \\ 0 & S(z) \end{bmatrix}, \tag{4.4}$$

where

$$\Gamma_{cl}(z) := \left[\frac{\partial G_{cl}(z)}{\partial \theta} \quad \frac{\partial H_{cl}(z)}{\partial \theta}\right], \quad \Gamma_{ol}(z) := \left[\frac{\partial G(z)}{\partial \theta} \quad \frac{\partial H(z)}{\partial \theta}\right].$$

Remark 4.5.1 *In the case where a reference prefilter, say F, is present, the expressions of this section can be easily adapted accordingly, by replacing Φ_r with $|F|^2\Phi_r$.*

In the sequel we assume that $(G(z,\theta_G), H(z,\theta_H))$ is *parameter identifiable* under Φ_r for the ML method (Söderström and Stoica 1989).

4.5.1 General Case

Theorem 4.5.1 (Limitations on closed loop identification) *In closed loop, i.e. where $\{u_t\}$ and $\{w_t\}$ are not necessarily independent, as well, G and H are not necessarily independently parameterised,*

with a common parameter vector $\theta \in \mathbb{R}^{n_\theta}$, we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left\{ \begin{bmatrix} \frac{\Phi_u^r(\omega)}{\Phi_v(\omega)} + |C(e^{j\omega})S(e^{j\omega})|^2 & -\frac{C(e^{j\omega})S(e^{j\omega})}{H^*(e^{j\omega})} \\ -\frac{C^*(e^{j\omega})S^*(e^{j\omega})}{H(e^{j\omega})} & \frac{1}{|H(e^{j\omega})|^2} \end{bmatrix} \mathrm{Cov}\begin{bmatrix} \hat{G}(e^{j\omega}) \\ \hat{H}(e^{j\omega}) \end{bmatrix} \right\} d\omega = \frac{n_\theta}{N}.$$

Proof. Note that

$$\begin{aligned} \mathrm{Cov}[\hat{\theta}] &= \frac{2\pi}{N}\left\{ \int_{-\pi}^{\pi} \frac{1}{|S(e^{j\tau})|^2\Phi_v(\tau)} \Gamma_{cl}(e^{j\tau}) \begin{bmatrix} \Phi_r(\tau) & 0 \\ 0 & \sigma^2 \end{bmatrix} \Gamma_{cl}^H(e^{j\tau}) d\tau \right\}^{-1} \\ &= \frac{2\pi}{N}\left\{ \int_{-\pi}^{\pi} \frac{1}{|S(e^{j\tau})|^2\Phi_v(\tau)} \Gamma_{ol}(e^{j\tau}) \begin{bmatrix} S^2(e^{j\tau}) & -H(e^{j\tau})C(e^{j\tau})S^2(e^{j\tau}) \\ 0 & S(e^{j\tau}) \end{bmatrix} \begin{bmatrix} \Phi_r(\tau) & 0 \\ 0 & \sigma^2 \end{bmatrix} \right. \\ &\qquad \left. \cdot \begin{bmatrix} S^2(e^{j\tau}) & -H(e^{j\tau})C(e^{j\tau})S^2(e^{j\tau}) \\ 0 & S(e^{j\tau}) \end{bmatrix}^H \Gamma_{ol}^H(e^{j\tau}) d\tau \right\}^{-1}. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{1}{|S(e^{j\omega})|^2\Phi_v(\omega)} \begin{bmatrix} S^2(e^{j\omega}) & -H(e^{j\omega})C(e^{j\omega})S^2(e^{j\omega}) \\ 0 & S(e^{j\omega}) \end{bmatrix} \begin{bmatrix} \Phi_r(\omega) & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} S^2(e^{j\omega}) & -H(e^{j\omega})C(e^{j\omega})S^2(e^{j\omega}) \\ 0 & S(e^{j\omega}) \end{bmatrix}^H \\ &\qquad = \begin{bmatrix} \frac{\Phi_u^r(\omega)}{\Phi_v(\omega)} + |C(e^{j\omega})S(e^{j\omega})|^2 & -\frac{C(e^{j\omega})S(e^{j\omega})}{H^*(e^{j\omega})} \\ -\frac{C^*(e^{j\omega})S^*(e^{j\omega})}{H(e^{j\omega})} & \frac{1}{|H(e^{j\omega})|^2} \end{bmatrix}. \end{aligned}$$

On the other hand, by the Gauss' approximation formula we have that

$$\mathrm{Cov}\begin{bmatrix} \hat{G}(e^{j\omega}) \\ \hat{H}(e^{j\omega}) \end{bmatrix} = \Gamma_{ol}^H(e^{j\omega})\,\mathrm{Cov}[\hat{\theta}]\Gamma_{ol}(e^{j\omega}).$$

The rest of the proof follows similar lines to Theorem 4.3.1. □

4.5.2 Indirect Identification

Theorem 4.5.2 (Limitations on indirect identification) *In indirect closed loop identification, i.e. where $\{u_t\}$ and $\{w_t\}$ are not necessarily independent, however G_{cl} and H_{cl} are independently parameterised with $n_{G_{cl}}$ and $n_{H_{cl}}$ parameters respectively and the closed loop is described by $y_t = G_{cl}(z)r_t + H_{cl}(z)w_t$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_u^r(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega = \frac{n_{G_{cl}}}{N}.$$

Proof. With G_{cl} and H_{cl} independently parameterised, we can essentially consider the open loop case (with Φ_r, $|S|^2\Phi_v$, G_{cl} and H_{cl} instead of Φ_u, Φ_v, G and H, respectively). Thus, from Theorem

4.3.1 we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_r(\omega)}{|S(e^{j\omega})|^2\Phi_v(\omega)}\,\mathrm{Var}[\hat{G_{cl}}(e^{j\omega})]d\omega=\frac{n_{G_{cl}}}{N}. \tag{4.5}$$

Now, from (4.4) and the Gauss' approximation formula we can relate the variance of $G_{cl}(e^{j\omega})$ and $G(e^{j\omega})$ as

$$\mathrm{Var}[\hat{G}_{cl}(e^{j\omega})]=|S(e^{j\omega})|^4\,\mathrm{Var}[\hat{G}(e^{j\omega})]. \tag{4.6}$$

Substituting (4.6) into (4.5) gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|S(e^{j\omega})|^2\Phi_r(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega=\frac{n_{G_{cl}}}{N},$$

which completes the proof. □

Corollary 4.5.1 (Tailor-made parametrisation) *For indirect closed loop identification with a tailor-made parametrisation (Donkelaar and Van den Hof 2000), i.e. when $\{u_t\}$ and $\{w_t\}$ are not necessarily independent, but when G and H_{cl} are independently parameterised with n_G and $n_{H_{cl}}$ parameters respectively (as in* (4.3)*), then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_u^r(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega=\frac{n_G}{N}.$$

Proof. Follows from Theorem 4.5.2 and the fact that G_{cl} and H_{cl} are independently parameterised, with $n_{G_{cl}}=n_G$ and $n_{H_{cl}}$ parameters, respectively. □

4.5.3 Direct Identification

In direct closed loop identification it is difficult to establish an exact integral constraint for the fundamental limitation. However, based on results from (Agüero and Goodwin 2007), the following bounds are established:

Theorem 4.5.3 (Limitation in direct identification) *In direct closed loop identification, i.e. when $\{u_t\}$ and $\{w_t\}$ are not necessarily independent, however G and H are independently parameterised with n_G and n_H parameters respectively, we have that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_u(\omega)}{\Phi_v(\omega)}\,\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega\geq\frac{n_G}{N},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\Phi_v(\omega)}\left[\Phi_u(\omega)-\frac{\Phi_{uw}(\omega)}{\sigma^2}\right]\mathrm{Var}[\hat{G}(e^{j\omega})]d\omega\leq\frac{n_G}{N}.$$

Proof. By applying the Cauchy-Schwarz inequality, the following inequalities can be obtained (see Theorem 1 of (Agüero and Goodwin 2007)):

$$\mathrm{Cov}[\hat{\theta}_G] \geq \left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma_G(e^{j\tau})\Gamma_G^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1},$$
$$\mathrm{Cov}[\hat{\theta}_G] \leq \left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma_G(e^{j\tau})\Gamma_G^H(e^{j\tau})\frac{1}{\Phi_v(\tau)}\left[\Phi_u(\tau)-\frac{\Phi_{uw}(\tau)}{\sigma^2}\right]d\tau\right]^{-1}.$$

Now, by the Gauss' approximation formula,

$$\mathrm{Var}[\hat{G}(e^{j\omega})] = \Gamma_G^H(e^{j\omega})\,\mathrm{Var}[\hat{\theta}_G]\Gamma_G(e^{j\omega}).$$

The rest of the proof follows similar lines to Theorem 4.3.1. □

Remark 4.5.2 *Notice that the inequalities of Theorem 4.5.3 are valid even if the controller C is nonlinear and/or time variant, provided* $\{u_t\}$ *is quasi-stationary. If the controller is linear and time invariant, the expression* $\Phi_u - \Phi_{uw}/\sigma^2$ *in the second inequality of Theorem 4.5.3 corresponds to* Φ_u^r*, as defined in* (4.3)*.*

Remark 4.5.3 *In the open loop case, i.e. when* $\Phi_{uw} = 0$*, the combination of both inequalities of Theorem 4.5.3 gives the result of Theorem 4.3.1.*

4.6 Bounds on the Variance

As an application of the above results, we show that for an input comprising multisines, the asymptotic variance expression (Ljung 1985)

$$\mathrm{Var}[\hat{G}(e^{j\omega})] = \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_u(\omega)}, \tag{4.7}$$

provides an upper bound on the variance of G, irrespective of the model structure.

It is important to notice that, as shown in (Ninness and Hjalmarsson 2004), better approximations exist to that given in (4.7), some of which are actually exact for finite model orders. Furthermore, there exist finite sample variance expressions for linearly parameterised model structures which hold for periodic signals; see (Hjalmarsson and Ninness 2006). However, here we focus on (4.7) since it has a simple form which can be related to the fundamental limitations developed in this chapter, as shown below.

Consider the open loop case, with a multisine input of the form

$$\Phi_u(\omega) = \sum_{i=1}^{m} 2\pi U_i \delta(\omega - \omega_i),$$

where $\omega_i \in [-\pi, \pi]$, $U_i > 0$ for every $i = 1, \ldots, m$, and Φ_u is even. For identifiability reasons, we assume that $m \geq n_G$, the number of parameters in G.

By Theorem 4.3.1, we have that

$$\sum_{i=1}^{m} \frac{U_i}{\Phi_v(\omega_i)} \mathrm{Var}[\hat{G}(e^{j\omega_i})] = \frac{n_G}{N}.$$

Since all the terms in the sum are nonnegative, we obtain

$$\mathrm{Var}[\hat{G}(e^{j\omega_i})] \leq \frac{n_G}{N} \frac{\Phi_v(\omega_i)}{U_i}, \quad i = 1, \ldots, m.$$

Similarly, in the closed loop case (either direct or indirect, assuming the controller is linear and time invariant), we have that

$$\mathrm{Var}[\hat{G}(e^{j\omega_i})] \leq \frac{n_{G_{cl}}}{N} \frac{\Phi_v(\omega_i)}{U_i^r}, \quad i = 1, \ldots, m.$$

where $n_{G_{cl}}$ is the number of parameters in G or G_{cl}, (dependent on direct or indirect identification), and

$$\Phi_r(\omega) = \sum_{i=1}^{m} 2\pi R_i \delta(\omega - \omega_i),$$

where $\omega_i \in [-\pi, \pi]$, $R_i > 0$ for every $i = 1, \ldots, m$, Φ_r is even, and $U_i^r := |S(e^{j\omega_i})|^2 R_i$, for $i = 1, \ldots, m$. Again, for identifiability reasons we assume that $m \geq n_G$, the number of parameters in G (or G_{cl}).

Hence, for multisine inputs (or reference signals), Ljung's asymptotic (in model order) variance expressions provide an upper bound on the true variance of the parametric models[1]. Note that these bounds, although asymptotic in the number of samples, hold for more general model structures than those treated in (Hjalmarsson and Ninness 2006), which deals with linearly parameterised models (but it provides finite sample results).

4.7 Numerical Example

Consider a system described by

$$G_0(z) = \frac{z^{-1}}{1 - a^0 z^{-1}}, \qquad H_0(z) = 1,$$

where $a^0 = 0.4$, and the model structures,

$$G_1(z, \theta_1) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1}}, \qquad H_1(z, \theta_1) = 1,$$

$$G_2(z, \theta_2) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}, \qquad H_2(z, \theta_2) = 1,$$

[1] Actually, several variance expressions, such as (4.7) and those in (Ninness and Hjalmarsson 2004), do not apply for multisine signals. In fact, the bounds obtained here are similar, but not exactly equal, to Ljung's variance expression, since their denominators involve only the amplitudes of the Dirac deltas of the input (or reference) spectrum.

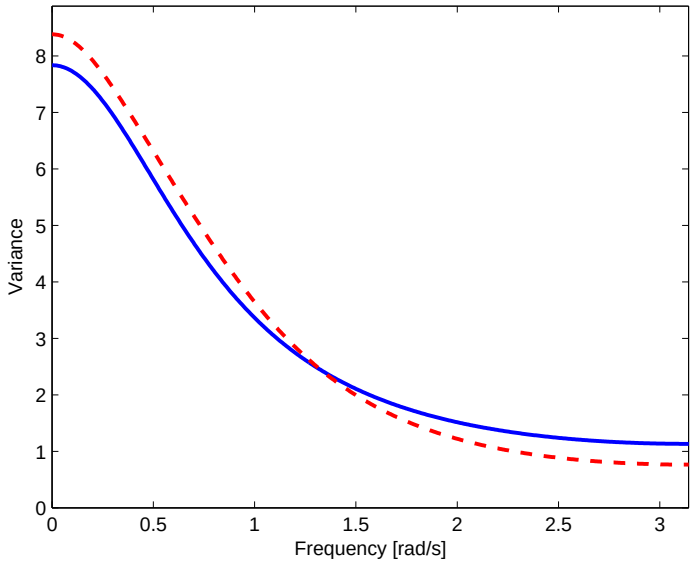

Figure 4.2. (Normalised) variance of the transfer function estimators of G, based on the model structures $(G_1(z,\theta_1),H_1(z,\theta_1))$ (solid) and $(G_2(z,\theta_2),H_2(z,\theta_2))$ (dashed), as functions of ω.

where $\theta_1 := [b_1\ b_2\ a_1]^T$ and $\theta_2 := [b_1\ a_1\ a_2]^T$. Notice that both model structures, $(G_1(z,\theta_1),H_1(z,\theta_1))$ and $(G_2(z,\theta_2),H_2(z,\theta_2))$, have 3 parameters and include the true system.

If $\sigma = 1$ and $\Phi_u(\omega) = 1$, the normalised (i.e. multiplied by N) variances of the transfer function estimators $\hat{G}_1(e^{j\omega})$ and $\hat{G}_2(e^{j\omega})$ are shown in Figure 4.2. From the figure, we see that the variances are different functions of frequency. In particular, $\mathrm{Var}[\hat{G}_1]$ is smaller than $\mathrm{Var}[\hat{G}_2]$ at low frequencies and is larger at high frequencies. This is consistent with the fundamental limitation derived in Theorem 4.3.1, namely that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Var}[\hat{G}_i(e^{j\omega})]d\omega = \frac{3}{N}, \quad i = 1,2.$$

This means that it is not possible to reduce the variance of $\hat{G}$ at all frequencies by choosing a suitable model structure, since if we reduce the variance at some frequencies, it will necessarily increase at others, hence illustrating the 'water-bed' effect, and demostrating the associated tradeoffs.

4.8 Summary

In this chapter we have established fundamental limitations on the variance of estimated parametric models, for both open and closed loop identification. Furthermore we have shown the relationship to previous results and established that the results presented in this chapter hold for more general sys-

tems (with exogenous signals). In addition, our results are based on Ljung's covariance expressions, while previous results are based on Whittle's formula. For the closed loop case, we have presented results for both direct and indirect identification methods. Based on these results, we have shown that for multisine inputs, a well known asymptotic (in model order) variance expression provides upper bounds on the actual variance of the estimated models of finite order. It can be clearly seen from the results that any over parameterisation results in an increase in the integrated variance of the transfer function estimators. Finally, we have presented an example which highlights the tradeoffs imposed by the fundamental limitations, and also illustrates the 'water-bed' effect in system identification.

4.9 Appendix: Proof of Theorem 4.4.1

The proof of Theorem 4.4.1 is based on the following lemmas.

Lemma 4.9.1 (Interchange of limits and integrals) *Let $\hat{F}_N := F(\hat{\theta}_N)$ be an asymptotically efficient estimator of $F(\theta_0) =: F_0 : [-\pi,\pi] \to \mathbb{C}$, where $\hat{\theta}_N$ is an asymptotically efficient estimator of $\theta_0 \in \mathbb{R}^n$ based on N samples. Then,*

$$\int_{-\pi}^{\pi} \lim_{N\to\infty} NE\{|\hat{F}_N(\omega) - F_0(\omega)|^2\}d\omega = \lim_{N\to\infty} N\int_{-\pi}^{\pi} E\{|\hat{F}_N(\omega) - F_0(\omega)|^2\}d\omega.$$

Proof. By the Gauss' Approximation Formula, we have that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \lim_{N\to\infty} NE\{|\hat{F}_N(\omega) - F_0(\omega)|^2\}d\omega \\
&\qquad = \int_{-\pi}^{\pi} \left(\frac{\partial F(\omega)}{\partial \theta}\right)^H \lim_{N\to\infty} NE\{[\hat{\theta}_N - \theta_0][|\hat{\theta}_N - \theta_0]^T\}\frac{\partial F(\omega)}{\partial \theta}d\omega \\
&\qquad = \int_{-\pi}^{\pi} \mathrm{tr}\left\{\left(\frac{\partial F(\omega)}{\partial \theta}\right)\left(\frac{\partial F(\omega)}{\partial \theta}\right)^H \lim_{N\to\infty} NE\{[\hat{\theta}_N - \theta_0][|\hat{\theta}_N - \theta_0]^T\}\right\}d\omega \\
&\qquad = \mathrm{tr}\left\{\lim_{N\to\infty} NE\{[\hat{\theta}_N - \theta_0][|\hat{\theta}_N - \theta_0]^T\}\int_{-\pi}^{\pi}\left(\frac{\partial F(\omega)}{\partial \theta}\right)\left(\frac{\partial F(\omega)}{\partial \theta}\right)^H d\omega\right\} \\
&\qquad = \lim_{N\to\infty} N\,\mathrm{tr}\left\{E\{[\hat{\theta}_N - \theta_0][|\hat{\theta}_N - \theta_0]^T\}\int_{-\pi}^{\pi}\left(\frac{\partial F(\omega)}{\partial \theta}\right)\left(\frac{\partial F(\omega)}{\partial \theta}\right)^H d\omega\right\} \\
&\qquad = \lim_{N\to\infty} N\int_{-\pi}^{\pi} \left(\frac{\partial F(\omega)}{\partial \theta}\right)^H E\{[\hat{\theta}_N - \theta_0][|\hat{\theta}_N - \theta_0]^T\}\frac{\partial F(\omega)}{\partial \theta}d\omega \\
&\qquad = \lim_{N\to\infty} N\int_{-\pi}^{\pi} E\{|\hat{F}_N(\omega) - F_0(\omega)|^2\}d\omega.
\end{aligned}$$

□

Lemma 4.9.2 (Variance of an estimator) *Let $\{z_N\}_{N\in\mathbb{N}}$ be a sequence of complex random variables such that $\sqrt{N}[\mathrm{Re}(z_N - 1)\ \mathrm{Im}(z_N - 1)]^T$ converges in distribution as $N \to \infty$ to a random vector of zero*

mean and covariance matrix Σ*. Then,*

$$\lim_{N\to\infty} NE\{(|z_N|^2-1)^2\} = \lim_{N\to\infty} NE\{(2\mathrm{Re}\{z_N-1\})^2\}.$$

Proof. This follows similar lines to the proof of the Delta Method (Lehmann and Casella 1998) (even though the Delta Method assumes asymptotic normality, which is not the case here). Let $z_N - 1 =: \varepsilon_N e^{j\theta_N}$, where $\varepsilon_N \geq 0$ and $\theta_N \in (-\pi,\pi]$ are real random variables. Then,

$$\begin{aligned}\sqrt{N}(|z_N|^2-1) &= \sqrt{N}(|1+\varepsilon_N e^{j\theta_N}|^2-1)\\ &= \sqrt{N}[1+2\varepsilon_N\cos\theta_N+\varepsilon_N^2-1]\\ &= 2\sqrt{N}\,\mathrm{Re}\{z_N-1\}+\sqrt{N}|z_N-1|^2.\end{aligned}$$

Due to the conditions of the Lemma, $N|z_N-1|^2 \xrightarrow{N\to\infty} \mathrm{tr}(\Sigma) > 0$ in probability, hence $\sqrt{N}|z_N-1|^2 \xrightarrow{N\to\infty} 0$. Therefore, $\sqrt{N}(|z_N|^2-1) - 2\sqrt{N}\,\mathrm{Re}\{z_N-1\} \xrightarrow{N\to\infty} 0$ in probability, which, by Slutsky's Theorem (Lehmann and Casella 1998, Theorem 8.10), establishes the result. □

Proof of Theorem 4.4.1: Note that

$$\begin{aligned}&\int_{-\pi}^{\pi} \lim_{N\to\infty} NE\{|\hat{H}_N(e^{j\omega})-H_0(e^{j\omega})|^2\}\frac{1}{|H_0(e^{j\omega})|^2}d\omega\\ &\quad= \lim_{N\to\infty} N\int_{-\pi}^{\pi} E\{|\hat{H}_N(e^{j\omega})-H_0(e^{j\omega})|^2\}\frac{1}{|H_0(e^{j\omega})|^2}d\omega \qquad (4.8)\\ &\quad= \lim_{N\to\infty} NE\left\{\int_{-\pi}^{\pi}\left|\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right|^2 d\omega\right\}\\ &\quad= \lim_{N\to\infty} NE\left\{\int_{-\pi}^{\pi}\left[\mathrm{Re}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2 d\omega\right\} + \lim_{N\to\infty} NE\left\{\int_{-\pi}^{\pi}\left[\mathrm{Im}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2 d\omega\right\}.\end{aligned}$$

The interchange of the limit and the integral in (4.8) comes from Lemma 4.9.1 and the continuity of H_0 on the unit circle $\{z\in\mathbb{C}:\,|z|=1\}$ (which implies that there exists a $\delta>0$ such that $|H_0(e^{j\omega})|\geq\delta$ for every $\omega\in[-\pi,\pi]$, thus ensuring that $\partial(H/H_0)/\partial\theta$ is square integrable). The interchange of the expectation operator and the integrals in (4.8) is due to Tonelli's Theorem (Bartle 1966).

Now, since H_0 and $\hat{H}_N$ are stable, minimum phase and such that $H_0(\bar{z}) = \overline{H_0(z)}$, $\hat{H}_N(\bar{z}) = \overline{\hat{H}_N(z)}$ and $H_0(\infty) = \hat{H}_N(\infty) = 1$, we have that $\hat{H}_N/H_0 - 1$ has a Laurent series expansion for $\{z\in\mathbb{C}:\,|z|\geq 1\}$ with real coefficients, i.e.,

$$\frac{\hat{H}_N(z)}{H_0(z)} - 1 = \sum_{k=1}^{\infty} a_k z^{-k}, \quad a_k\in\mathbb{R}, \quad |z|\geq 1,$$

Therefore, we have that

$$\int_{-\pi}^{\pi}\left[\mathrm{Re}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2 d\omega = \frac{1}{2}\sum_{k=1}^{\infty} a_k^2 = \int_{-\pi}^{\pi}\left[\mathrm{Im}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2 d\omega.$$

Furthermore, by Lemmas 4.9.1 and 4.9.2 (with the continuity of H_0 on the unit circle),

$$\lim_{N\to\infty} N\int_{-\pi}^{\pi} E\left\{\left[\mathrm{Re}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2\right\}d\omega = \int_{-\pi}^{\pi}\lim_{N\to\infty} NE\left\{\left[\mathrm{Re}\left\{\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}-1\right\}\right]^2\right\}d\omega$$
$$= \frac{1}{4}\int_{-\pi}^{\pi}\lim_{N\to\infty} NE\left\{\left[\left|\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}\right|^2-1\right]^2\right\}d\omega. \quad (4.9)$$

Hence, the combination of (4.8)-(4.9) yields

$$\int_{-\pi}^{\pi}\lim_{N\to\infty} NE\{|\hat{H}_N(e^{j\omega})-H_0(e^{j\omega})|^2\}\frac{1}{|H_0(e^{j\omega})|^2}d\omega$$
$$= \frac{1}{2}\int_{-\pi}^{\pi}\lim_{N\to\infty} NE\left\{\left[\left|\frac{\hat{H}_N(e^{j\omega})}{H_0(e^{j\omega})}\right|^2-1\right]^2\right\}d\omega$$
$$= \frac{1}{2}\int_{-\pi}^{\pi}\lim_{N\to\infty} NE\{||\hat{H}_N(e^{j\omega})|^2-|H_0(e^{j\omega})|^2|^2\}\frac{1}{|H_0(e^{j\omega})|^4}d\omega.$$

The result is then obtained by dividing this expression by 2π. □

CHAPTER 5

EXPERIMENT DESIGN CONSIDERING MODELS OF FINITE ORDER

5.1 Introduction

In this chapter we return to the problem of experiment design when only prior information is available. In this context we need to design input signals which provide a relatively good estimation performance over a large number of systems and model structures. Based on the fundamental limitations derived in Chapter 4, a closed form expression is obtained for the input spectrum which minimises the maximum value of a weighted integral of the variance of the frequency response estimator, over all model structures with a given number of parameters. The technique to acomplish this result uses the fundamental limitations as constraints to reformulate the optimisation as a simple constrained variational problem, which can be solved with Lagrange multipliers.

Chapter 3 analysed the problem of finding a class of cost functions, dependant on the relative variance of the frequency response estimator, that give an optimal input independent of the true system and noise dynamics. The resulting cost functions were established asymptotically in model order. In this chapter we extend the validity of these results to finite order models.

Finally, the robust optimal input obtained in this chapter is compared with Yuan & Ljung's unprejudiced optimal input design (Yuan and Ljung 1985), where the effect of both bias and variance errors in experiment design was considered. Utilising the fundamental limitations of Chapter 4 we reconsider their approach, which is based on an asymptotic (in model order) variance expression, and derive an unprejudiced optimal input for finite order models.

5.2 Problem Description

Consider a SISO linear system given by

$$y_t = G_0(z)u_t + H_0(z)w_t,$$

where $\{u_t\}$ is a quasi-stationary input signal (Ljung 1999), $\{y_t\}$ is the output signal, $\{w_t\}$ is a zero mean Gaussian white noise sequence with variance σ^2, and H_0 is assumed to be a stable minimum phase transfer function with $H_0(\infty) = 1$. To simplify notation, we denote $H_0(z)w_t$ by v_t.

Given N input-output data pairs $\{u_t, y_t\}_{t=1}^N$, a model of the form

$$y_t = G(z,\theta)u_t + H(z,\theta)\varepsilon_t,$$

will be estimated. We assume that the estimators of G_0 and H_0 are asymptotically efficient (e.g., ML or PEM).

Let $\mathbb{E} := \{z \in \mathbb{C} : \ |z| \geq 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}$. The Hardy space of analytic functions f on $\mathbb{E}$ taking values on $\mathbb{C}^n$ such that $\lim_{r\to 1_+} \int_{-\pi}^{\pi} \|f(re^{j\omega})\|_2^2 d\omega < \infty$ is denoted as $\mathscr{H}_2^n$ (Duren 1970, Koosis 1998). Define $C^1(X,Y)$ as the space of all functions from $X \subseteq \mathbb{R}$ to $Y \subseteq \mathbb{R}$ having a continuous derivative, and $C(\mathbb{T}, \mathbb{R}_0^+)$ as the space of all continuous functions $f : \mathbb{T} \to \mathbb{R}_0^+$ such that $f(z^*) = [f(z)]^*$ for every $z \in \mathbb{T}$.

Recall from Chapter 4 that the parameters involved in the fundamental limitations must be evaluated at their true values.

5.3 Preliminaries

The following result is a converse to Theorem 4.3.1, of Chapter 4, which will prove useful in the sequel.

Theorem 5.3.1 (Converse of the fundamental limitations on open loop identification) *Let $\Phi_u, \Phi_v : [-\pi, \pi] \to \mathbb{R}^+$ be continuous and even. Also, let $V \in C(\mathbb{T}, \mathbb{R}_0^+)$ such that*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega}) d\omega = \frac{n}{N},$$

where $n, N \in \mathbb{N}$. Then, there exists a function $\Gamma \in \mathscr{H}_2^n$ such that

$$\Gamma^H(z) \left[\frac{N}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(z) = V(z),$$

for every $z \in \mathbb{T}$.

Proof. See Appendix 5.8. □

5.4 Min Max Robust Experiment Design

Utilising the results of the previous section, we consider the problem of designing an input signal which is robust against all possible model structures (and the true values of the system parameters). We begin by rewriting the experiment design problem in terms of a function which satisfies the fundamental limitation of Theorem 4.3.1. In this section we assume no undermodeling, i.e. there exists a $\theta = \theta_0$ such that $G_0(z) = G(z, \theta_0)$.

Theorem 5.4.1 (Min Max Robust experiment design) *Consider the experiment design problem*[1]*:*

$$\min_{\Phi_u \geq 0} \sup_{\Gamma \in \mathscr{H}_2^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Var}[\hat{G}(e^{j\omega})] W(e^{j\omega}) d\omega,$$
$$\text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1,$$

where $W \in C(\mathbb{T}, \mathbb{R}_0^+)$ *and*

$$\mathrm{Var}[\hat{G}(z)] := \Gamma^H(z) \left[\frac{N}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(z),$$

for $z \in \mathbb{T}$*. The solution of this problem is given by*

$$\Phi_u^{opt}(\omega) := \frac{\Phi_v(\omega) W(e^{j\omega})}{\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_v(\tau) W(e^{j\tau}) d\tau}, \tag{5.1}$$

and the optimal cost is

$$\min_{\substack{\Phi_u \geq 0 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1}} \sup_{\Gamma \in \mathscr{H}_2^n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Var}[\hat{G}(e^{j\omega})] W(e^{j\omega}) d\omega = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \Phi_v(\omega) W(e^{j\omega}) d\omega.$$

Proof. By Theorems 4.3.1 and 5.3.1, the experiment design problem is equivalent to

$$\min_{\Phi_u \geq 0} \sup_{V \in C(\mathbb{T}, \mathbb{R}_0^+)} \frac{1}{2\pi} \int_{-\pi}^{\pi} V(e^{j\omega}) W(e^{j\omega}) d\omega,$$
$$\text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega = 1, \tag{5.2}$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega}) d\omega = \frac{n}{N}.$$

The idea here is that every $\Gamma \in \mathscr{H}_2^n$ gives rise to a variance $V \in C(\mathbb{T}, \mathbb{R}_0^+)$ which satisfies the integral constraint established in Theorem 4.3.1, and conversely, any $V \in C(\mathbb{T}, \mathbb{R}_0^+)$ which satisfies that

[1] Notice that the input power has been normalised to be less that or equal to 1. In case the input power is constrained to be below some other value, it suffices in the problems considered in the chapter to scale the optimal solution to satisfy that constraint. For other experiment design problems, the reader is referred to Chapter 6 which provides methods to renormalise the optimal input.

integral constraint can be related, by Theorem 5.3.1, to at least[2] one $\Gamma \in \mathscr{H}_2^n$. Therefore, the maximisation with respect to $\Gamma \in \mathscr{H}_2^n$ can be replaced by a maximisation with respect to $V \in C(\mathbb{T}, \mathbb{R}_0^+)$ (imposing the integral constraint of Theorem 4.3.1).

Note that in (5.2) we have changed the $\leq$ sign in the input power constraint to an equality, since it is an active constraint. This problem is now more amenable to the tools of the calculus of variations (Gelfand and Fomin 1963).

Let us fix Φ_u and define

$$\tilde{V}(e^{j\omega}) := \frac{N}{2\pi n} \frac{\Phi_v(\omega)}{\Phi_u(\omega)} V(e^{j\omega}).$$

Then, problem (5.2) for $\tilde{V}$ becomes

$$\sup_{\tilde{V} \in C(\mathbb{T}, \mathbb{R}_0^+)} \frac{n}{N} \int_{-\pi}^{\pi} \frac{\Phi_v(\omega) W(e^{j\omega})}{\Phi_u(\omega)} \tilde{V}(e^{j\omega}) d\omega,$$
$$\text{s.t.} \quad \int_{-\pi}^{\pi} \tilde{V}(e^{j\omega}) d\omega = 1.$$

This is a mass distribution problem, hence its optimal cost is

$$\sup_{\substack{\tilde{V} \in C(\mathbb{T}, \mathbb{R}_0^+) \\ \int_{-\pi}^{\pi} \tilde{V}(e^{j\omega}) d\omega = 1}} \frac{n}{N} \int_{-\pi}^{\pi} \frac{\Phi_v(\omega) W(e^{j\omega})}{\Phi_u(\omega)} \tilde{V}(e^{j\omega}) d\omega = \frac{n}{N} \max_{\omega \in [-\pi, \pi]} \frac{\Phi_v(\omega) W(e^{j\omega})}{\Phi_u(\omega)}.$$

Now, if $\Phi_u \neq \Phi_u^{opt}$ for almost every $\omega \in [-\pi, \pi]$, as defined in (5.1), then $\Phi_u(\omega) < \Phi_u^{opt}(\omega)$ for some $\omega = \omega^* \in [-\pi, \pi]$. Otherwise, we would have that $\Phi_u > \Phi_u^{opt}$ in a set of positive measure, hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega > \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_v(\omega) W(e^{j\omega}) d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_v(\tau) W(e^{j\tau}) d\tau} = 1,$$

which contradicts the constraint in Φ_u. Therefore,

$$\sup_{\substack{\tilde{V} \in C(\mathbb{T}, \mathbb{R}_0^+) \\ \int_{-\pi}^{\pi} \tilde{V}(e^{j\omega}) d\omega = 1}} \frac{n}{N} \int_{-\pi}^{\pi} \frac{\Phi_v(\omega) W(e^{j\omega})}{\Phi_u(\omega)} \tilde{V}(e^{j\omega}) d\omega = \frac{n}{N} \max_{\omega \in [-\pi, \pi]} \frac{\Phi_v(\omega) W(e^{j\omega})}{\Phi_u(\omega)}$$
$$\geq \frac{n}{N} \frac{\Phi_v(\omega^*) W(e^{j\omega^*})}{\Phi_u(\omega^*)}$$
$$> \frac{n}{2\pi N} \int_{-\pi}^{\pi} \Phi_v(\omega) W(e^{j\omega}) d\omega,$$

and the cost is minimised with $\Phi_u = \Phi_u^{opt}$. □

[2]The possibility of having more than one Γ associated to the same variance V is not an issue here, since the cost function of the experiment design problem depends on Γ only through $\text{Var}[\hat{G}(z)]$.

Remark 5.4.1 *The desire to obtain a robust input with respect to* Γ *comes from the fact that this quantity is the gradient of G with respect to* θ*, evaluated at the true value of* θ*. Therefore, for a nonlinearly parameterised model, even though the user knows the model structure (since it is a design variable),* Γ *will depend on the true system, hence it will be unknown prior to the experiment. Of course,* Γ *cannot take any possible value in* $\mathscr{H}_2^n$ *for some particular model structures (e.g. linearly parameterised models, for which* Γ *is actually independent of* θ*). However, and in the sense of a fundamental limitation, the result of Theorem 5.4.1 establishes a lower bound (and an input spectrum which achieves it) on the performance of the estimation of the system, even before the selection of the model structure.*

5.5 Unprejudiced Input Design for Finite Model Order

The solution of the min max robust experiment problem obtained in Section 5.4 is now used to obtain an improved unprejudiced open loop input design, in Yuan and Ljung's sense (Yuan and Ljung 1985). First we recall the concept of an unprejudiced input design.

Yuan and Ljung considered an experiment design problem of the form

$$\begin{aligned} &\min_{\Phi_u \geq 0} \int_{-\pi}^{\pi} \mathrm{E}\{|\hat{G}(e^{j\omega}) - G_0(e^{j\omega})|^2\} W(e^{j\omega}) d\omega, \\ &\text{s.t. } \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1, \end{aligned}$$

where $W \in C(\mathbb{T}, \mathbb{R}_0^+)$, and undermodelling, i.e. bias in G can exist. To solve this problem, the mean square error in the estimation of G_0 can be decomposed into bias and variance terms,

$$\mathrm{E}\{|\hat{G}(e^{j\omega}) - G_0(e^{j\omega})|^2\} = |G_0(e^{j\omega}) - G^*(e^{j\omega})|^2 + \mathrm{Var}[\hat{G}(e^{j\omega})],$$

where $G^*(e^{j\omega}) := \lim_{N\to\infty} \hat{G}(e^{j\omega})$ almost surely. This decomposition holds asymptotically in N, in the sense that for finite N, the bias term should consider $\mathrm{E}\{\hat{G}(e^{j\omega})\}$ instead of the limit estimate $G^*(e^{j\omega})$. This approximation, however, allows further simplifications in the calculation of the optimal experiment. Minimisation of the bias term leads to the following solution (Yuan and Ljung 1985):

$$\Phi_u^{opt}(\omega) = c_1 W(e^{j\omega}) |H^*(e^{j\omega})|^2, \tag{5.3}$$

where $H^*(e^{j\omega}) := \lim_{N\to\infty} \hat{H}(e^{j\omega})$ almost surely, and $c_1 > 0$ is a normalisation constant. Notice that this solution is independent of both G_0 and G^*.

With respect to the variance term, (Yuan and Ljung 1985) used an asymptotic (in model order) variance expression (Ljung 1985), which is minimised for the following input spectrum:

$$\Phi_u^{opt}(\omega) = c_2 \sqrt{W(e^{j\omega}) |H_0(e^{j\omega})|^2},$$

where $c_2 > 0$ is a normalisation constant. Note that the asymptotic (in model order) variance expression used to develop this expression for the input spectrum does not consider the effect of bias.

In order to reconcile both expressions for Φ_u^{opt}, H^* is considered as a prefilter (designed by the user), such that

$$|H^*(e^{j\omega})|^2 = c_3 \sqrt{\frac{|H_0(e^{j\omega})|^2}{W(e^{j\omega})}},$$

where $c_3 > 0$. This solution appears to be dimensionally inconsistent, since it forces the noise prefilter to be proportional to the square root of the true noise spectrum, hence creating a paradox.

This paradox arises due to the use of an asymptotic (in model order) variance expression, which only holds approximately for model sets with a shift structure (Ljung 1999, Section 9.4).

To solve this dilemma, we consider the following experiment design problem:

$$\begin{aligned} \min_{\Phi_u \geq 0} \sup_{G \in \mathscr{M}_n} & \int_{-\pi}^{\pi} \mathrm{E}\{|\hat{G}(e^{j\omega}) - G_0(e^{j\omega})|^2\} W(e^{j\omega}) d\omega, \\ \text{s.t.} \quad & \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1, \end{aligned}$$

where $\mathscr{M}_n$ is the set of all stable model structures with n parameters, i.e., $\mathscr{M}_n := \{G : \mathbb{C} \times \Theta \to \mathbb{C} : G(z,\cdot)$ is differentiable in the connected open set $\Theta \subseteq \mathbb{R}^n$ for all $z \in \mathbb{T}$, and $G(\cdot,\theta) \in \mathscr{H}_2$ for all $\theta \in \Theta\}$.

In this problem formulation we are considering the worst case of the (weighted) mean square error over all model structures of a given order. Again, the cost function can be decomposed into both bias and variance terms. The bias term is minimised by (5.3), since the solution is independent of G_0 and G^*. This implies that by taking the supremum over all model structures in $\mathscr{M}_n$ does not affect the previous solution. This argument is formalised in the following theorem.

Theorem 5.5.1 (Optimality of dominant strategies) *Let $J : X \times Y \to \mathbb{R}$ be an arbitrary function, where X and Y are any sets. Assume that there exists an $x^* \in X$ such that*

$$J(x^*, y) = \min_{x \in X} J(x, y) =: C_y \in \mathbb{R}, \quad y \in Y.$$

Then,

$$\sup_{y \in Y} J(x^*, y) = \min_{x \in X} \sup_{y \in Y} J(x, y),$$

therefore x^* is an optimal solution of the min-max problem[3] $\min_{x \in X} \sup_{y \in Y} J(x, y)$.

[3] In game-theoretical terms, Theorem 5.5.1 establishes that a dominating strategy is an equilibrium strategy.

Proof. By definition of the infimum of a function, we have that

$$\inf_{x\in X}\sup_{y\in Y} J(x,y) \leq \sup_{y\in Y} J(x^*,y) = \sup_{y\in Y} C_y. \tag{5.4}$$

On the other hand, by the definition of the supremum,

$$\sup_{y\in Y} J(x,y) \geq J(x,y^\circ), \quad x\in X, y^\circ \in Y.$$

Thus, by taking the infimum over $x \in X$, we obtain

$$\inf_{x\in X}\sup_{y\in Y} J(x,y) \geq \inf_{x\in X} J(x,y^\circ) = \min_{x\in X} J(x,y^\circ) = C_{y^\circ}, \quad y^\circ \in Y. \tag{5.5}$$

Since (5.5) holds for every $y^\circ \in Y$, we can take the supremum over this quantity, which gives

$$\inf_{x\in X}\sup_{y\in Y} J(x,y) \geq \sup_{y^\circ\in Y} C_{y^\circ}. \tag{5.6}$$

The combination of (5.4) and (5.6) shows that inf can be replaced by max (since the infimum is attained with $x = x^*$). This establishes the result. □

For the variance term, we consider the true variance expression

$$\mathrm{Var}[\hat{G}(e^{j\omega})] = \Gamma^H(e^{j\omega})\left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma(e^{j\tau})\Gamma^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma(e^{j\omega}), \tag{5.7}$$

which is asymptotic only in the model order. Notice, however, that we are still not considering the effect of bias on the variance of $\hat{G}$.

The variance term, based on expression (5.7), corresponds exactly to the min-max robust optimal experiment design problem considered in Section 5.4, hence the solution (from Theorem 5.4.1) is

$$\Phi_u^{opt}(\omega) = c_4 W(e^{j\omega})|H_0(e^{j\omega})|^2, \tag{5.8}$$

where $c_4 > 0$ is a normalisation constant.

Remark 5.5.1 *Note that* (5.8) *and* (5.3) *can be naturally combined by letting* $H^* = H_0$*!*

Since H_0 *is typically unknown, and there is usually bias in* H*, it is not always possible to obtain* $H^* = H_0$*. However, the previous paragraph tells us that a good experiment is given by* $\Phi_u^{opt}(\omega) \propto W(e^{j\omega})|\tilde{H}(e^{j\omega})|^2$*, where* $\tilde{H}$ *is a good estimate of* H_0*.*

The optimal input obtained here has a nice interpretation, i.e. it is naturally chosen such that the signal-to-noise ratio is proportional at each frequency to the weighting function W*.*

5.6 Experiment Design with Diffuse Prior Information

In Chapter 3, the problem of designing a 'good' input signal with only diffuse prior information was examined. In that chapter, Ljung's asymptotic variance expression (Ljung 1985) was utilised. In this section, we show that the results of Chapter 3 are also valid for finite model orders. We assume no undermodeling, i.e. there exists a $\theta = \theta_0$ such that $G_0(z) = G(z, \theta_0)$.

Our aim, as stated in Chapter 3, is to design an experiment which is 'good' for a very broad class of systems. This means that we need a measure of 'goodness' of an experiment which is system independent. As argued in Chapters 2 and 3, absolute variances are not particularly useful when one wants to design an experiment that applies to a broad class of systems. Hence, it seems preferable to work with relative errors. Rather than look at a single frequency ω, we will look at an 'average' measure over a range of frequencies. This leads to a general measure of the 'goodness' of an experiment,

$$J(\Phi_u) = \int_a^b F(\mathrm{Var}[\hat{G}(e^{j\omega})]/|G(e^{j\omega})|^2) W(\omega) d\omega, \tag{5.9}$$

where

$$\mathrm{Var}[\hat{G}(e^{j\omega})] = \Gamma^H(e^{j\omega}) \left[\frac{N}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(e^{j\omega}),$$

F and W are functions to be specified later, and $0 < a < b < 2\pi$, as discussed in Chapter 3.

Recall that it is argued in Chapter 3 that the functions F and W should satisfy the following criteria:

A.1) The optimal experiment, Φ_u^{opt}, which minimises $\sup_{\Gamma \in \mathscr{H}_2^n} J$ in (5.9), should be independent of the system G and the noise dynamics Φ_v.

A.2) The integrand in (5.9) should increase if the variance $\mathrm{Var}[\hat{G}(e^{j\omega})]$ increases at any frequency. This implies that F should be a monotonically increasing function.

B) The weighting function W should satisfy the following: for every $0 < \alpha < \beta < 2\pi$ and every $k > 0$ such that $0 < k\alpha < k\beta < 2\pi$,

$$\int_\alpha^\beta W(\omega) d\omega = \int_{k\alpha}^{k\beta} W(\omega) d\omega.$$

Criteria A.1 and A.2 are based on the desire to design an input signal which is independent of the system and the noise variance. Note that Criterion A.1 is not exactly the same as stated in Chapter 3, since we are considering the worst case of J over all possible systems and model structures (of order n).

Criterion B on the other hand, is based on the observation that many properties of linear systems depend on the *ratio* of poles and zeros rather than on their absolute locations in the frequency domain.

Lemma 3.4.2, from Chapter 3, shows how W must be chosen to satisfy Criterion B. However, Criteria A.1 and A.2 force W to have a very particular form, as shown in the following Lemma:

Lemma 5.6.1 (Experiment design with diffuse prior information) *Consider the experiment design problem:*

$$\begin{aligned} \min_{\Phi_u \geq 0} \sup_{\Gamma \in \mathscr{H}_2^n} & \int_a^b F\left(\frac{\mathrm{Var}[\hat{G}(e^{j\omega})]}{|G(e^{j\omega})|^2}\right) W(\omega) d\omega \\ s.t. \quad & \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1, \end{aligned} \tag{5.10}$$

where $0 < a < b < 2\pi$, $F \in C^1([a,b], \mathbb{R}_0^+)$, $W \in C^1([a,b], \mathbb{R}^+)$, $|G|^2$ *is differentiable on* $\mathbb{T}$, *and*

$$\mathrm{Var}[\hat{G}(z)] = \Gamma^H(z)\left[\frac{N}{2\pi}\int_{-\pi}^{\pi} \Gamma(e^{j\tau})\Gamma^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau\right]^{-1}\Gamma(z)$$

for $z \in \mathbb{T}$. *Let* Φ_u^{opt} *be a stationary point of this problem. If* Φ_u^{opt} *does not depend on G, then there exist constants* $\alpha, \beta \in \mathbb{R}$ *such that*

$$F(y) = \alpha \ln y + \beta, \quad \inf_{\omega \in [a,b]} \frac{\mathrm{Var}[\hat{G}(e^{j\omega})]}{|G(e^{j\omega})|^2} \leq y \leq \sup_{\omega \in [a,b]} \frac{\mathrm{Var}[\hat{G}(e^{j\omega})]}{|G(e^{j\omega})|^2},$$

and

$$\Phi_u^{opt}(\omega) := \begin{cases} \frac{W(\omega)}{\frac{1}{2\pi}\int_a^b W(\tau) d\tau}, & \omega \in [a,b], \\ 0, & otherwise. \end{cases}$$

Proof. As in the proof of Theorem 5.4.1, by Theorems 4.3.1 and 5.3.1, the experiment design problem is equivalent to

$$\begin{aligned} \min_{\Phi_u \geq 0} \sup_{V \in C(\mathbb{T}, \mathbb{R}_0^+)} & \int_{-\pi}^{\pi} F\left(\frac{V(e^{j\omega})}{|G(e^{j\omega})|^2}\right) W(\omega) d\omega, \\ \text{s.t.} \quad & \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) d\omega = 1, \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega}) d\omega = \frac{n}{N}. \end{aligned} \tag{5.11}$$

Let G and W be fixed, and assume that Φ_u^{opt} exists.

The Lagrangian of problem (5.11) is

$$L(V, \Phi_u, \lambda_1, \lambda_2) := \frac{1}{2\pi}\int_a^b \left[2\pi F\left(\frac{V(e^{j\omega})}{|G(e^{j\omega})|^2}\right) W(\omega) + \lambda_1 \Phi_u(\omega) + \lambda_2 \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega})\right] d\omega,$$

where λ_1 and λ_2 are Lagrange multipliers. By (Luenberger 1969, Section 7.7, Theorem 2) there are constants $\lambda_1, \lambda_2 \in \mathbb{R}$ for which (V^{opt}, Φ_u^{opt}), the solution of (5.11), is a stationary point of $L(V, \Phi_u, \lambda_1, \lambda_2)$.

Thus, for any $h_1, h_2 \in C^1([a,b], \mathbb{R}_0^+)$ we have that $\delta L(V^{opt}, \Phi_u^{opt}, \lambda_1, \lambda_2; [h_1\ h_2]^T) = 0$, which means (Luenberger 1969, Section 7.5) that

$$\int_a^b \left\{ \left[\frac{W(\omega)}{|G(e^{j\omega})|^2} F' \left(\frac{V^{opt}(e^{j\omega})}{|G(e^{j\omega})|^2} \right) + \lambda_1 \frac{\Phi_u^{opt}(\omega)}{\Phi_v(\omega)} \right] h_1'(\omega) + \left[\lambda_1 \frac{V^{opt}(e^{j\omega})}{\Phi_v(\omega)} + \lambda_2 \right] h_2'(\omega) \right\} d\omega = 0.$$

Thus, by (Luenberger 1969, Section 7.5, Lemma 1),

$$\frac{2\pi W(\omega)}{|G(e^{j\omega})|^2} F' \left(\frac{V^{opt}(e^{j\omega})}{|G(e^{j\omega})|^2} \right) + \lambda_1 \frac{\Phi_u^{opt}(\omega)}{\Phi_v(\omega)} = 0, \tag{5.12}$$

$$\lambda_1 \frac{V^{opt}(e^{j\omega})}{\Phi_v(\omega)} + \lambda_2 = 0, \quad \omega \in [a,b]. \tag{5.13}$$

From (5.13) we have that

$$V^{opt}(e^{j\omega}) = -\frac{\lambda_2}{\lambda_1} \Phi_v(\omega). \tag{5.14}$$

Thus, if we substitute (5.14) into (5.12), and let $l(e^{j\omega}) := V^{opt}(e^{j\omega})/|G(e^{j\omega})|^2$, we have

$$l(e^{j\omega}) F'(l(e^{j\omega})) = \frac{\lambda_2}{2\pi} \frac{\Phi_u^{opt}(\omega)}{W(\omega)}, \quad \omega \in [a,b]. \tag{5.15}$$

The left side of (5.15) depends on G (through l), however the right side does not (due to the assumption on the independence of Φ_u^{opt} upon G). Thus, both sides are equal to a constant, say, $\alpha \in \mathbb{R}$ which implies that

$$F'(l(e^{j\omega})) = \frac{\alpha}{l(e^{j\omega})}, \quad \omega \in [a,b].$$

Now, by integrating both sides with respect to l between $\inf_{\omega \in [a,b]} l(e^{j\omega})$ and $\sup_{\omega \in [a,b]} l(e^{j\omega})$, we obtain

$$F(l(e^{j\omega})) = \alpha \ln l(e^{j\omega}) + \beta, \quad \omega \in [a,b],$$

for some constant $\beta \in \mathbb{R}$.

On the other hand, we have that

$$\frac{\lambda_2}{2\pi} \frac{\Phi_u^{opt}(\omega)}{W(\omega)} = \alpha, \quad \omega \in [a,b],$$

hence Φ_u^{opt} is proportional to W in $[a,b]$, and can be made equal to 0 outside this interval. This concludes the proof. □

In the following lemma we establish that for the choice of F given by Lemma 5.6.1, Φ_u^{opt} actually corresponds to the global optimum of the experiment design problem (5.10).

Lemma 5.6.2 (Global optimality of the experiment design problem with diffuse prior information) *Consider the experiment design problem:*

$$\min_{\Phi_u \geq 0} \sup_{\Gamma \in \mathscr{H}_2^n} \int_a^b \ln \left[\frac{\mathrm{Var}[\hat{G}(e^{j\omega})]}{|G(e^{j\omega})|^2} \right] W(\omega) d\omega$$
$$\text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1,$$

where $0 < a < b < 2\pi$, $W \in C^1([a,b], \mathbb{R}^+)$, $|G|^2$ *is differentiable on* $\mathbb{T}$, *and*

$$\mathrm{Var}[\hat{G}(z)] = \Gamma^H(z) \left[\frac{N}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(z)$$

for $z \in \mathbb{T}$. *The solution to this problem is given by*

$$\Phi_u^{opt}(\omega) = \begin{cases} \frac{W(\omega)}{\frac{1}{2\pi}\int_a^b W(\tau) d\tau}, & \omega \in [a,b], \\ 0, & \textit{otherwise}. \end{cases} \tag{5.16}$$

Proof. To simplify the development, we extend W to a periodic function of period 2π in $\mathbb{R}$ by making it equal to 0 in $[0, 2\pi] \setminus [a,b]$. Then, as in the proof of Lemma 5.6.1, the experiment design problem is equivalent to

$$\min_{\Phi_u \geq 0} \sup_{V \in C(\mathbb{T}, \mathbb{R}_0^+)} \int_{-\pi}^{\pi} \ln \left[\frac{V(e^{j\omega})}{|G(e^{j\omega})|^2} \right] W(\omega) d\omega,$$
$$\text{s.t.} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega = 1,$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega}) d\omega = \frac{n}{N}.$$

Let Φ_u be fixed. Then, the cost function can be written as

$$\int_{-\pi}^{\pi} \ln \left[\frac{V(e^{j\omega})}{|G(e^{j\omega})|^2} \right] W(\omega) d\omega = C_1 + \int_{-\pi}^{\pi} \ln[\tilde{V}(e^{j\omega})] W(\omega) d\omega,$$

where C_1 is a constant, independent of V, given by

$$C_1 = -\int_{-\pi}^{\pi} \ln \left[\frac{N|G(e^{j\omega})|^2 \Phi_u(\omega)}{2\pi n \Phi_v(\omega)} \right] W(\omega) d\omega,$$

and

$$\tilde{V}(e^{j\omega}) = \frac{N\Phi_u(\omega)}{2\pi n \Phi_v(\omega)} V(e^{j\omega}), \quad \omega \in [-\pi, \pi].$$

Note that due to the constraint in V, $\tilde{V}$ should satisfy

$$\int_{-\pi}^{\pi} \tilde{V}(e^{j\omega}) d\omega = 1. \tag{5.17}$$

Let $\tilde{V}^{opt} = cW$, where c is chosen to satisfy (5.17), and let $\tilde{V}$ be any function in $C(\mathbb{T}, \mathbb{R}_0^+)$ which satisfies (5.17). Then, since $\ln(1+x) \leq x$ for all $x \in (-1, \infty)$, with equality if and only if $x = 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \ln[\tilde{V}(e^{j\omega})]W(\omega)d\omega &= \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega}) + (\tilde{V}(e^{j\omega}) - \tilde{V}^{opt}(e^{j\omega}))]W(\omega)d\omega \\
&= \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega})]W(\omega)d\omega + \int_{-\pi}^{\pi} \ln\left[1 + \frac{\tilde{V}(e^{j\omega}) - \tilde{V}^{opt}(e^{j\omega})}{\tilde{V}^{opt}(e^{j\omega})}\right] W(\omega)d\omega \\
&\leq \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega})]W(\omega)d\omega + \int_{-\pi}^{\pi} \frac{\tilde{V}(e^{j\omega}) - \tilde{V}^{opt}(e^{j\omega})}{\tilde{V}^{opt}(e^{j\omega})} W(\omega)d\omega \\
&= \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega})]W(\omega)d\omega + \frac{1}{c}\int_{-\pi}^{\pi} [\tilde{V}(e^{j\omega}) - \tilde{V}^{opt}(e^{j\omega})]d\omega \qquad (5.18) \\
&= \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega})]W(\omega)d\omega,
\end{aligned}$$

with equality if and only if $\tilde{V} = \tilde{V}^{opt}$.

The previous derivation implies that for a given Φ_u, we have that

$$\begin{aligned}
\sup_{V \in C(\mathbb{T}, \mathbb{R}_0^+)} \int_{-\pi}^{\pi} \ln\left[\frac{V(e^{j\omega})}{|G(e^{j\omega})|^2}\right] W(\omega)d\omega &= C_1 + \int_{-\pi}^{\pi} \ln[\tilde{V}^{opt}(e^{j\omega})]W(\omega)d\omega \\
&= C_2 - \int_{-\pi}^{\pi} \ln[\Phi_u(\omega)]W(\omega)d\omega,
\end{aligned}$$

where the supremum is taken over all V satisfying the integral constraint of the experiment design problem, and C_2 is given by

$$C_2 = \int_{-\pi}^{\pi} \ln\left[\frac{2\pi nc\Phi_v(\omega)}{N|G(e^{j\omega})|^2}\right] W(\omega)d\omega.$$

Now, take Φ_u^{opt} as in (5.16). Then, following a similar derivation to that in (5.18), we have

$$C_2 - \int_{-\pi}^{\pi} \ln[\Phi_u(\omega)]W(\omega)d\omega \leq C_2 - \int_{-\pi}^{\pi} \ln[\Phi_u^{opt}(\omega)]W(\omega)d\omega,$$

with equality if and only if $\Phi_u = \Phi_u^{opt}$. This proves that Φ_u^{opt} is the optimal solution of the experiment design problem. □

Notice that, by Lemmas 3.4.2 and 5.6.1, Criteria A.1, A.2 and B imply that a reasonable experiment design problem, when only diffuse prior information regarding the system and the noise is available, can be stated as

$$\begin{aligned}
&\min_{\Phi_u \geq 0} \sup_{\Gamma \in \mathscr{H}_2^n} \int_a^b \ln\left[\frac{\mathrm{Var}[\hat{G}(e^{j\omega})]}{|G(e^{j\omega})|^2}\right] \frac{1}{\omega} d\omega \\
&\qquad \text{s.t.} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \leq 1.
\end{aligned}$$

Moreover, by Lemma 5.6.2, the optimal input spectrum is given by

$$\Phi_u^{opt}(\omega) = \frac{1/\omega}{\int_a^b \frac{d\omega}{\omega}} = \frac{1/\omega}{\ln b - \ln a}, \quad \omega \in [a, b],$$

which is bandlimited '$1/f$' noise! See Chapter 2. Hence, this extends the results of Chapter 3 to finite model orders.

5.7 Summary

In this chapter we have used the fundamental limitation results developed in Chapter 4 to solve several robust experiment design problems.

First we derived a closed form expression for the input spectrum which minimises the maximum value of a weighted integral of the variance of the frequency response estimator, over all model structures with a given number of parameters. Based on this solution, we revisited the problem studied in Chapter 3, which consisted of finding a class of cost functions that give an optimal input independent of the true system and noise dynamics. The solution, which is valid for finite order models, turns out to be the same as that obtained in Chapter 3, namely bandlimited '$1/f$' noise.

Finally, we analysed Yuan & Ljung's unprejudiced optimal input design. With the approach developed in this chapter, we obtained an unprejudiced optimal input that is valid for finite order models, and solves the apparent paradox present in Yuan & Ljung's original result.

It is important to note that neither the variance expression utilised in Yuan & Ljung's approach, nor the expression used here, take into account the effect of undermodelling. Furthermore, both approaches consider the noise dynamics as known. However, we believe our result goes a step further towards a truly unprejudiced input design.

5.8 Appendix: Proof of Theorem 5.3.1

To prove Theorem 5.3.1, we first need to establish the following lemma.

Lemma 5.8.1 (Uniform approximation of the variance expression) *Let $\Phi_u, \Phi_v : [-\pi, \pi] \to \mathbb{R}^+$ be continuous and even. Also, let $V \in C(\mathbb{T}, \mathbb{R}_0^+)$ such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_u(\omega)}{\Phi_v(\omega)} V(e^{j\omega}) d\omega = \frac{n}{N},$$

where $n, N \in \mathbb{N}$. Then, for every $\varepsilon > 0$ there exists a vector-valued polynomial in z^{-1}, $\Gamma \in \mathscr{H}_2^n$, such that

$$\left| \Gamma^H(z) \left[\frac{N}{2\pi} \int_{-\pi}^{\pi} \Gamma(e^{j\tau}) \Gamma^H(e^{j\tau}) \frac{\Phi_u(\tau)}{\Phi_v(\tau)} d\tau \right]^{-1} \Gamma(z) - V(z) \right| < \varepsilon,$$

for every $z \in \mathbb{T}$.

Proof. Defining

$$\tilde{\varepsilon} := \frac{N}{2\pi}\left[\min_{\omega\in[-\pi,\pi]}\frac{\Phi_u(\omega)}{\Phi_v(\omega)}\right]\varepsilon$$
$$\tilde{\Gamma}(e^{j\omega}) := \Gamma(e^{j\omega})\sqrt{\frac{\Phi_u(\omega)}{\Phi_v(\omega)}}$$
$$\tilde{V}(e^{j\omega}) := \frac{N}{2\pi}\frac{\Phi_u(\omega)}{\Phi_v(\omega)}V(e^{j\omega}), \quad \omega\in[-\pi,\pi],$$

it is readily seen that Theorem 5.3.1 would follow from establishing the existence of a function $\tilde{\Gamma}\in\mathscr{H}_2^n$ such that

$$\left|\tilde{\Gamma}^H(z)\left[\int_{-\pi}^{\pi}\tilde{\Gamma}(e^{j\tau})\tilde{\Gamma}^H(e^{j\tau})d\tau\right]^{-1}\tilde{\Gamma}(z)-\tilde{V}(z)\right|<\tilde{\varepsilon}, \quad z\in\mathbb{T}. \tag{5.19}$$

Now, let $m\geq 2n$ such that $|\tilde{V}(e^{j\omega_1})-\tilde{V}(e^{j\omega_2})|<\tilde{\varepsilon}/6$ whenever $|\omega_1-\omega_2|\leq 2\pi/m$ (for $\omega_1,\omega_2\in[-\pi,\pi]$). According to the Lemma of (Lieb 1981), there are n orthonormal vectors $v^i\in\mathbb{R}^m$ such that[4]

$$\sum_{i=1}^{n}|v_k^i|^2=\frac{2\pi}{m}\tilde{V}\left(\frac{2\pi}{m}[k-1/2]-\pi\right)+\frac{\eta}{m}, \quad k=1,\ldots,m, \tag{5.20}$$

where

$$\eta := n-\frac{2\pi}{m}\sum_{k=1}^{m}\tilde{V}\left(\frac{2\pi}{m}[k-1/2]-\pi\right).$$

Thus, if we define the function $\Gamma_1:\ \mathbb{T}\to\mathbb{R}^n$ by $[\Gamma_1(e^{j\omega})]_i=\sqrt{m/2\pi}v_k^i$ for $\omega\in[2\pi(k-1)/m-\pi,2\pi k/m-\pi)$ and $i=1,\ldots,n$, then it holds that

$$\left[\int_{-\pi}^{\pi}\Gamma_1(e^{j\omega})\Gamma_1^H(e^{j\omega})d\omega\right]_{il}=\sum_{k=1}^{m}v_k^i(v_k^l)^*\frac{m}{2\pi}\frac{2\pi}{m}=(v^l)^Hv^i=\delta_{i,l}, \quad i,l=1,\ldots,n,$$

where $\delta_{i,l}$ is the Kronecker Delta function, so

$$\int_{-\pi}^{\pi}\Gamma_1(e^{j\omega})\Gamma_1^H(e^{j\omega})d\omega=I.$$

On the other hand,

$$\Gamma_1^H(e^{j\omega})\Gamma_1(e^{j\omega})=\sum_{i=1}^{m}|v_k^i|^2\frac{m}{2\pi}=\tilde{V}\left(\frac{2\pi}{m}[k-1/2]-\pi\right)+\frac{\eta}{2\pi}, \quad \omega\in[2\pi(k-1)/m-\pi,2\pi k/m-\pi).$$

Thus,

$$\begin{aligned}\left|\Gamma_1^H(e^{j\omega})\left[\int_{-\pi}^{\pi}\Gamma_1(e^{j\tau})\Gamma_1^H(e^{j\tau})d\tau\right]^{-1}\Gamma_1(e^{j\omega})-\tilde{V}(e^{j\omega})\right| &= \left|\Gamma_1^H(e^{j\omega})\Gamma_1(e^{j\omega})-\tilde{V}(e^{j\omega})\right| \\ &= \left|\tilde{V}\left(\frac{2\pi}{m}[k-1/2]-\pi\right)-\tilde{V}(e^{j\omega})+\frac{\eta}{2\pi}\right| \\ &< \frac{\varepsilon}{6}+\frac{|\eta|}{2\pi} \\ &< \frac{\varepsilon}{6}+\frac{\varepsilon}{6} \\ &= \frac{\varepsilon}{3}, \quad \omega\in[-\pi,\pi],\end{aligned} \tag{5.21}$$

[4]The term η/m in (5.20) is due to the requirement that $\sum_{k=1}^{m}\sum_{i=1}^{n}|v_k^i|^2=\sum_{i=1}^{n}\|v^i\|_2^2=n$.

since

$$\frac{|\eta|}{2\pi} = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\tilde{V}(e^{j\omega})d\omega - \frac{2\pi}{m}\sum_{k=1}^{m}\tilde{V}\left(\frac{2\pi}{m}[k-1/2]-\pi\right)\right| < \frac{1}{2\pi}m\frac{2\pi}{m}\frac{\varepsilon}{6} = \frac{\varepsilon}{6}.$$

Now, let $\Gamma_2 : \mathbb{T} \to \mathbb{R}^n$ be a continuous function such that $\Gamma_2(z^*) = [\Gamma_2(z)]^*$ for all $z \in \mathbb{T}$, and

$$\left|\Gamma_1^H(z)\Gamma_1(z) - \Gamma_2^H(z)\left[\int_{-\pi}^{\pi}\Gamma_2(e^{j\tau})\Gamma_2^H(e^{j\tau})d\tau\right]^{-1}\Gamma_2(z)\right| < \frac{\varepsilon}{2}, \quad z \in \mathbb{T}. \tag{5.22}$$

Here we replace Γ_1, for a given $\alpha > 0$, by a piecewise linear function Γ_2 such that

$$\left\|\left[\int_{-\pi}^{\pi}\Gamma_2(e^{j\tau})\Gamma_2^H(e^{j\tau})d\tau\right]^{-1} - I\right\|_{\infty} < \alpha$$

and $|\Gamma_2^H(z)\Gamma_2(z) - \Gamma_1^H(z)\Gamma_1(z)| < \varepsilon/6$ for every $z \in \mathbb{T}$, the later being possible since $|\tilde{V}(e^{j\omega_1}) - \tilde{V}(e^{j\omega_2})| < \tilde{\varepsilon}/6$ whenever $|\omega_1 - \omega_2| \leq 2\pi/m$. Thus, we can choose α small enough to make (5.22) hold.

Finally, since $\Gamma \mapsto \Gamma^H(\int_{-\pi}^{\pi}\Gamma(e^{j\tau})\Gamma^H(e^{j\tau})d\tau)^{-1}\Gamma$ is continuous with respect to the uniform norm of $C(\mathbb{T},\mathbb{R}^n)$ in a neighbourhood of Γ_2, by Weierstrass' Second Theorem (Achieser 1956) there exists a (vector-valued) trigonometric polynomial

$$\Gamma_3(e^{j\omega}) = \sum_{i=-p}^{p} a_{|i|}e^{j\omega i}, \quad \omega \in [-\pi,\pi],$$

with $a_i \in \mathbb{R}^n$ for $i = -p,\ldots,p$, such that

$$\left|\Gamma_3^H(z)\left[\int_{-\pi}^{\pi}\Gamma_3(e^{j\tau})\Gamma_3^H(e^{j\tau})d\tau\right]^{-1}\Gamma_3(z) - \Gamma_2^H(z)\left[\int_{-\pi}^{\pi}\Gamma_2(e^{j\tau})\Gamma_2^H(e^{j\tau})d\tau\right]^{-1}\Gamma_2(z)\right| < \frac{\varepsilon}{6}, \tag{5.23}$$

for every $z \in \mathbb{T}$. Therefore, the function $\tilde{\Gamma} \in \mathscr{H}_2^n$ given by $\tilde{\Gamma}(z) := \Gamma_3(z)z^{-p}$ satisfies (5.19), as seen by combining (5.21)–(5.23). □

Proof of Theorem 5.3.1. To proceed, we use Lemma 5.8.1 to construct a sequence of functions in $\mathscr{H}_2^n$, $\{\Gamma_k\}_{k=1}^{\infty}$, such that

$$\left|\Gamma_k^H(z)\left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma_k(e^{j\tau})\Gamma_k^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma_k(z) - V(z)\right| < \frac{1}{k}, \quad z \in \mathbb{T}.$$

Since the Γ_k's are polynomials in z^{-1}, they are analytic in the set $\mathbb{E}_{1/2} := \{z \in \mathbb{C} : |z| > 1/2\}$, and in particular they are bounded in this set. This, together with the fact that

$$\Gamma \mapsto \Gamma^H\left[\frac{N}{2\pi}\int_{-\pi}^{\pi}\Gamma(e^{j\tau})\Gamma^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma$$

is invariant under scaling of its argument, implies that we can assume that

$$\lim_{r\to\frac{1}{2}+}\max_{z\in\mathbb{T}}\|\Gamma_k(rz)\|_2 = 1, \quad k \in \mathbb{N}. \tag{5.24}$$

Furthermore, by applying a suitable constant unitary linear transformation to each Γ_k, we can assume that

$$\int_{-\pi}^{\pi} \Gamma_k(e^{j\tau})\Gamma_k^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau = \alpha_k I, \quad k \in \mathbb{N},$$

where $\alpha_k > 0$ for every $k \in \mathbb{N}$.

From the Maximum Modulus Theorem (Rudin 1987), it follows that $\{\Gamma_k\}_{k=1}^{\infty}$ is uniformly bounded (by 1) in $\mathbb{E}_{1/2}$. Therefore, by (Rudin 1987, Theorem 14.6) we have that $\{\Gamma_k\}_{k=1}^{\infty}$ is a *normal family* in $\mathbb{E}_{1/2}$, i.e., there exists a subsequence $\{\Gamma_{k_i}\}_{i=1}^{\infty}$ which converges uniformly on compact subsets of $\mathbb{E}_{1/2}$. Let Γ_∞ be the limit of this subsequence. Note that Γ_∞ is analytic in $\mathbb{E}_{1/2}$ by (Rudin 1987, Theorem 10.28), and belongs to $\mathscr{H}_2^n$ because $\sup_{z\in\mathbb{E}_{1/2}} \|\Gamma_\infty(z)\|_2 \leq 1$.

Since $\mathbb{T} \subset \mathbb{E}_{1/2}$ is compact, $\Gamma_{k_i} \to \Gamma_\infty$ uniformly in $\mathbb{T}$. Also, the function

$$\Gamma \mapsto \Gamma^H \left[\frac{N}{2\pi}\int_{-\pi}^{\pi} \Gamma(e^{j\tau})\Gamma^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma$$

is continuous in a neighbourhood of Γ_∞ if

$$\int_{-\pi}^{\pi} \Gamma_\infty(e^{j\tau})\Gamma_\infty^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau = \lim_{i\to\infty}\alpha_{k_i} I > 0.$$

Therefore, we have that

$$\begin{aligned}
&\Gamma_\infty^H(z)\left[\frac{N}{2\pi}\int_{-\pi}^{\pi} \Gamma_\infty(e^{j\tau})\Gamma_\infty^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma_\infty(z) \\
&\qquad = \lim_{i\to\infty}\Gamma_{k_i}^H(z)\left[\frac{N}{2\pi}\int_{-\pi}^{\pi} \Gamma_{k_i}(e^{j\tau})\Gamma_{k_i}^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma_{k_i}(z) \\
&\qquad = V(z), \quad z \in \mathbb{T}.
\end{aligned}$$

Thus, in order to show that Γ_∞ satisfies the condition of the Theorem, we need to show that $\lim_{i\to\infty}\alpha_{k_i} > 0$. This can be seen from the expression

$$V_{k_i}(z) := \Gamma_{k_i}^H(z)\left[\frac{N}{2\pi}\int_{-\pi}^{\pi} \Gamma_{k_i}(e^{j\tau})\Gamma_{k_i}^H(e^{j\tau})\frac{\Phi_u(\tau)}{\Phi_v(\tau)}d\tau\right]^{-1}\Gamma_{k_i}(z) = \alpha_{k_i}^{-1}\Gamma_{k_i}^H(z)\Gamma_{k_i}(z) = \alpha_{k_i}^{-1}\|\Gamma_{k_i}(z)\|_2^2,$$

for $i \in \mathbb{N}$ and $z \in \mathbb{T}$, where $V_{k_i} \to V$ uniformly in $\mathbb{T}$ as $i \to \infty$. Therefore, by taking the maximum over $\mathbb{T}$ and letting $i \to \infty$, we obtain

$$\max_{z\in\mathbb{T}} V(z) = \lim_{i\to\infty}\max_{z\in\mathbb{T}} V_{k_i}(z) = \lim_{i\to\infty}\alpha_{k_i}^{-1}\max_{z\in\mathbb{T}}\|\Gamma_{k_i}(z)\|_2^2 = \lim_{i\to\infty}\alpha_{k_i}^{-1}\max_{z\in\mathbb{T}}\|\Gamma_\infty(z)\|_2^2.$$

This implies that

$$\lim_{i\to\infty}\alpha_{k_i} = \frac{\max_{z\in\mathbb{T}}\|\Gamma_\infty(z)\|_2^2}{\max_{z\in\mathbb{T}} V(z)} > 0,$$

since otherwise (5.24) would not hold. This concludes the proof. □

CHAPTER 6

LEAST COSTLY AND TRADITIONAL EXPERIMENT DESIGN FOR CONTROL

6.1 Introduction

The least costly and traditional approaches to experiment design differ in the way the input/output power is considered as part of the optimisation. Specifically, we say that a given experiment design formulation is in the *traditional* framework if the input/output power is included as a constraint in the optimisation problem, such that the purpose of the optimal experiment is to maximise a given quantity related to the model quality, i.e. a function of the parameter covariance matrix. On the other hand, a *least costly* experiment design formulation is defined as an optimisation problem where the input/output power is minimised subject to a model quality constraint, given as a function of the parameter covariance matrix.

In (Bombois et al. 2006) the least costly paradigm is stated, but not established, to be a 'dual approach' to the traditional optimal experiment design problem. Here we establish equivalence between the two paradigms and hence show that they are indeed dual problems. Specifically, we show equivalence between the traditional optimal experiment design problem and the results for the least costly approach for both open and closed loop systems. In particular we establish equivalence for each of the cases analysed in the least costly framework (Bombois et al. 2004a;b; 2006).

In open loop equivalence is established for three cases relating to different parametrisations of the covariance expression (finite and high order approximations) and model structure (dependent and independently parameterised system and noise models). In the closed loop setting, only finite order covariance expressions are considered. Furthermore, by using $\mathscr{H}_\infty$ performance specifications for control, bounds on the covariance expression, for both the open and closed loop cases, are determined.

Essentially, the results show that solutions of several experiment design problems in the least costly framework are equivalent to scaled versions of solutions to corresponding traditional experiment design problems. This implies that it is possible to make use of computationally efficient algorithms developed for one framework in the other framework. Additionally, the equivalence allows the incor-

poration of additional constraints into the least costly framework. It also permits the results obtained in one framework to be interpreted in the other. For example, there are usually hard constraints on the input power or amplitude, due to actuator limitations. Thus, if the power of the optimal least costly input (where the cost has been measured in terms of the input power) exceeds the maximum allowed value, the equivalence results show that this is due to an excessively tight constraint on the model quality. Thus, by translating the problem into the traditional framework and then reverting back to the least costly framework, it is possible to modify this constraint in an appropriate way so as to satisfy the hard input power constraint.

6.2 Technical Preliminaries

In this section we develop several preliminary results which will be utilised in the sequel to establish the equivalence between least costly and traditional experiment design problems for control. Theorem 6.2.1 below provides a duality result between two optimisation problems, where the roles of the cost function and constraints are exchanged. This result will be used repeatedly to show the equivalence between the two experiment design frameworks.

Let X be a cone[1], and $f, g : X \to \mathbb{R}_0^+$ be[2] such that for every $x \in X$ and $\alpha > 0$, $f(\alpha x) = \alpha f(x)$ and $g(\alpha x) = \alpha^{-1} g(x)$.

Now define the following optimisation problems:

$$\text{Problem A: } \min_{x \in X} f(x) \text{ s.t. } g(x) \leq 1.$$

$$\text{Problem B: } \min_{y \in X} g(y) \text{ s.t. } f(y) \leq 1.$$

Lemma 6.2.1 (Strengthening of Problem A) *Assume that Problem A has a solution $x^{opt} \in X$. Then x^{opt} is also a solution of the following problem:*

$$\text{Problem A': } \min_{x \in X} f(x) \text{ s.t. } g(x) = 1.$$

Proof. If $g(x^{opt}) < 1$, take $x = g(x^{opt})x^{opt}$. Then, $g(x) = [1/g(x^{opt})]g(x^{opt}) = 1$ and $f(x) = g(x^{opt})f(x^{opt}) < f(x^{opt})$. This contradicts the optimality of x^{opt}. Thus, $g(x^{opt}) = 1$. □

[1] A set X is a *cone* if it is subset of a vector space, and $x \in X$ implies that $\tau x \in X$ for all $\tau \geq 0$ (Rockafellar 1970).

[2] f and g are called *positively homogeneous functions of degree* 1 and -1, respectively (Lasserre and Hiriart-Urruty 2002).

Corollary 6.2.1 (Strengthening of Problem B) *Problem B is equivalent to*

$$\text{Problem B': } \min_{y \in X} g(y) \text{ s.t. } f(y) = 1.$$

Proof. Follows from proof of Lemma 6.2.1. □

Theorem 6.2.1 (Equivalence between problems A and B) *Problem A has a solution, $x^{opt} \in X$, if and only if Problem B has a solution, $y^{opt} \in X$. Moreover, if a solution x^{opt} of Problem A exists, then $x^{opt} = Ky^{opt}$, where $K = f(x^{opt}) = g(y^{opt})$ and y^{opt} is a solution of Problem B, and vice versa.*

Proof. Let $x^{opt} \in X$ be a solution of Problem A. Then, by Lemma 6.2.1, $g(x^{opt}) = 1$. Take $K = f(x^{opt})$. Then, $y' = K^{-1}x^{opt}$ satisfies

$$\begin{aligned} f(y') &= f(K^{-1}x^{opt}) = K^{-1}f(x^{opt}) = [1/f(x^{opt})]f(x^{opt}) = 1, \\ g(y') &= g(K^{-1}x^{opt}) = Kg(x^{opt}) = f(x^{opt})g(x^{opt}) = f(x^{opt}). \end{aligned} \tag{6.1}$$

Now, $g(y') \geq \inf_{y \in X, f(y)=1} g(y)$. Assume that y' is not a solution of Problem B, i.e. that $g(y') > \inf_{y \in X, f(y)=1} g(y)$. Then there exists a $y'' \in X$ such that $g(y'') < g(y')$ and $f(y'') = 1$. Thus, taking $K' = g(y'')$ and $x' = K'y''$, we have, by (6.1), that

$$\begin{aligned} g(x') &= g(K'y'') = (K')^{-1}g(y'') = [1/g(y'')]g(y'') = 1, \\ f(x') &= K'f(y'') = g(y'')f(y'') = g(y'') < g(y') = f(x^{opt}). \end{aligned}$$

This contradicts the optimality of x^{opt}. Hence $g(y') = \inf_{y \in X, f(y)=1} g(y)$, thus y' is a solution of Problem B.

The converse can be established analogously. □

Remark 6.2.1 *To the best of our knowledge, Theorem 6.2.1 has not been stated in its full generality in the literature. However, a particular instance of Corollary 6.2.1 has been used in (Hildebrand and Gevers 2003b).*

6.3 Basic Definitions in Experiment Design

Here we provide some basic definitions related to experiment design. For the motivation behind these definitions, the reader is referred to the least costly experiment design literature (Bombois et al. 2004a;b; 2006).

Consider the true system to be given by

$$y_t = G(z,\theta_T)u_t + H(z,\theta_T)w_t, \quad \theta_T \in \mathbb{R}^n,$$

where $\{u_t\}$ is a quasi-stationary input signal having spectrum Φ_u, and $\{w_t\}$ is white noise of variance σ^2.

Now consider a locally identifiable model structure given by $\mathscr{M} := \{(G(z,\theta),H(z,\theta)) : \theta \in D_{\mathscr{M}} \subseteq \mathbb{R}^n\}$ (Ljung 1999), which includes the true system $(G(z,\theta_T),H(z,\theta_T))$. Here, $G(z,\theta)$ and $H(z,\theta)$ are rational transfer functions, where the number of poles of $G(z,\theta)$ is $n' \in \mathbb{N}$ (called the *model order*[3]), $H(z,\theta)$ is stable and minimum phase, and $H(\infty,\theta) = 1$. Let $\theta =: [\theta_G^T\ \theta_{GH}^T\ \theta_H^T]^T$, where θ_G, θ_H and θ_{GH} contain the parameters which are exclusively in G, exclusively in H, and common to both G and H, respectively.

As remarked earlier, it is typical in experiment design that the optimal experiment depends on the true system, which is a-priori unknown. In the least costly framework, an initial experiment is required. Therefore, in the sequel we assume that θ_0 is an initial estimate of the true parameter vector θ_T (see Section 2.2.2), that has been obtained from a previous open loop experiment using an input signal with spectrum $\Phi_{u,init}(\omega)$.

6.3.1 Open Loop Experiment Design

Utilising PEM to estimate the parameters in θ, based on N observations, it has been shown that, under mild conditions (Ljung 1999),

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{d} \mathrm{N}(0,\bar{P}_{\theta_0})$$

where $\hat{\theta}_N$ is the PEM estimator of θ, and

$$\begin{aligned}
\bar{P}_{\theta_0}^{-1} &:= \frac{1}{2\pi\sigma^2}\int_{-\pi}^{\pi} F_u(e^{j\omega},\theta_0)F_u^H(e^{j\omega},\theta_0)\Phi_u(\omega)d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} F_w(e^{j\omega},\theta_0)F_w^H(e^{j\omega},\theta_0)d\omega \\
F_u(z,\theta) &:= H^{-1}(z,\theta)\Gamma_G(z,\theta) =: [F_{u,G}^T(z,\theta)\ \ F_{u,GH}^T(z,\theta)\ \ 0]^T \\
F_w(z,\theta) &:= H^{-1}(z,\theta)\Gamma_H(z,\theta) =: [0\ \ F_{w,GH}^T(z,\theta)\ \ F_{w,H}^T(z,\theta)]^T \\
\Gamma_G(z,\theta) &:= \frac{\partial G(z,\theta)}{\partial\theta} =: [\Gamma_{G,G}^T(z,\theta)\ \ \Gamma_{G,GH}^T(z,\theta)\ \ 0]^T \\
\Gamma_H(z,\theta) &:= \frac{\partial H(z,\theta)}{\partial\theta} =: [0\ \ \Gamma_{H,GH}^T(z,\theta)\ \ \Gamma_{H,H}^T(z,\theta)]^T.
\end{aligned}$$

Note that F_u, F_w, Γ_G and Γ_H have been partitioned with respect to the parameters of $\theta = [\theta_G^T\ \ \theta_{GH}^T\ \ \theta_H^T]^T$.

[3]Notice that n and n' are not necessarily equal, since n is the number of parameters in θ, and n' is the number of poles of G.

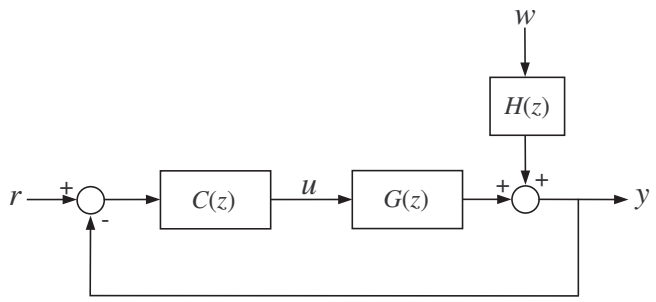

Figure 6.1. Block diagram describing the closed loop system.

Thus, the variance of θ, P_{θ_0}, for large N, is asymptotically given by

$$\begin{aligned} P_{\theta_0}^{-1} &\approx N\bar{P}_{\theta_0}^{-1} \\ &= \frac{N}{2\pi\sigma^2}\int_{-\pi}^{\pi} F_u(e^{j\omega},\theta_0)F_u^H(e^{j\omega},\theta_0)\Phi_u(\omega)d\omega + \frac{N}{2\pi}\int_{-\pi}^{\pi} F_w(e^{j\omega},\theta_0)F_w^H(e^{j\omega},\theta_0)d\omega. \end{aligned} \tag{6.2}$$

6.3.2 Closed Loop Experiment Design

For a system operating under linear feedback, the input is generated by

$$u_t = C(z)[r_t - y_t]$$

where C denotes a controller transfer function and $\{r_t\}$ is a quasi-stationary reference signal. Figure 6.1 shows the closed loop system. In this case, under mild conditions, the asymptotic covariance of the PEM estimator of θ based on N observations, P_{θ_0}, satisfies (Ljung 1999)

$$P_{\theta_0}^{-1} := NP_r^{-1} + NP_w^{-1}, \tag{6.3}$$

where $N \in \mathbb{N}$ is the length of the experiment, $P_w^{-1} \in \mathbb{R}^{n\times n}$ is a fixed positive semi-definite symmetric matrix related to the influence of noise on P_{θ_0},

$$P_r^{-1} := \frac{1}{2\pi\sigma^2}\int_{-\pi}^{\pi} F_r(e^{j\omega},\theta_0)F_r^H(e^{j\omega},\theta_0)\Phi_r(\omega)d\omega$$

is the part of P_{θ_0} due to the reference signal, Φ_r is the reference spectrum,

$$F_r(z,\theta_0) := H^{-1}(z,\theta_0)S_{id}(z)\Gamma_G(z,\theta_0),$$

and S_{id} is an a-priori estimate of the loop sensitivity, $(1+GC)^{-1}$, applicable during the experiment.

6.3.3 Model Confidence Regions

By utilising the covariance P_{θ_0}, one is able to obtain the following confidence region of possible system models (Bombois et al. 2001):

$$\mathscr{D} := \left\{ G(z,\theta) = \frac{Z_N(z)\theta}{1+Z_D(z)\theta} : \quad [\theta-\theta_0]^T P_{\theta_0}^{-1}[\theta-\theta_0] < \chi, \theta \in D_{\mathscr{M}} \right\},$$

where Z_N and Z_D are row-vector-valued transfer functions related to $\mathscr{M}$, and $\chi > 0$ is related to the confidence level of $\mathscr{D}$.

Now, define

$$r_u(\omega) := \sqrt{\chi \lambda_{\max}\{T(e^{j\omega},\theta_0) P_{\theta_0} T(e^{j\omega},\theta_0)^T\}}, \tag{6.4}$$

where[4] $T(z,\theta) := [\mathrm{Re}\,\Gamma_G(z,\theta) \;\; \mathrm{Im}\,\Gamma_G(z,\theta)]^T$. Notice that $T(e^{j\omega},\theta_0) P_{\theta_0} T(e^{j\omega},\theta_0)^T$ is positive semidefinite. Hence r_u is a measure of the size of $\mathscr{D}$ (Bombois et al. 2004a), since[5]

$$G \in \mathscr{D} \Rightarrow |G(e^{j\omega},\theta) - G(e^{j\omega},\theta_0)| < r_u(\omega), \quad \omega \in [-\pi,\pi], \theta \in D_{\mathscr{M}}. \tag{6.5}$$

Let $r_{\mathrm{adm}} : [-\pi,\pi] \to \mathbb{R}^+$ be the largest admissible size of this region to guarantee a minimum level of closed loop performance, based on an initial system estimate (and a pre-selected fixed control design method, to be used with the model obtained in the estimation stage to design a controller). See (Bombois et al. 2004a;b) for details on how r_{adm} can be computed from an estimate of the controller that will be designed with the model obtained from the estimation stage. From the initial experiment, let $r_{\Phi,init}(\omega)$ be an estimate of the size of $\mathscr{D}$ obtained using the input spectrum $\Phi_{u,init}(\omega)$.

6.3.4 Input/Output Power

In the least costly approach to experiment design, the following cost function is considered:

$$\mathscr{J}_r := \frac{1}{2\pi}\int_{-\pi}^{\pi} (\alpha_u \left|S_{id}(e^{j\omega})\right|^2 + \alpha_y \left|G(e^{j\omega},\theta_0) S_{id}(e^{j\omega})\right|^2) \Phi_r(\omega) d\omega,$$

where $\mathscr{J}_r$ is a weighted sum of the input and output power, with α_u and α_y the corresponding weights. This represents the power of the perturbations induced by the excitation signal ($\{u_t\}$ or $\{r_t\}$, depending on whether the identification is performed in open or closed loop) on the normal operating signals.

The input power is given by the first term in the expression for $\mathscr{J}_r$. The output power is given by the second term, which is determined by S_{id}, Φ_r and the initial estimate of the system, $G(e^{j\omega},\theta_0)$. In

[4] $\lambda_{\max}\{A\}$ denotes the *largest eigenvalue*, or *spectral radius*, of a positive semi-definite matrix A (Bernstein 2005).

[5] Expression (6.5) is in fact valid only asymptotically as $N \to \infty$, since it is based on a Taylor approximation of the model $G(z,\theta)$ around $\theta = \theta_0$.

open loop experiment design, we take $\alpha_u = 1$ and $\alpha_y = 0$, i.e. we focus on the problem of minimising the input power.

6.3.5 Model Quality

In order to relate experiment design to robust control, model quality is defined in terms of the following functional

$$J(G) := \left\| W_l \begin{bmatrix} \frac{GC}{1+GC} & \frac{G}{1+GC} \\ \frac{C}{1+GC} & \frac{1}{1+GC} \end{bmatrix} W_r \right\|_\infty ,$$

where J is an $\mathscr{H}_\infty$ performance measure determined by the system, G, the controller, C, and the diagonal frequency-dependent weighting transfer matrices W_l and W_r. This definition will be considered only for the closed loop experiment design case. It should be noted that it could also be used in the open loop case.

6.3.6 Signal Spaces

In the sequel we work with three spaces of input (or reference) spectra as defined below:

1. $\mathscr{U}_1$ is the space of all input spectra Φ_u which are uniformly bounded from above and below (i.e., such that $C \geq \Phi_u(\omega) \geq \delta > 0$ for some C and δ). This requirement is necessary in order to apply the asymptotic covariance formula (Yuan and Ljung 1984).

2. $\mathscr{U}_2$ is the space of all input spectra Φ_u on $[-\pi, \pi]$ for which the open loop *information matrix* $P_{\theta_0}^{-1}$ defined by the right hand side of (6.2) is nonsingular and

$$\frac{N}{2\pi\sigma^2} \int_{-\pi}^{\pi} \frac{\partial G(e^{j\omega}, \theta_0)}{\partial \rho} \left[\frac{\partial G(e^{j\omega}, \theta_0)}{\partial \rho} \right]^H \frac{\Phi_u(\omega)}{|H(e^{j\omega}, \theta_0)|^2} d\omega > 0, \tag{6.6}$$

   where $\rho := [\theta_G^T \ \theta_{GH}^T]^T$. These conditions are equivalent to requiring that $(G(z, \theta_0), H(z, \theta_0))$ be *parameter identifiable* under $\mathscr{M}$ and Φ_u for the maximum likelihood method (Söderström and Stoica 1989), and also that the parameters of G (θ_G and θ_{GH}) should be identifiable for zero noise (i.e. $(G(z, \theta_0), 0)$ should be *parameter identifiable* under the modified model structure $\tilde{\mathscr{M}} := \{(G(z, \theta), 0) : \ \theta \in D_{\mathscr{M}}\}$ and Φ_u, for the maximum likelihood method). Notice that these conditions impose restrictions on both the model structure and the input signal.

   For model structures $\mathscr{M}$, where G and H are independently parameterised, equation (6.6) is implied by the condition that the open loop information matrix $P_{\theta_0}^{-1}$ is nonsingular, since P_{θ_0} is

block diagonal and its upper block is the inverse of the integral that appears in (6.6) (see e.g. Section 2.2).

In the case of standard model structures where G and H have parameters in common, condition (6.6) usually reduces to requiring $\{u_t\}$ to be *persistently exciting* of a sufficiently high order (Ljung 1999). For example, an ARMAX model structure having polynomial orders n_a, n_b and n_c, (6.6) requires $\{u_t\}$ to be persistently exciting of order $n_a + n_b$.

3. $\mathscr{U}_3$ is the set of all reference spectra Φ_r on $[-\pi, \pi]$ for which the closed loop information matrix $P_{\theta_0}^{-1}$ defined by the right side of (6.3) is nonsingular.

Spaces $\mathscr{U}_1$ and $\mathscr{U}_2$ will be considered for the case of open loop experiment design (i.e. when $\{u_t\}$ and $\{w_t\}$ are independent), and $\mathscr{U}_3$ will be used for the case of closed loop experiment design.

Remark 6.3.1 *It is well known (see e.g. Wold's Theorem (Priestley 1981)) that a generalised function (Lighthill 1959) on $[-\pi, \pi]$ is the spectrum of a stationary stochastic process if and only if its antiderivative F is an ordinary non-decreasing function. From this fact it can be readily seen that $\mathscr{U}_1$, $\mathscr{U}_2$ and $\mathscr{U}_3$ are cones, but not linear spaces (because multiplication by negative scalars is not allowed). This observation will be utilised in later sections when Theorem 6.2.1 is applied to establish the equivalence between the least costly and traditional experiment design approaches.*

Remark 6.3.2 *For computational reasons, the spectra space is typically approximated by a subset of a finite dimensional space. With this approximation, the results presented in the sequel are equally valid, since the parametrisations are usually cones, as in the case of the 'finite dimensional spectrum' and 'partial correlation' parametrisations (Jansson and Hjalmarsson 2005a). Moreover, condition* (6.6) *is automatically satisfied (if G is locally identifiable under zero noise (Ljung 1999)) when the input spectrum is parameterised by a finite linear combination of rational basis functions, since the corresponding spectra are nonzero for almost all $\omega \in [0, \pi]$, i.e.* persistently exciting of infinite order *(Ljung 1999).*

6.4 Cheapest Open Loop Experiment Design for Control

In this section we consider the system to be operating in open loop. Three specific least costly problems are examined and are shown to be equivalent to traditional experiment design problems.

6.4.1 High-Order Model Approximation Approach

In this approach (Bombois et al. 2004b), the modeling error is assumed to be due to variance only and is hence approximated by the following asymptotic (in model order) variance expression (Ljung 1999):

$$P_G(e^{j\omega}) := \mathrm{Var}[G(e^{j\omega}, \hat{\theta}_N)] \approx \frac{n'}{N} \frac{\sigma^2 |H(e^{j\omega}, \theta_0)|^2}{\Phi_u(\omega)}. \tag{6.7}$$

where n' is the model order, as defined in Section 6.3. Notice that the variance is asymptotically inversely proportional to Φ_u. From this observation, it is possible to derive an expression for the variance based on data obtained from an initial experiment, i.e. $r_{\Phi,init}$ and $\Phi_{u,init}$. This is considered in (Bombois et al. 2004b), where it is noted that

$$r_u^2(\omega) \approx r_{\Phi,init}^2(\omega) \frac{\Phi_{u,init}(\omega)}{\Phi_u(\omega)},$$

from which the model quality constraint '$r_u(\omega) \leq r_{\mathrm{adm}}(\omega)$ for all $\omega \in [-\pi, \pi]$' gives rise to the following problem:

Cheapest open loop experiment design for control (based on a high-order model approximation):

$$\begin{aligned} &\min_{\Phi_u \in \mathscr{U}_1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \\ &\text{s.t. } r_{\Phi,init}^2(\omega) \frac{\Phi_{u,init}(\omega)}{\Phi_u(\omega)} \leq r_{\mathrm{adm}}^2(\omega), \quad \omega \in [-\pi, \pi]. \end{aligned} \tag{6.8}$$

where the definitions of $\mathscr{U}_1$, Φ_u, $\Phi_{u,init}$, $r_{\Phi,init}$ and r_{adm} are given in Section 6.3.

The equivalence between problem (6.8) and a traditional experiment design problem is established in the following result:

Theorem 6.4.1 (Least costly high-order open loop problem) *The cheapest open loop experiment design for control problem, based on a high-order model approximation, is equivalent to the following traditional experiment design problem:*

$$\begin{aligned} &\min_{\tilde{\Phi}_u \in \mathscr{U}_1} \left\| \frac{r_{\Phi,init}^2(\omega) \Phi_{u,init}(\omega)}{r_{\mathrm{adm}}^2(\omega) \tilde{\Phi}_u(\omega)} \right\|_\infty \\ &\text{s.t. } \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Phi}_u(\omega) d\omega \leq 1, \end{aligned}$$

in[6] *the sense that the solutions Φ_u^{opt} and $\tilde{\Phi}_u^{opt}$, if they exist, are related by $\Phi_u^{opt}(\omega) = K\tilde{\Phi}_u^{opt}(\omega)$ for every $\omega \in [-\pi, \pi]$, where $K := (2\pi)^{-1} \int_{-\pi}^{\pi} \Phi_u^{opt}(\omega) d\omega$.*

[6] If $A : [-\pi, \pi] \to \mathbb{C}^{n \times n}$ is (essentially) bounded, then $\|A\|_\infty := \operatorname{ess\,sup}_{\omega \in [-\pi,\pi]} \bar{\sigma}[A(\omega)]$, where $\bar{\sigma}[A(\omega)]$ denotes the *largest singular value* of $A(\omega)$, and ess sup is the *essential supremum*; see e.g. (Zhou et al. 1996).

Proof. Define $f, g : \mathscr{U}_1 \to \mathbb{R}_0^+$ as

$$f(\Phi_u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega,$$
$$g(\Phi_u) = \sup_{\omega\in[-\pi,\pi]} \frac{r^2_{\Phi,init}(\omega)\Phi_{u,init}(\omega)}{r^2_{\text{adm}}(\omega)\Phi_u(\omega)} = \left\| \frac{r^2_{\Phi,init}(\omega)\Phi_{u,init}(\omega)}{r^2_{\text{adm}}(\omega)\Phi_u(\omega)} \right\|_\infty .$$

Application of Theorem 6.2.1 immediately provides the result, since for every $\Phi_u \in \mathscr{U}_1$ and $\alpha > 0$, $f(\alpha\Phi_u) = \alpha f(\Phi_u)$, and $g(\alpha\Phi_u) = \alpha^{-1} g(\Phi_u)$. □

Remark 6.4.1 *Notice that the equivalent traditional problem can be stated in terms of the variance of G, P_G, as:*

$$\min_{\tilde{\Phi}_u\in\mathscr{U}_1} \left\| \frac{1}{r^2_{\text{adm}}(\omega)} P_G(e^{j\omega}) \right\|_\infty$$
$$\text{s.t. } \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Phi}_u(\omega)d\omega \le 1.$$

This is due to $r^2_{\Phi,init}(\omega)\Phi_{u,init}(\omega)/\Phi_u(\omega)$ being proportional to $P_G(e^{j\omega})$, according to (6.7)*. This last problem reflects the traditional purpose of designing an input signal to minimise some scalar function of the covariance of G or θ, subject to input and/or output power constraints.*

6.4.2 Finite-Order Model Approach

Here we consider the use of covariance expressions based on a finite model order (Ljung 1999). In this case, the experiment design problem can be stated in the least costly framework (Bombois et al. 2004a) as,

Cheapest Experiment Design Problem for Control (for finite-order models):

$$\min_{\Phi_u\in\mathscr{U}_2} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega$$
$$\text{s.t. } r_u(\omega) \le r_{\text{adm}}(\omega), \quad \omega \in [-\pi,\pi],$$

where the definitions of $\mathscr{U}_2$, Φ_u, r_u and r_{adm} are given in Section 6.3.

In this problem, it can be seen that one seeks to minimise the input power subject to a measure of the model uncertainty being less than a control based performance constraint. The equivalence between this particular cheapest experiment design problem and traditional open loop experiment design is established in the following result:

Theorem 6.4.2 (Least costly open loop problem, for G and H independently parameterised) *If the model structure $\mathscr{M}$ is such that G and H are independently parameterised (i.e. for Box-Jenkins,*

Output-Error and FIR models), then the cheapest experiment design problem for control is equivalent to the following problem:

$$\min_{\tilde{\Phi}_u \in \mathscr{U}_2} \left\| \frac{1}{r_{\text{adm}}^2(\omega)} T(e^{j\omega},\theta_0) P_{\theta_0} T^T(e^{j\omega},\theta_0) \right\|_\infty$$
$$\text{s.t. } \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\Phi}_u(\omega) d\omega \leq 1,$$

in the sense that the solutions Φ_u^{opt} and $\tilde{\Phi}_u^{opt}$, if they exist, are related by $\Phi_u^{opt}(\omega) = K\tilde{\Phi}_u^{opt}(\omega)$ for every $\omega \in [-\pi,\pi]$, where $K := (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi_u^{opt}(\omega)d\omega$.

Proof. As in the proof of Theorem 6.4.1, define

$$f(\Phi_u) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega,$$
$$g(\Phi_u) := \sup_{\omega\in[-\pi,\pi]} \frac{1}{r_{\text{adm}}^2(\omega)} r_u^2(\omega),$$

Note that by the definition of r_u in (6.4),

$$\begin{aligned} r_u^2(\omega) &= \chi\lambda_{\max}\{T(e^{j\omega},\theta_0)P_{\theta_0}T^T(e^{j\omega},\theta_0)\} \\ &= \frac{2\pi\chi\sigma^2}{N}\lambda_{\max}\Bigg\{ T(e^{j\omega},\theta_0)\Bigg[\int_{-\pi}^{\pi} F_u(e^{j\tau},\theta_0)F_u^H(e^{j\tau},\theta_0)\Phi_u(\tau)d\tau + \\ &\qquad \sigma^2\int_{-\pi}^{\pi} F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau\Bigg]^{-1} T^T(e^{j\omega},\theta_0)\Bigg\}. \end{aligned} \tag{6.9}$$

Given that G and H are independently parameterised then, according to the partition $\theta = [\theta_G^T\ \theta_H^T]^T$,

$$\int_{-\pi}^{\pi} F_u(e^{j\tau},\theta_0)F_u^H(e^{j\tau},\theta_0)\Phi_u(\tau)d\tau = \left[\begin{array}{c|c} \displaystyle\int_{-\pi}^{\pi} F_{u,G}(e^{j\tau},\theta_0)F_{u,G}^T(e^{-j\tau},\theta_0)\Phi_u(\tau)d\tau & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\sigma^2\int_{-\pi}^{\pi} F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma^2\displaystyle\int_{-\pi}^{\pi} F_{w,H}(e^{j\tau},\theta_0)F_{w,H}^T(e^{-j\tau},\theta_0)d\tau \end{array}\right]$$

$$T^T(e^{j\omega},\theta_0) = \left[\begin{array}{c} \text{Re}\,\Gamma_{G,G}(e^{j\omega},\theta_0)\ \text{Im}\,\Gamma_{G,G}(e^{j\omega},\theta_0) \\ \hline 0 \end{array}\right].$$

Hence the sum of the integrals in (6.9) is a block-diagonal matrix, whose inverse is block-diagonal as well (Bernstein 2005). Pre- and post-multiplication by T and T^T respectively shows that the cost function takes into account only the upper-left block of the sum, which is related to the integral of $F_u F_u^T \Phi_u$. Thus, r_u can be written in terms of the Moore-Penrose generalised inverse, denoted $(\,\cdot\,)^\dagger$,

(Bernstein 2005) as

$$r_u^2(\omega) = \frac{2\pi\chi\sigma^2}{N}\lambda_{\max}\left\{T(e^{j\omega},\theta_0)\left[\begin{array}{c|c}\left(\int_{-\pi}^{\pi}F_{u,G}(e^{j\tau},\theta_0)F_{u,G}^T(e^{-j\tau},\theta_0)\Phi_u(\tau)d\tau\right)^{-1} & 0\\ \hline 0 & 0\end{array}\right]\cdot T^T(e^{j\omega},\theta_0)\right\}$$
$$= \frac{2\pi\chi\sigma^2}{N}\lambda_{\max}\left\{T(e^{j\omega},\theta_0)\left[\int_{-\pi}^{\pi}F_u(e^{j\tau},\theta_0)F_u^H(e^{j\tau},\theta_0)\Phi_u(\tau)d\tau\right]^{\dagger}T^T(e^{j\omega},\theta_0)\right\}.$$

Therefore $r_u^2(\omega)|_{\alpha\Phi_u} = \alpha^{-1}r_u^2(\omega)|_{\Phi_u}$ for every $\omega\in[-\pi,\pi]$, and $g(\alpha\Phi_u)=\alpha^{-1}g(\Phi_u)$. We can now proceed as in the proof of Theorem 6.4.1, since g can be written in terms of the infinity norm, and the constant χ is irrelevant. □

In the case of model structures $\mathscr{M}$, where G and H are not independently parameterised, it is more difficult, but still possible, to find a traditional equivalent of the cheapest experiment design problem for control. To this end, we introduce the following definition:

Definition 6.4.1 *Let $A\in\mathbb{R}^{n\times n}$ be symmetric. Then, if $V\in\mathbb{R}^{n\times n}$ such that $A=V^TDV$, where $D\in\mathbb{R}^{n\times n}$ is diagonal (Bernstein 2005), we define $[A]_+$ as a Cholesky Factor of $(1/2)V^T(D+|D|)V$, i.e.*[7],

$$[A]_+^T[A]_+ = \frac{1}{2}V^T(D+|D|)V.$$

Note that $[A]_+$ is not uniquely defined in Definition 6.4.1. This is not an issue in the following results as it is not a requirement for the equivalence to be unique. The following result establishes the equivalence between the cheapest experiment design and traditional methodologies in the general case.

Theorem 6.4.3 (Least costly open loop problem, general case) *Consider a model structure, $\mathscr{M}$, where G and H are not necessarily independently parameterised (i.e. for ARX and ARMAX models). In this case the cheapest experiment design problem for control is equivalent to the following problem:*

$$\min_{\tilde{\Phi}_u\in\mathscr{U}_2}\left\|W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0)\right\|_\infty$$
$$\text{s.t. }\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\Phi}_u(\omega)d\omega\le 1,$$

where

$$W(e^{j\omega},\theta_0) := \left[\frac{\chi}{r_{adm}^2(\omega)}T^T(e^{j\omega},\theta_0)T(e^{j\omega},\theta_0) - \frac{N}{2\pi}\int_{-\pi}^{\pi}F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau\right]_+$$
$$P_u := \left[\frac{N}{2\pi\sigma^2}\int_{-\pi}^{\pi}F_u(e^{j\tau},\theta_0)F_u^H(e^{j\tau},\theta_0)\Phi_u(\tau)d\tau\right]^{\dagger}. \qquad (6.10)$$

[7] $|D|$ is the *absolute value* of a matrix D (Bernstein 2005).

The equivalence holds in the sense that the solutions Φ_u^{opt} and $\tilde{\Phi}_u^{opt}$, if they exist, are related by $\Phi_u^{opt}(\omega) = K\tilde{\Phi}_u^{opt}(\omega)$ for every $\omega \in [-\pi,\pi]$, where $K := (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi_u^{opt}(\omega)d\omega$.

Proof. Notice that for each $\omega \in [-\pi,\pi]$, the definition of r_u in (6.4) implies

$$\begin{aligned} r_u(\omega) \leq r_{\text{adm}}(\omega) \quad &\Leftrightarrow \quad \chi\lambda_{\max}\{T(e^{j\omega},\theta_0)P_{\theta_0}T^T(e^{j\omega},\theta_0)\} \leq r_{\text{adm}}^2(\omega) \\ &\Leftrightarrow \quad \frac{r_{\text{adm}}^2(\omega)}{\chi}I - T(e^{j\omega},\theta_0)P_{\theta_0}T^T(e^{j\omega},\theta_0) \geq 0 \\ &\Leftrightarrow \quad P_{\theta_0}^{-1} - \frac{\chi}{r_{\text{adm}}^2(\omega)}T^T(e^{j\omega},\theta_0)T(e^{j\omega},\theta_0) \geq 0. \end{aligned}$$

The equivalence between lines 1 and 2 is due to a characterization of positive definiteness in terms of the spectral radius (Bernstein 2005, fact 8.15.4). Lines 2 and 3 are equivalent due to a property of *Schur complements* for nonstrict inequalities (Bernstein 2005, proposition 8.2.3; Boyd et al. 1994, page 28). Now, by (6.2) and (6.10), we have that

$$P_{\theta_0}^{-1} = P_u^{\dagger} + \frac{N}{2\pi}\int_{-\pi}^{\pi}F_w(e^{j\omega},\theta_0)F_w^H(e^{j\omega},\theta_0)d\omega.$$

Thus, we obtain

$$\begin{aligned} r_u(\omega) \leq r_{\text{adm}}(\omega) \Leftrightarrow \quad & P_u^{\dagger} \geq \frac{\chi}{r_{\text{adm}}^2(\omega)}T^T(e^{j\omega},\theta_0)T(e^{j\omega},\theta_0) - \frac{N}{2\pi}\int_{-\pi}^{\pi}F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau \\ \Leftrightarrow \quad & P_u^{\dagger} \geq W^T(e^{j\omega},\theta_0)W(e^{j\omega},\theta_0) \qquad (6.11) \\ \Leftrightarrow \quad & I - W(e^{j\omega},\theta_0)P_u(\Phi_u)W^T(e^{j\omega},\theta_0) \geq 0, \quad P_u^{\dagger} \geq 0, \quad [I - P_u^{\dagger}P_u]W^T(e^{j\omega},\theta_0) = 0 \\ \Leftrightarrow \quad & W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0) \leq I, \quad [I - P_u^{\dagger}P_u]W^T(e^{j\omega},\theta_0) = 0 \\ \Leftrightarrow \quad & \lambda_{\max}\{W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0)\} \leq 1, \quad [I - P_u^{\dagger}P_u]W^T(e^{j\omega},\theta_0) = 0. \end{aligned}$$

Here, the equivalence between lines 1 and 2 follows from Lemma 6.8.1 of Appendix 6.8, since $P_u^{\dagger} \geq 0$. Also, the equivalence between lines 2 and 3 is due to a property of *Schur complements* for nonstrict inequalities, and the equivalence between lines 4 and 5 comes from the characterization of positive definiteness in terms of the spectral radius.

By Lemma 6.8.2 of Appendix 6.8, $[I - P_u^{\dagger}P_u]\ W^T(e^{j\omega},\theta_0) = 0$. Since (6.11) holds for every $\omega \in [-\pi,\pi]$, then

$$\begin{aligned} r_u(\omega) \leq r_{\text{adm}}(\omega), \quad \omega \in [-\pi,\pi] \quad &\Leftrightarrow \quad \sup_{\omega\in[-\pi,\pi]} \lambda_{\max}\{W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0)\} \leq 1 \\ &\Leftrightarrow \quad \left\|W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0)\right\|_{\infty} \leq 1. \end{aligned}$$

Hence, taking

$$\begin{aligned} f(\Phi_u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega \\ g(\Phi_u) &= \left\|W(e^{j\omega},\theta_0)P_uW^T(e^{j\omega},\theta_0)\right\|_{\infty}, \end{aligned}$$

the rest of the proof follows as in Theorem 6.4.1. □

Theorems 6.4.2 and 6.4.3 show explicitly the equivalence between the least costly framework and traditional open loop experiment design problems for the most commonly used model structures.

6.5 Least Costly Closed Loop Experiment Design for Control

Now we consider the case when the system is operating under closed loop control. In this setting, Bombois et al. (2005b; 2006) considered the use of covariance expressions based on a finite model order (Ljung 1999). In this framework, the following experiment design problem has been stated:

Least Costly Identification Experiment for Control:

$$\min_{\Phi_r \in \mathscr{U}_3} \mathscr{J}_r$$
$$\text{s.t. } J(G) \leq 1, \quad G \in \mathscr{D}.$$

where Φ_r, $\mathscr{U}_3$, $\mathscr{J}_r$, $J(G)$ and $\mathscr{D}$ are as defined in Section 6.3.

The following result establishes the equivalence between the closed loop least costly framework and traditional experiment design, based on covariance expressions related to a finite model order.

Theorem 6.5.1 (Least costly closed loop problem) *The least costly identification experiment for control problem is equivalent to the following problem*[8]*:*

$$\min_{\tilde{\Phi}_r \in \mathscr{U}_3} \sup_{\{\theta: J(G(\theta))>1\}} \frac{\chi/N - [\theta-\theta_0]^T P_w^{-1} [\theta-\theta_0]}{[\theta-\theta_0]^T P_r^{-1} [\theta-\theta_0]}$$
$$\text{s.t. } \mathscr{J}_r \leq 1,$$

(where $G(\theta) := Z_N\theta/(1+Z_D\theta)$) in the sense that the solutions Φ_r^{opt} and $\tilde{\Phi}_r^{opt}$, if they exist, are related by $\Phi_r^{opt}(\omega) = K\tilde{\Phi}_r^{opt}(\omega)$ for every $\omega \in [-\pi,\pi]$, where $K := (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi_r^{opt}(\omega)d\omega$.

Proof. Notice that the condition '$J(G) \leq 1,\ G \in \mathscr{D}$' can be written as

$$\begin{aligned}
\{G \in \mathscr{D} \Rightarrow J(G) \leq 1\} &\Leftrightarrow \{J(G) > 1 \Rightarrow G \notin \mathscr{D}\} \\
&\Leftrightarrow \{J(G(\theta)) > 1 \Rightarrow [\theta-\theta_0]^T P_{\theta_0}^{-1} [\theta-\theta_0] \geq \chi\} \\
&\Leftrightarrow \{J(G(\theta)) > 1 \Rightarrow [\theta-\theta_0]^T P_r^{-1} [\theta-\theta_0] \geq \chi/N - [\theta-\theta_0]^T P_w^{-1} [\theta-\theta_0]\} \\
&\Leftrightarrow \left\{J(G(\theta)) > 1 \Rightarrow \frac{\chi/N - [\theta-\theta_0]^T P_w^{-1} [\theta-\theta_0]}{[\theta-\theta_0]^T P_r^{-1} [\theta-\theta_0]} \leq 1\right\} \\
&\Leftrightarrow \sup_{\{\theta: J(G(\theta))>1\}} \frac{\chi/N - [\theta-\theta_0]^T P_w^{-1} [\theta-\theta_0]}{[\theta-\theta_0]^T P_r^{-1} [\theta-\theta_0]} \leq 1.
\end{aligned}$$

[8] Here we adopt the convention that $a/0 = +\infty$ if $a > 0$, $a/0 = -\infty$ if $a < 0$, and $0/0 = 0$.

Thus, setting

$$f(\Phi_u) = \mathscr{J}_r,$$
$$g(\Phi_u) = \sup_{\{\theta: J(G(\theta))>1\}} \frac{\chi/N - [\theta-\theta_0]^T P_w^{-1}[\theta-\theta_0]}{[\theta-\theta_0]^T P_r^{-1}[\theta-\theta_0]},$$

we can proceed as in the proof of Theorem 6.4.1. □

Corollary 6.5.1 (Further equivalence to the least costly closed loop problem) *The least costly identification experiment for control problem is also equivalent to the following problem:*

$$\max_{\tilde{\Phi}_r \in \mathscr{U}_3} \inf_{\{\theta: J(G(\theta))>1\}} \operatorname{tr}\{W(\theta)P_{\theta_0}^{-1}\} - g(\theta)$$
$$s.t.\ \mathscr{J}_r \leq 1,$$

where

$$W(\theta) := \frac{[\theta-\theta_0][\theta-\theta_0]^T}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]},$$
$$g(\theta) := \frac{\chi}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]}.$$

Proof. This can be seen from Theorem 6.5.1, by inverting the cost function and noting that

$$\begin{aligned}
\frac{[\theta-\theta_0]^T P_r^{-1}[\theta-\theta_0]}{\chi/N - [\theta-\theta_0]^T P_w^{-1}[\theta-\theta_0]} &= \frac{[\theta-\theta_0]^T N P_r^{-1}[\theta-\theta_0]}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]} \\
&= \frac{[\theta-\theta_0]^T P_{\theta_0}^{-1}[\theta-\theta_0] - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]} \\
&= \frac{[\theta-\theta_0]^T P_{\theta_0}^{-1}[\theta-\theta_0]}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]} - \frac{[\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]} \\
&= \operatorname{tr}\left\{ \frac{[\theta-\theta_0][\theta-\theta_0]^T}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]} P_{\theta_0}^{-1} \right\} + 1 \\
&\quad - \frac{\chi}{\chi - [\theta-\theta_0]^T N P_w^{-1}[\theta-\theta_0]},
\end{aligned} \tag{6.12}$$

where the last line follows from the fact that $\operatorname{tr}[AB] = \operatorname{tr}[BA]$ for any matrices A, B such that AB is square (Bernstein 2005, page 22). The constant term 1 in (6.12) can be omitted from the optimisation problem. □

Remark 6.5.1 *The problem formulated in Corollary 6.5.1 is similar to a popular traditional experiment design problem described in (Mehra 1974b), except for the inclusion of the term $-g$ and the minimisation with respect to θ, which can be interpreted as a form of 'robustification' with respect to uncertainty regarding the prior knowledge of θ (see Section 2.2.3).*

6.6 An Illustrative Example

The results presented in the previous sections show that solutions of several least costly experiment design problems are essentially scaled versions of solutions to corresponding traditional experiment design problems. This implies that it is possible to use computationally efficient algorithms developed for use in one framework in the other.

These equivalence results also allow the incorporation and interpretation of additional constraints into the least costly framework. This is illustrated below by a simple example.

Consider the problem of designing an input signal of minimum power to identify (in open loop) a Finite Impulse Response (FIR) system of order n, such that the asymptotic variance of the transfer function estimator is uniformly bounded by a constant $1/(N\gamma)$ in a given bandwidth $[-\omega_B, \omega_B]$, where N is the data length. This is a least costly experiment design formulation, which can be written as the following optimization problem:

$$\begin{aligned} &\min_{\Phi_u} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \\ &\text{s.t.} \quad \Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi \\ &\qquad \lim_{N\to\infty} N \operatorname{Var}\{G(e^{j\omega}, \hat{\theta}_{N,n})\} \leq \frac{1}{\gamma}, \quad |\omega| \leq \omega_B, \end{aligned}$$

where, by the *Gauss' approximation formula* (Ljung 1999),

$$\lim_{N\to\infty} N \operatorname{Var}\{G(e^{j\omega}, \hat{\theta}_{N,n})\} = \sigma^2 \Gamma_n^H(e^{j\omega}) T_n^{-1} \Gamma_n(e^{j\omega}),$$

which will be denoted as the *normalised variance* of G. Here T_n is a Toeplitz matrix of the vector $[r_0\, r_1\, \cdots\, r_{n-1}]$ of the first n lags of the autocovariance sequence of $\{u_t\}$, and $\Gamma_n(q) := [1\; q^{-1}\; \cdots\; q^{-(n-1)}]^T$.

This problem is studied in Chapter 7, where the techniques of (Jansson and Hjalmarsson 2005a) are used to formulate it as a semidefinite program. Notice that the optimal input is not unique, since the problem depends only on the first n lags of the autocorrelation sequence of u.

Figure 6.2 shows the normalised variance of the frequency response obtained with the optimal inputs, designed for $n = 7$, $\sigma^2 = \gamma = 1$ and $\omega_B = 0.8\pi \approx 2.51$. In this case, the minimum input power required is approximately 6.87.

Now generally in practice there is always a constraint, say P, on the maximum input power. To analyse the effect of this constraint, notice that an equivalent traditional experiment design problem,

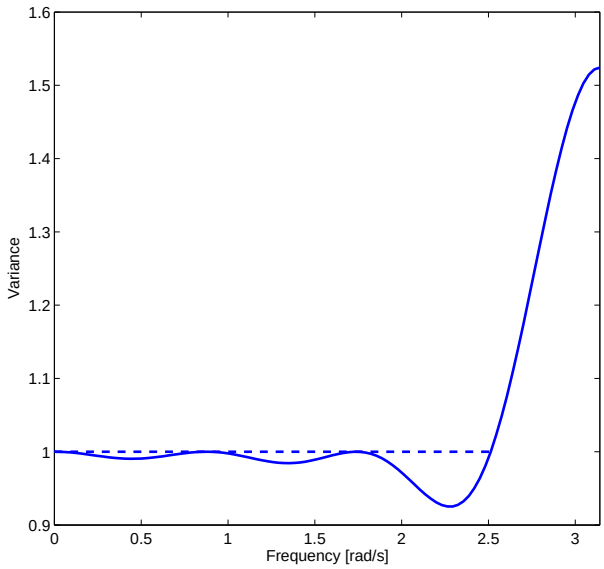

Figure 6.2. Normalised variance of the frequency response of an FIR model of order 10, identified using an optimal input signal (solid). Required upper bound on the normalised variance (dashed).

according to Theorem 6.2.1, is

$$\begin{aligned} \min_{\Phi_u} \max_{\omega\in[-\omega_B,\omega_B]} & \gamma \lim_{N\to\infty} N \text{ Var } \{G(e^{j\omega},\hat{\theta}_{N,n})\} \\ \text{s.t.} \quad & \Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq P. \end{aligned} \tag{6.13}$$

If $P > 6.87$, then there is some extra input power available which can be used to obtain an even better model, if possible. To this end, the duality between the least costly and traditional problems shows that it suffices to scale the optimal inputs by $P/6.87 > 1$ to obtain an input for which the obtained model's quality is the best possible (in the sense described by the cost function of (6.13)) while taking advantage of the full input power available.

On the other hand, if $P < 6.87$, it means that there is less input power available than necessary to satisfy the model quality constraint. We then need to determine what is the best that one can now do.

From the equivalent traditional experiment design problem, it is readily seen that if $P < 6.87$, we can simply scale the optimal solutions of the least costly problem by $P/6.87 < 1$ to obtain inputs using the full power available to give the best possible model (according to the model quality criterion described by the cost function of (6.13)).

Notice that when hard constraints exist on both the input and output power, the results of this chapter show that the traditional experiment design problem is actually equivalent to a min-max least costly problem. For example, in the open loop case, notice that hard constraints on both the input and output power can be combined into a constraint of the form

$$\max\left\{\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega,\int_{-\pi}^{\pi}\left|G(e^{j\omega},\theta_0)\right|^2\Phi_u(\omega)d\omega\right\}\leq 1,$$

where Φ_u and G have been properly scaled. The left hand side of this constraint is proportional to Φ_u, hence an equivalent least costly problem consists of minimising this quantity, subject to a model quality constraint.

6.7 Summary

We have shown, via a duality result, that least costly experiment design for control can be formulated in a more traditional setting, by stating it as an $\mathscr{H}_\infty$ or minimax optimisation problem, depending on the particular constraints being taken into account.

Specifically, we have examined 4 problems from the least costly framework. In the open loop case, equivalence to the traditional experiment design problems has been established using high order and finite order approximations of the covariance, where both independently and non-independently parameterised system and noise models are considered. For systems operating in closed loop, equivalence has been shown using covariance expressions which are non asymptotic in the model order.

The duality between least costly and traditional experiment design provides new insights into both frameworks. It also offers practical advantages e.g. by allowing the computational tools developed for each problem to be used in either framework.

6.8 Appendix: Technical Lemmas

Lemma 6.8.1 *Let $A,B\in\mathbb{R}^{n\times n}$ be symmetric, where $A\geq 0$. Then, it follows that $A\geq B$ if and only if $A\geq[B]_+^T[B]_+$.*

Proof. Let $V\in\mathbb{R}^{n\times n}$ be such that $B=V^TDV$, where $D\in\mathbb{R}^{n\times n}$ is diagonal (Bernstein 2005, Fact 5.8.16). Then, $A\geq B$ if and only if $V^{-T}AV^{-1}\geq D$. Let $U\in\mathbb{R}^{n\times n}$ be unitary and such that $U^TV^{-T}AV^{-1}U$ is diagonal (Bernstein 2005, Fact 5.8.16). Thus, $A\geq B$ if and only if $U^TV^{-T}AV^{-1}U\geq U^TDU=D$. This last condition holds if and only if the elements of $U^TV^{-T}AV^{-1}U$ are not less than

those of D. However, $A \geq 0$ implies that the elements of $U^T V^{-T} A V^{-1} U$ are nonnegative. Therefore,

$$\begin{aligned} A \geq B \quad &\Leftrightarrow \quad U^T V^{-T} A V^{-1} U \geq D \\ &\Leftrightarrow \quad U^T V^{-T} A V^{-1} U \geq \frac{1}{2}(D + |D|) \\ &\Leftrightarrow \quad A \geq \frac{1}{2} V^T U (D + |D|) U^T V = \frac{1}{2} V^T (D + |D|) V = [B]_+^T [B]_+. \end{aligned}$$

□

Lemma 6.8.2 *If condition* (6.6) *holds, where* $\{u_t\}$ *and* $\{w_t\}$ *are independent (i.e. in open loop), then* $[I - P_u^\dagger P_u] W^T(e^{j\omega}, \theta_0) = 0$.

Proof. Notice that, for every[9] $\omega \in [-\pi, \pi]$,

$$\begin{aligned} [I - P_u^\dagger P_u] W^T(e^{j\omega}, \theta_0) = 0 \quad &\Leftrightarrow \quad \mathscr{N}\{P_u^\dagger\} \perp \mathscr{R}\{W^T(e^{j\omega}, \theta_0)\} \\ &\Leftrightarrow \quad \mathscr{N}\{P_u^\dagger\} \perp \mathscr{R}\{W^T(e^{j\omega}, \theta_0) W(e^{j\omega}, \theta_0)\}. \end{aligned} \tag{6.14}$$

The first equivalence follows from properties of the Moore-Penrose generalised inverse (Bernstein 2005, Theorems 2.4.3 and 6.1.6), and the second equivalence is due to the fact that the range of a matrix A is equal to the range of AA^H (Bernstein 2005, Theorem 2.4.3).

We need to show that condition (6.14) holds automatically for every $\omega \in [-\pi, \pi]$. To this end, notice that, according to the partition $\theta = [\theta_G^T \ \theta_{GH}^T \ \theta_H^T]^T$,

$$P_u^\dagger =: \left[\begin{array}{c|c|c} A_{11} & A_{12} & 0 \\ \hline A_{21} & A_{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

$$\frac{\chi}{r_{\text{adm}}^2(\omega)} T^T(e^{j\omega}, \theta_0) T(e^{j\omega}, \theta_0) =: \left[\begin{array}{c|c|c} B_{11} & B_{12} & 0 \\ \hline B_{21} & B_{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

$$\frac{N}{2\pi} \int_{-\pi}^{\pi} F_w(e^{j\tau}, \theta_0) F_w^H(e^{j\tau}, \theta_0) d\tau =: \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & C_{22} & C_{23} \\ \hline 0 & C_{32} & C_{33} \end{array}\right].$$

Now, by condition (6.6), the upper left $(2,2)$-block of $P_u^\dagger$ is non-singular, so $\mathscr{N}\{P_u^\dagger\} = \mathscr{R}\{[0 \mid 0 \mid I]^T\}$. Therefore, we need to show that every vector $v = [0 \mid 0 \mid \tilde{v}^T]^T$ is an eigenvector of $W^T(e^{j\omega}, \theta_0) W(e^{j\omega}, \theta_0)$ associated with the eigenvalue 0. Let $\tilde{v}$ be an eigenvector of C_{33}, associated with an eigenvalue, say,

[9] $\mathscr{R}\{A\}$ and $\mathscr{N}\{A\}$ denote the *range* and *null space* of a matrix A, respectively (Bernstein 2005).

λ. Since $C_{33} \geq 0$ (as $\int_{-\pi}^{\pi} F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau \geq 0$), we have that $\lambda \geq 0$, hence

$$\left[\frac{\chi}{r_{\text{adm}}^2(\omega)}T^T(e^{j\omega},\theta_0)T(e^{j\omega},\theta_0) - \frac{N}{2\pi}\int_{-\pi}^{\pi} F_w(e^{j\tau},\theta_0)F_w^H(e^{j\tau},\theta_0)d\tau\right] v = \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline -C_{33}v \end{array}\right] = -\lambda v,$$

where $-\lambda \leq 0$. Thus, by the definition of $W(e^{j\omega},\theta_0)$, we have that $W^T(e^{j\omega},\theta_0)W(e^{j\omega},\theta_0)v = 0v = 0$. Since the vectors $[0 \mid 0 \mid \tilde{v}^T]^T$, where $\tilde{v}$ is an eigenvector of C_{33}, span $\mathscr{R}\{[0 \mid 0 \mid I]^T\}$, this proves (6.14) and concludes the proof. □

CHAPTER 7

THE COST OF COMPLEXITY IN FINITE IMPULSE RESPONSE SYSTEMS

7.1 Introduction

It is well known that one reason why system identification can work in practice lies in the nature of the input signal. It is noted that experiment design can emphasise system properties of interest, while properties of little or no interest can be 'hidden' (Hjalmarsson 2005, Hjalmarsson et al. 2006). As remarked in (Hjalmarsson et al. 2006), some properties can be more easily estimated than others, in the sense that the amount of input power required to estimate them, with a given level of accuracy, does not depend on the complexity of the model considered. For example, it has been shown that the cost of estimating the transfer function at a particular frequency, or one non-minimum phase zero, is independent of the model order (Hjalmarsson et al. 2006). However, some properties do depend on the model order.

In this chapter we extend the study of the work in (Hjalmarsson et al. 2006). Specifically, we investigate the minimum amount of input power required to estimate a given linear system with a prescribed degree of accuracy, as a function of the model complexity. This we define as the 'cost of complexity'. The degree of accuracy considered is the maximum variance of the discrete-time transfer function estimator over a frequency range $[-\omega_B, \omega_B]$. For simplicity, in this chapter we restrict the model class to systems described by Finite Impulse Response (FIR) models. Also, we assume there exists no undermodelling, i.e., the true system belongs to the model structure.

The major contribution of this chapter consists of establishing several properties for the dependence of the cost on the model complexity. Some of these seem self-evident, but others are quite unexpected. For example, if ω_B is close (but not necessarily equal) to π, the optimal input actually satisfies the model quality constraint for all frequencies!

The results developed here provide a better understanding of the relationship between the amount of information that we ask to be extracted from a system, and the sensitivity of the cost of the identification with respect to the model complexity. This appears to be essential for understanding why system

identification works also for very complex systems, since the cost of estimating some features of the true system can be kept low, even if the true system is very complex, by focusing only on particular properties.

7.2 Problem Description

Consider a FIR system with input $\{u_t\}$ and output $\{y_t\}$,

$$y_t = G(z, \theta_{n_o}^o)u_t + w_t = [\theta_{n_o}^o]^T \Lambda_{n_o}(z)u_t + w_t,$$

where $\Lambda_n(z) := [1\ z^{-1}\ \cdots\ z^{-(n-1)}]^T$ and $\theta_{n_o}^o = [b_0^o\ \cdots\ b_{n_o-1}^o]^T$. Furthermore, $\{w_t\}$ is zero mean white noise with variance σ_o^2, and the input signal is considered to be wide-sense stationary. The model to be estimated for this system is given by

$$y_t = \sum_{k=0}^{n-1} b_k u_{t-k} + \varepsilon_t = [\theta_n]^T \Lambda_n(z)u_t + \varepsilon_t$$

where $n \geq n_o$. Also, we consider the following autocovariance representation for the power spectrum of $\{u_t\}$,

$$\Phi_u(\omega) := \sum_{k=-\infty}^{\infty} r_{|k|} e^{-j\omega k}.$$

The (normalised) asymptotic covariance matrix of the estimated parameter vector is

$$\lim_{N\to\infty} N\mathrm{E}[(\hat{\theta}_{N,n} - \theta_n^o)(\hat{\theta}_{N,n} - \theta_n^o)^T] = \sigma_o^2 T_n^{-1},$$

where $\hat{\theta}_{N,n}$ is the Prediction Error Method (PEM) parameter estimator of order n based on N observations of input/output data, $\theta_n^o := [b_0^o\ \cdots\ b_{n_o-1}^o\ 0\ \cdots\ 0]^T$ and $T_n := T(\{r_k\}_{k=0}^{n-1})$ is a symmetric Toeplitz matrix of the vector $[r_0\ r_1\ \cdots\ r_{n-1}]$ (Ljung 1999). Note that r_0 corresponds to the input power.

For Φ_u to define a spectrum, it must satisfy

$$\Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi. \tag{7.1}$$

Now in designing the sequence $\{r_0, r_1, \ldots, r_{n-1}\}$, we must ensure that there exists an extension $r_n, r_{n+1}, \ldots$ such that the nonnegativity constraint (7.1) holds. A necessary and sufficient condition for the existence of such an extension is that $T_n \geq 0$ (Byrnes et al. 2001, Grenander and Szegö 1958, Lindquist and Picci 1996).

The input design problem we consider is,

$$\begin{aligned} &\min_{\Phi_u} \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega \\ &\text{s.t. } \Phi_u(\omega)\geq 0, \quad |\omega|\leq\pi \\ &\lim_{N\to\infty} N \text{ Var } \{G(e^{j\omega},\hat{\theta}_{N,n})\}\leq\frac{1}{\gamma}, \quad |\omega|\leq\omega_B. \end{aligned} \tag{7.2}$$

In other words, the problem is to find an input of minimum power for which the variance of the estimated model is bounded by $1/\gamma$ in the frequency range $[-\omega_B,\omega_B]$.

By the *Gauss' approximation formula* (Ljung 1999),

$$\lim_{N\to\infty} N \text{ Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} = \sigma_o^2\Lambda_n^H(e^{j\omega})T_n^{-1}\Lambda_n(e^{j\omega}). \tag{7.3}$$

This formula is valid only when T_n is non-singular, i.e. when $T_n>0$ (since T_n must be positive semidefinite in order to define a proper spectrum Φ_u). Assuming $T_n>0$ and applying Schur complements (Boyd et al. 1994), the second constraint in (7.2) can be written as

$$T_n-\sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\geq 0, \quad |\omega|\leq\omega_B.$$

Thus, problem (7.2) can be reformulated, for $\omega_B\in(0,\pi]$, as:

$$\begin{aligned} &\min_{r_0,\ldots,r_{n-1}} r_0 \\ &\text{s.t. } T_n-\sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\geq 0, \quad |\omega|\leq\omega_B \end{aligned} \tag{7.4}$$

(see e.g., (Hjalmarsson et al. 2006)). The constraint $T_n>0$ has not been included in (7.4), as it can be shown (see Lemma 7.9.1 in Appendix 7.9) that $T_n>0$ holds for any solution of (7.4) if $\omega_B>0$.

The case where $\omega_B=0$ is treated separately (see Remark 7.3.4 of Section 7.3) as the optimal solution produces a singular matrix T_n.

Let us denote by r_0^{opt} the solution to (7.4). The focus here is to study the dependence of r_0^{opt} on the model order n, bandwidth ω_B and precision γ, by analyzing the frequency-wise Linear Matrix Inequality (LMI)

$$T_n-\sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\geq 0, \quad |\omega|\leq\omega_B. \tag{7.5}$$

Remark 7.2.1 *Note that* (7.4) *only depends on the first n terms of the autocovariance sequence of* $\{u_t\}$. *This means that the solution of this problem is in general not unique, unless* T_n *is singular (Byrnes et al. 2001, Grenander and Szegö 1958, Lindquist and Picci 1996). This special case only arises when* $\omega_B=0$, *as will be seen in Remark 7.3.4, where it is concluded that the optimal input is a constant.*

7.3 Main Results

In this section the main results of the chapter are presented. Properties associated with monotonicity and continuity are established. As well, upper and lower bounds on the cost are developed which are asymptotic in the model order.

7.3.1 Monotonicity and continuity

The implication of Theorems 7.3.1 and 7.3.2 below is that the more information we require for the model, the larger the cost. In particular, Theorem 7.3.1 shows that the optimal cost is a non-decreasing function of n, whilst Theorem 7.3.2 reveals that the cost is a non-decreasing function of the frequency range. Theorem 7.3.3 establishes continuity of the cost with respect to the frequency range.

Theorem 7.3.1 (Monotonicity of r_0^{opt} with respect to n) *The optimal cost of* (7.4)*, r_0^{opt}, is a monotonically non-decreasing function of n.*

Proof. Note that

$$T_{n+1} - \sigma_o^2 \gamma \Lambda_{n+1}(e^{j\omega}) \Lambda_{n+1}^H(e^{j\omega}) = \begin{bmatrix} A_n & B_{n+1} \\ B_{n+1}^H & r_0 - \sigma_o^2 \gamma \end{bmatrix},$$

where

$$A_n := T_n - \sigma_o^2 \gamma \Lambda_n(e^{j\omega}) \Lambda_n^H(e^{j\omega})$$

$$B_{n+1} := \begin{bmatrix} r_n - \sigma_o^2 \gamma e^{jn\omega} \\ \vdots \\ r_1 - \sigma_o^2 \gamma e^{j\omega} \end{bmatrix}.$$

Thus, if Φ_u satisfies $A_{n+1} \geq 0$, it also satisfies $A_n \geq 0$, for every $\omega \in [-\omega_B, \omega_B]$. This means that r_0^{opt} is monotonically non-decreasing in n. □

Theorem 7.3.2 (Monotonicity of r_0^{opt} with respect to ω_B) *Let $r_0^{opt,1}$ and $r_0^{opt,2}$ be the optimal costs of the input design problem* (7.2) *for $\omega_B = \omega_{B1}$ and $\omega_B = \omega_{B2}$, respectively, and a fixed model order n. If $0 \leq \omega_{B1} < \omega_{B2} \leq \pi$, then $r_0^{opt,1} \leq r_0^{opt,2}$.*

Proof. Follows directly from the fact that the set of allowable input spectra Φ_u decreases with increasing ω_B. □

The following result establishes the continuity of the optimal cost with respect to ω_B.

Theorem 7.3.3 (Continuity of r_0^{opt} with respect to ω_B) *The optimal cost of* (7.4)*, r_0^{opt}, is a continuous function of ω_B.*

Proof. The basis of this proof utilises the results of (Dantzig et al. 1967). First, we fix σ_o^2, γ and n, and rewrite problem (7.2) as

$$\min_{(r_0,\ldots,r_{n-1})\in H(\omega_B)} r_0,$$

where

$$\begin{aligned} H(\Omega) &:= \left\{ r\in\mathbb{R}^n :\ \sigma_o^2 \Lambda_n^H(e^{j\omega}) T_n^{-1} \Lambda_n(e^{j\omega}) \le \frac{1}{\gamma}, \quad |\omega|\le\Omega \right\} \\ &= \left\{ r\in\mathbb{R}^n :\ \sup_{|\omega|\le\Omega} \Lambda_n^H(e^{j\omega}) T_n^{-1} \Lambda_n(e^{j\omega}) \le \frac{1}{\sigma_o^2\gamma} \right\}. \end{aligned}$$

From (Dantzig et al. 1967, Theorem I.3.3), the continuity of r_0^{opt} follows if for every sequence $\{\Omega_k\}_{k=1}^{\infty}$ such that $H(\Omega_k)\neq\emptyset$ for every $k\in\mathbb{N}$, and $\Omega_k\to\Omega_0$, with $H(\Omega_0)\neq\emptyset$, it holds that $\lim_{k\to\infty} H(\Omega_k) = H(\Omega_0)$ in a set-theoretical sense.

Now, we note that $H(\Omega)$ is closed for every $\Omega\in[-\pi,\pi]$, and $H(\Omega_1)\subseteq H(\Omega_2)$ whenever $\Omega_1\ge\Omega_2$. Thus, by definition (Dantzig et al. 1967, Section I.1),

$$\overline{\lim_{k\to\infty}} H(\Omega_k) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{m=k}^{\infty} H(\Omega_m)} = \bigcap_{k=1}^{\infty} H\left(\inf_{m\ge k}\Omega_m\right) = H\left(\overline{\lim_{k\to\infty}}\,\Omega_k\right) = H(\Omega_0),$$

where for the second equality we used the fact that $\Omega\mapsto\sup_{|\omega|\le\Omega}\Lambda_n^H(e^{j\omega})T_n^{-1}\Lambda_n(e^{j\omega})$ is continuous for every T_n. Similarly, if we let K denote the set of all infinite subsequences $\{n_i\}_{i=1}^{\infty}$ in $\mathbb{N}$,

$$\overline{\lim_{k\to\infty}} H(\Omega_k) = \bigcup_{\{n_i\}_{i=1}^{\infty}\in K} \overline{\lim_{i\to\infty}} H(\Omega_{n_i}) = \bigcup_{\{n_i\}_{i=1}^{\infty}\in K} H(\Omega_0) = H(\Omega_0).$$

This proves that $\lim_{k\to\infty} H(\Omega_k) = H(\Omega_0)$, from which the result follows. □

7.3.2 Cost for a white noise input

In the next theorem, we derive r_0^{opt} for the case when the input is restricted to white noise spectra. This in fact constitutes an upper bound for r_0^{opt}.

Theorem 7.3.4 (White noise input spectrum) *For the case of a white noise input spectrum, we have*

$$r_0^{opt} = r_{whitenoise}^{opt} := n\sigma_o^2\gamma.$$

Proof. White noise corresponds to $r_k = r_0\delta_k$. From (7.3) we obtain $\frac{\sigma_o^2}{r_0}\Lambda_n^H(e^{j\omega})\Lambda_n(e^{j\omega})\le 1/\gamma$. Since $\Lambda_n^H(e^{j\omega})\Lambda_n(e^{j\omega}) = n$, we obtain $r_0^{opt} = n\sigma_o^2\gamma$. □

Based on this theorem we conclude that $r^{opt}_{whitenoise}$ is proportional to the model order n and the precision γ, but independent of the bandwidth ω_B.

7.3.3 Cost for $\omega_B = 0$ and $\omega_B \to \pi$

The next theorem is a particular case of Theorem 3.1 in (Hjalmarsson et al. 2006). It shows that if we are only interested in estimating the static gain, the optimal input is independent of the model order.

Theorem 7.3.5 *When $\omega_B = 0$, the optimal cost is given by $r_0^{opt} = \sigma_o^2\gamma$.*

Proof. This proof is a particular case of the proof presented in Theorem 3.1 of (Hjalmarsson et al. 2006). Here we have

$$T_n - \sigma_o^2\gamma\Lambda_n(1)\Lambda_n^H(1) \geq 0 \quad \Leftrightarrow \quad \begin{bmatrix} r_0 - \sigma_o^2\gamma & r_1 - \sigma_o^2\gamma & \cdots & r_{n-1} - \sigma_o^2\gamma \\ r_1 - \sigma_o^2\gamma & r_0 - \sigma_o^2\gamma & \cdots & r_{n-2} - \sigma_o^2\gamma \\ \vdots & \vdots & \ddots & \vdots \\ r_{n-1} - \sigma_o^2\gamma & r_{n-2} - \sigma_o^2\gamma & \cdots & r_0 - \sigma_o^2\gamma \end{bmatrix} \geq 0. \qquad (7.6)$$

A necessary condition for (7.6) to hold is that $r_0 \geq \sigma_o^2\gamma$, hence $r_0^{opt} \geq \sigma_o^2\gamma$. On the other hand, if we take $\Phi_u(\omega) = (\sigma_o^2\gamma)\delta(\omega)$ (e.g. by taking $\{u_t\}$ to be a constant equal to $\sigma_o^2\gamma$), we have $r_i = \sigma_o^2\gamma$ for $i = 0, \ldots, n-1$, which implies that $r_0^{opt} \leq \sigma_o^2\gamma$. This then implies that $r_0^{opt} = \sigma_o^2\gamma$. □

The next theorem considers the case when ω_B is very close (but not necessarily equal) to π. It is shown that in this case, white noise in an optimal input.

Theorem 7.3.6 (Behaviour of r_0^{opt} as $\omega_B \to \pi$) *When $\omega_B > (n-1)\pi/n$, we have that $r_0^{opt} = r^{opt}_{whitenoise}$.*

Proof. See Appendix 7.7.1. □

An important implication of this theorem is that, since r_0^{opt} equals $r^{opt}_{whitenoise}$, extracting more information from the system (i.e., increasing ω_B to π) does not cost more if ω_B is sufficiently close to π.

7.3.4 Asymptotic behaviour (in n)

The next observation exploits the asymptotic variance formula (Ljung 1985) to provide a heuristic derivation of the fact that, for large n, r_0^{opt} is proportional to n and ω_B.

Observation 7.3.1 *Utilising the asymptotic variance formula (Ljung 1985), the condition*

$$\lim_{N\to\infty} N\ Var\ G(e^{j\omega},\hat{\theta}_{N,n}) \leq 1/\gamma$$

can be approximated by

$$n\frac{\sigma_o^2}{\Phi_u(\omega)} \leq 1/\gamma.$$

This implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega \geq \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B}\Phi_u(\omega)d\omega \geq n\frac{\sigma_o^2\omega_B\gamma}{\pi}, \tag{7.7}$$

however if we take

$$\Phi_u(\omega) = \begin{cases} n\sigma_o^2\gamma, & \omega\in[-\omega_B,\omega_B] \\ 0, & otherwise, \end{cases}$$

(7.7) *turns into an equality. This suggests that* r_0^{opt} *is asymptotically proportional to the model order, n, to the accuracy,* γ*, and to the bandwidth* ω_B*, that is,*

$$r_0^{opt} = n\frac{\sigma_o^2\omega_B\gamma}{\pi}.$$

In the following two theorems, the heuristic argument of Observation 7.3.1 is established in a rigorous fashion by providing asymptotic bounds on r_0^{opt}. These bounds are both proportional to n, ω_B and γ. Theorem 7.3.7 presents an asymptotic lower bound for r_0^{opt} and Theorem 7.3.8 an upper bound.

Theorem 7.3.7 (Lower bound for the asymptotic cost) *Assume that* $\omega_B\in(0,\pi)$*. Then, for every* $\varepsilon>0$ *there exists an* $n_{as}\in\mathbb{N}$*, depending on* σ_o^2*,* γ*,* ω_B *and* ε*, such that, for all* $n\geq n_{as}$*,*

$$r_0^{opt} \geq \left[n\frac{\omega_B}{\pi}+1-\varepsilon\right]\sigma_o^2\gamma.$$

Proof. See Appendix 7.7.2. □

Theorem 7.3.8 (Upper bound for the asymptotic cost) *Assume that* $\omega_B\in(0,\pi]$*. Then, there exists an* $n_{as}\in\mathbb{N}$*, depending on* σ_o^2*,* γ *and* ω_B*, such that, for all* $n\geq n_{as}$*,*

$$r_0^{opt} \leq \left[n\frac{\omega_B}{\pi}+1+\frac{25}{\pi}\ln\left(\frac{2n\omega_B}{\pi}\right)\right]\sigma_o^2\gamma.$$

Proof. See Appendix 7.7.3. □

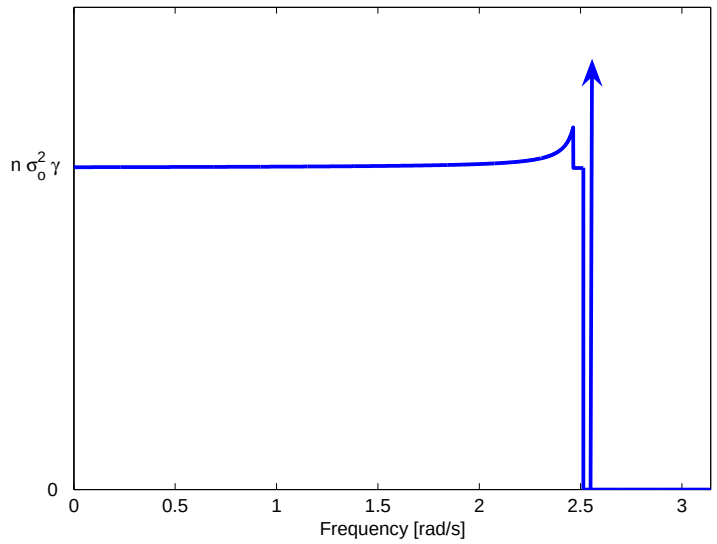

Figure 7.1. Input spectrum described in Remark 7.3.1, for $\omega_B = 0.8\pi \approx 2.51$. The arrow represents a Dirac delta of area $\sigma_o^2\gamma\pi$.

Remark 7.3.1 *From the proof of Theorem 7.3.8, in Appendix 7.7.3 (in particular from* (7.24) *and* (7.20)*), it follows that an input spectrum with power equal to the asymptotic upper bound, which satisfies the model quality constraint of problem* (7.2) *for sufficiently large n, is*

$$\Phi_u(\omega) = n\sigma_o^2\gamma\left\{\pi\delta(n[\omega-\omega_B]-1) - \frac{25}{n(\omega-\omega_B)}\mu(-n[\omega-\omega_B]-\pi/2) + \mu(-n[\omega-\omega_B])\right\},$$
$$\omega \in [0,\omega_B],$$

where μ *is the Heaviside step function. This input spectrum is shown in Figure 7.1.*

Remark 7.3.2 *As mentioned in Remark 7.2.1, the solution of the original input design problem is not unique, and this lack of uniqueness does not disappear after the normalisation in the proof of Theorem 7.3.8. To see this, notice that the left side of* (7.23) *is the convolution of* $\tilde{\Phi}_u^{0*}$ *and a kernel whose Fourier transform is*

$$\mathscr{F}\left\{\frac{\sin^2(t/2)}{(t/2)^2}\right\}(\omega) = \begin{cases} \omega+1, & \omega \in [-1,0] \\ 1-\omega, & \omega \in (0,1] \\ 0, & \textit{otherwise}. \end{cases}$$

Since this Fourier transform is zero for $\omega \notin (-1,1)$*, there are several functions* $\tilde{\Phi}_u^{0*}$ *for which the left side of* (7.23) *is the same function of* τ*. However, the first n autocovariance terms of the functions* Φ_u^n *associated with these solutions will be the same.*

Note that the difference between the lower and upper bounds given by Theorems 7.3.7 and 7.3.8 is essentially a term which increases logarithmically with n.

In the next theorem we again restrict attention to a particular structure on Φ_u, namely bandlimited white noise. It is shown that r_0^{opt}, as in the two previous theorems, is proportional to n, ω_B and γ, however in this case the constant of proportionality is larger.

Theorem 7.3.9 (Optimal bandlimited white noise) *Consider the case when Φ_u is restricted to be bandlimited white noise:*

$$\Phi_u(\omega) = \begin{cases} \alpha, & |\omega| \leq \omega_B \\ 0, & \omega_B < |\omega| \leq \pi. \end{cases} \tag{7.8}$$

Then, there exists an $n_{as} \in \mathbb{N}$, depending on σ_o^2, γ and ω_B, such that, for all $n \geq n_{as}$, Φ_u satisfies the model quality constraint (7.5) *if and only if*

$$\begin{aligned} \alpha &\geq n\sigma_o^2\gamma \max_{x\geq 0} \frac{8\pi \sin^2(x/2)}{x[\pi x + 2 - 2\cos(x) - 2\,\mathrm{Si}(x)x]} \\ &\approx 3.6072 n\sigma_o^2\gamma, \end{aligned}$$

where Si *is the* Sine Integral *(Abramowitz and Stegun 1964):*

$$\mathrm{Si}(x) := \int_0^x \frac{\sin(t)}{t} dt, \quad x \in \mathbb{C}.$$

Thus, the optimal power for this class of signals is

$$r_0^{opt} \approx 3.6072 n \frac{\omega_B \sigma_o^2 \gamma}{\pi}.$$

Proof. We proceed as in the proof of Theorem 7.3.8, by performing the change of variables (7.20). Expression (7.8) is then equivalent to

$$\tilde{\Phi}_u(\omega) = \begin{cases} \tilde{\alpha}, & \omega \leq 0 \\ 0, & \omega > 0, \end{cases} \tag{7.9}$$

where, for simplicity, we are focusing only on the positive frequencies of Φ_u, and

$$\tilde{\alpha} := \frac{\alpha}{n\sigma_o^2\gamma}.$$

Now, substituting (7.9) into (7.23), we obtain

$$\tilde{\alpha} g(\tau) \geq \begin{cases} 1, & \tau \in (-\infty, 0] \\ \sup\limits_{\tau < x < \infty} \frac{\sin^2(x/2)}{(x/2)^2}, & \tau \in (0, \infty), \end{cases} \tag{7.10}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined as

$$g(\tau) := \frac{\pi\tau + 2 - 2\cos(\tau) - 2\,\mathrm{Si}(\tau)\tau}{2\pi\tau}.$$

This function has a derivative,

$$g'(\tau) = \frac{\cos(\tau) - 1}{\pi\tau^2} \leq 0, \quad \tau \in \mathbb{R},$$

hence g is monotonically non-increasing in $\mathbb{R}$. This means that (7.10) holds if and only if

$$\tilde{\alpha} g(\tau) \geq \frac{\sin^2(\tau/2)}{(\tau/2)^2}, \quad \tau \geq 0, \tag{7.11}$$

This comes from the fact that, by the monotonicity of g, $\tilde{\alpha} g(\tau) \geq 1$ for $\tau \leq 0$ if and only if

$$\tilde{\alpha} g(0) \geq 1 = \lim_{\tau \to 0} \frac{\sin^2(\tau/2)}{(\tau/2)^2}.$$

Furthermore, if (7.11) holds, then $\tilde{\alpha} g(\tau) \geq \sup_{\tau < x < \infty} \sin^2(x/2)/[x/2]^2$; otherwise, if

$$\tilde{\alpha} g(\tau) < \frac{\sin^2(x/2)}{(x/2)^2}$$

for some $x > \tau$, then by (7.11) we would have that $g(\tau) < g(x)$, thus contradicting the monotonicity of g. Now, (7.11) is also equivalent to

$$\tilde{\alpha} \geq \frac{\sin^2(\tau/2)}{g(\tau)(\tau/2)^2} = \frac{8\pi \sin^2(\tau/2)}{\tau[\pi\tau + 2 - 2\cos(\tau) - 2\,\mathrm{Si}(\tau)\tau]}, \quad \tau \geq 0.$$

□

To summarise, the results of Theorems 7.3.1 and 7.3.2 are consistent with the fact that all bounds derived for r_0^{opt} are asymptotically affine in n, ω_B and γ. The results presented in Theorems 7.3.7 and 7.3.8 can be seen as refinements of Observation 7.3.1 and Theorems 7.3.4, 7.3.5 and 7.3.9. From the proof of Theorem 7.3.8 it is clear that the optimal input spectrum is not bandlimited white noise, but a more intelligently designed input spectrum. A relaxed upper bound can be obtained if we restrict the input to be bandlimited white noise in the frequency range $[-\omega_B, \omega_B]$, which is the optimal input signal according to Observation 7.3.1, when n is very large. However, according to Theorem 7.3.9, the power of the bandlimited white noise has to be more than 3 times that of the optimal signal to satisfy the model quality constraints.

Remark 7.3.3 *In simple terms, from Theorems 7.3.7 and 7.3.8 it can be established that, asymptotically in n,*

$$r_0^{opt} \propto n\omega_B\sigma_o^2\gamma.$$

Remark 7.3.4 *The solution of* (7.2) *for* $\omega_B = 0$ *does not give a non-singular matrix* T_n*. However, if we add a small perturbation, say* $\varepsilon > 0$*, to* r_0*, we obtain a non-singular* T_n*. Thus,* $r_0^{opt} = \sigma_o^2\gamma$ *is the* infimum *value of* r_0*, however it is not actually attainable, in the sense that the right hand side of* (7.3) *is not defined for* $\det T_n = 0$*, even though the variance of* $G(e^{j\omega}, \hat{\theta}_{N,n})$ *is meaningful in this case. In fact, in engineering terms, it is possible to generate the solution by using a constant signal, which will give a consistent estimator of the steady state gain of the system.*

7.4 Computational Issues

The constraint (7.5) is infinite dimensional due to the dependence on the continuous variable ω. However, by utilising the Generalised Kalman-Yakubovich-Popov (KYP) Lemma (Iwasaki and Hara 2005), the dependence on ω is eliminated and thus (7.5) can be written as a finite dimensional problem. The associated trade-offs are that we add two new matrix variables and the dimension of the semidefinite program increases.

For the sake of clarity in this section, we will write the matrices 0 and I with subindices to indicate their dimensions.

The LMI constraint (7.5) can be written as

$$T_n - \sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega}) > 0 \quad \Leftrightarrow \quad [I_n \; \Lambda_n(e^{j\omega})] \begin{bmatrix} -T_n & 0_{n,1} \\ 0_{1,n} & \sigma_o^2\gamma \end{bmatrix} [I_n \; \Lambda_n(e^{j\omega})]^H < 0.$$

Now,

$$\Lambda_n(e^{j\omega}) = \left[\begin{array}{c|c} \begin{bmatrix} 0_{1,n-2} & 0 \\ I_{n-2} & 0_{n-2,1} \end{bmatrix} & \begin{bmatrix} 1 \\ 0_{n-2,1} \end{bmatrix} \\ \hline \begin{bmatrix} 0_{1,n-1} \\ I_{n-1} \end{bmatrix} & \begin{bmatrix} 1 \\ 0_{n-1,1} \end{bmatrix} \end{array}\right],$$

where we use the notation $\left[\frac{A|B}{C|D}\right] := C(e^{j\omega}I - A)^{-1}B + D$ (Zhou et al. 1996). Hence,

$$[I_n \; \Lambda_n(e^{j\omega})] = \left[\begin{array}{c|c} \begin{bmatrix} 0_{1,n-2} & 0 \\ I_{n-2} & 0_{n-2,1} \end{bmatrix} & \begin{bmatrix} 0_{1,n} & 1 \\ 0_{n-2,n} & 0_{n-2,1} \end{bmatrix} \\ \hline \begin{bmatrix} 0_{1,n-1} \\ I_{n-1} \end{bmatrix} & \begin{bmatrix} 1 & 0_{1,n-1} & 1 \\ 0_{n-1,1} & I_{n-1} & 0_{n-1,1} \end{bmatrix} \end{array}\right].$$

Thus, the constraint (7.5) is equivalent to

$$[I_n \Lambda_n(\lambda)] \begin{bmatrix} -T_n & 0_{n,1} \\ 0_{1,n} & \sigma_o^2 \gamma \end{bmatrix} [I_n \Lambda_n(\lambda)]^H < 0,$$

$$\forall \lambda \in \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^H \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^H \begin{bmatrix} 0 & 1 \\ 1 & -2\cos\omega_B \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0 \right\}.$$

Then, by the Generalised KYP Lemma (Hara et al. 2006), (7.5) is equivalent to

'There exist matrices $P = P^H, Q = Q^H \geq 0$ such that

$$H \operatorname{diag}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P + \begin{bmatrix} 0 & 1 \\ 1 & -2\cos\omega_B \end{bmatrix} \otimes Q, \begin{bmatrix} -T_n & 0_{n,1} \\ 0_{1,n} & \sigma_o^2 \gamma \end{bmatrix} \right) H^H \leq 0', \tag{7.12}$$

where $\otimes$ denotes the Kronecker product (Bernstein 2005), and

$$H := \begin{bmatrix} \begin{bmatrix} 0_{1,n-2} & 0 \\ I_{n-2} & 0_{n-2,1} \end{bmatrix} & I_{n-1} & \begin{bmatrix} 0_{1,n} & 1 \\ 0_{n-2,n} & 0_{n-2,1} \end{bmatrix} \\ \begin{bmatrix} 0_{1,n-1} \\ I_{n-1} \end{bmatrix} & 0_{n,n-1} & \begin{bmatrix} 1 & 0_{1,n-1} & 1 \\ 0_{n-1,1} & I_{n-1} & 0_{n-1,1} \end{bmatrix} \end{bmatrix}.$$

The optimal input spectrum can thus be computed numerically by solving the semidefinite program (7.4), with the constraint (7.5) being replaced by (7.12). An example is provided in the next section.

7.5 Numerical Example

Let $\sigma_o^2 = 1$ and $\gamma = 1$. First, we illustrate the result of Theorem 7.3.6. In Figure 7.2, the optimal cost r_0^{opt} is plotted for different ω_B. Also note that the white noise solution $r_{whitenoise}^{opt}$ is plotted. It is seen that if ω_B is close to π, r_0^{opt} equals $r_{whitenoise}^{opt}$ for finite n. Then, as n increases, there is a 'knee' in the curve (i.e., the slope decreases). In other words, the incremental cost of increasing the model order is larger for small n.

Next, we illustrate the asymptotic results derived in Section 7.3. Consider a fixed bandwidth $\omega_B = 0.15\pi$. In Figure 7.3, the optimal solution r_0^{opt} is plotted together with the asymptotic upper and lower bounds given in Theorems 7.3.7 and 7.3.8 respectively (taking $\varepsilon \ll 1$). It is clear that these bounds are closer to r_0^{opt} than the simple bounds given in Theorems 7.3.4 and 7.3.5 for n sufficiently large.

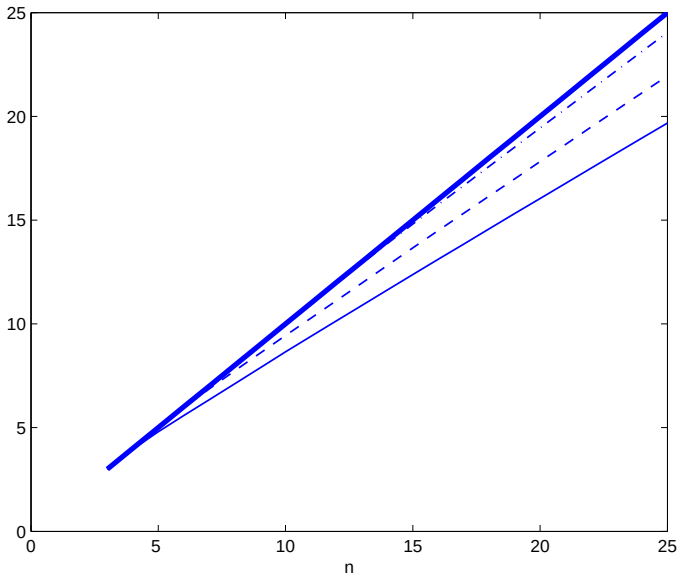

Figure 7.2. The optimal cost (r_0^{opt}) from (7.4) versus model order n for $\omega_B = 0.7\pi$ (thin solid); r_0^{opt} for $\omega_B = 0.8\pi$ (dashed); r_0^{opt} for $\omega_B = 0.9\pi$ (dash-dotted); the white noise solution $r_{whitenoise}^{opt}$ (thick solid).

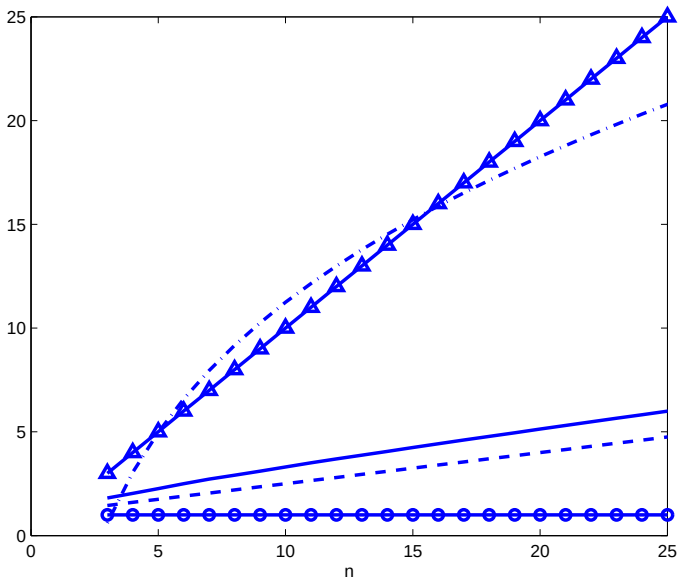

Figure 7.3. The optimal cost (r_0^{opt}) from (7.4) versus model order n (solid); asymptotic lower bound for r_0^{opt}, c.f. Theorem 7.3.7 (dashed); asymptotic upper bound for r_0^{opt}, c.f. Theorem 7.3.8 (dash-dotted); lower bound given by Theorem 7.3.5 ($\ominus$); the white noise solution $r_{whitenoise}^{opt}$ ($\preccurlyeq$).

7.6 Summary

In this chapter we have studied the minimum amount of input power, r_0^{opt}, needed to estimate an FIR model with prescribed precision γ over the frequency range $[-\omega_B, \omega_B]$, as a function of the model order n. It is assumed that n is large enough to capture the true system. Several properties of r_0^{opt} are derived. Firstly it is shown that if n is large, r_0^{opt} is asymptotically proportional to n, ω_B and γ. This is first deduced in a heuristic way from Ljung's asymptotic variance expression in Observation 7.3.1, then justified in a rigorous fashion by establishing asymptotic bounds on r_0^{opt}. A loose upper bound for r_0^{opt} is given by white noise input spectra, in Theorem 7.3.4. We also show in Theorem 7.3.6 that if ω_B is sufficiently close to (but not necessarily equal to) π, then r_0^{opt} equals the white noise solution. Furthermore a loose lower bound for r_0^{opt} is obtained by considering the solution for $\omega_B = 0$. This is given in Theorem 7.3.5. Results have also been developed which provide tighter asymptotic lower and upper bounds for r_0^{opt}, in Theorems 7.3.7 and 7.3.8, respectively. These bounds quantify the cost of extracting more information about the system and overmodelling. From these asymptotic bounds it can be concluded that, asymptotically in n,

$$r_0^{opt} \propto n\omega_B\sigma_o^2\gamma.$$

This expression shows that the amount of system information to be extracted (as measured by ω_B), the accuracy (as measured by γ) and the noise power are all on an equal footing with the model complexity n with respect to the cost of complexity. Returning to our comments made at the beginning of Section 7.1 regarding the capability of identifying complex systems, it shows us specifically that with a limited input power budget and limited time at our disposal, we can still identify highly complex systems to within a certain accuracy, however over a more limited bandwidth than for a less complex system. Notice that this expression holds asymptotically on the number of samples, N, although this quantity does not appear explicitly. The reason is that N is actually related to the accuracy γ, which is a bound on the normalised variance of the frequency response, that is, on the variance multiplied by N.

It is also important to notice that the excitation has to be carefully designed; recall that the cost of complexity is the minimum required input power to meet the model quality specifications. In particular, broadband excitation may not be suitable when only a limited frequency range is of interest, c.f. Theorem 7.3.4 that specifies that the cost for white noise excitation is $n\sigma_o^2\gamma$ regardless of ω_B.

The results above illustrate that the amount of information that we ask to be extracted from a system determines how sensitive the cost of the identification experiment is with respect to the system (and model) complexity.

7.7 Appendix: Proofs

7.7.1 Proof of Theorem 7.3.6

The basic idea is to use the Lagrangian dual of problem (7.4) (see Lemma 7.9.3 of Appendix 7.9), and to postulate a feasible solution of this dual problem, which turns out to be optimal, thus giving the optimal cost r_0^{opt}.

Let $\omega_B > (n-1)\pi/n$. Now, take $\tilde{\delta}$ as an approximate delta function with support on $[-\varepsilon,\varepsilon]$, where $\varepsilon \in (0,(\omega_B-(n-1)\pi/n)/2)$, i.e., a nonnegative function such that $\int_{-\infty}^{\infty}\tilde{\delta}(x)dx = 1$[1]. Now, define:

$$Q^o(\omega) := \frac{1}{n^2}\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\sum_{i=0}^{n-1}\tilde{\delta}\left(\omega-\left[\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]\right), \quad |\omega|\le\pi.$$

Notice that $\Lambda_n(e^{j\omega_1})$ and $\Lambda_n(e^{j\omega_2})$ are orthogonal if and only if $\omega_1-\omega_2$ is an integer multiple of $2\pi/n$, and $\|\Lambda_n(e^{j\omega})\|_2 = \sqrt{n}$ for every $\omega\in[-\pi,\pi]$. Then the matrix

$$\frac{1}{\sqrt{n}}\left[\Lambda_n\left(e^{j\left[\omega-\frac{(n-1)\pi}{n}\right]}\right)\cdots\Lambda_n\left(e^{j\left[\omega+\frac{(n-1)\pi}{n}\right]}\right)\right]$$

is unitary for every $\omega\in[-\pi,\pi]$. Thus,

$$\begin{aligned}
&\sum_{i=0}^{n-1}\Lambda_n\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\Lambda_n^H\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\\
&\qquad=\left[\Lambda_n\left(e^{j\left[\omega-\frac{(n-1)\pi}{n}\right]}\right)\cdots\Lambda_n\left(e^{j\left[\omega+\frac{(n-1)\pi}{n}\right]}\right)\right]\left[\Lambda_n\left(e^{j\left[\omega-\frac{(n-1)\pi}{n}\right]}\right)\cdots\Lambda_n\left(e^{j\left[\omega+\frac{(n-1)\pi}{n}\right]}\right)\right]^H\\
&\qquad=nI,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{-\omega_B}^{\omega_B}Q^o(\omega)d\omega &= \frac{1}{n^2}\int_{-\omega_B}^{\omega_B}\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\sum_{i=0}^{n-1}\tilde{\delta}\left(\omega-\left[\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]\right)d\omega\\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\int_{-\omega_B}^{\omega_B}\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\tilde{\delta}\left(\omega-\left[\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]\right)d\omega\\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\int_{-\varepsilon}^{\varepsilon}\Lambda_n\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\Lambda_n^H\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\tilde{\delta}(\omega)d\omega \qquad (7.13)\\
&= \frac{1}{n^2}\int_{-\varepsilon}^{\varepsilon}\sum_{i=0}^{n-1}\Lambda_n\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\Lambda_n^H\left(e^{j\left[\omega+\frac{2\pi i}{n}-\frac{(n-1)\pi}{n}\right]}\right)\tilde{\delta}(\omega)d\omega\\
&= \frac{1}{n}I\int_{-\varepsilon}^{\varepsilon}\tilde{\delta}(\omega)d\omega\\
&= \frac{1}{n}I.
\end{aligned}$$

Note that in this calculation we have used the fact that $\omega_B > (n-1)\pi/n$, otherwise not all of the $\tilde{\delta}$'s would be included in the integration range $[-\omega_B,\omega_B]$.

[1] We can consider as $\tilde{\delta}$, e.g., an element of an *approximate identity on* $\mathbb{R}^1$ (Rudin 1973, Definition 6.31).

By Lemma 7.9.3 of Appendix 7.9, expression (7.13) implies that $Q = Q^o$ is a feasible solution of the Lagrangian dual of problem (7.4), hence,

$$\begin{aligned} r_0^{opt} &\geq \sigma_o^2\gamma \int_{-\omega_B}^{\omega_B} \Lambda_n^H(e^{j\omega}) Q^o(\omega) \Lambda_n(e^{j\omega}) d\omega \\ &= \frac{\sigma_o^2\gamma}{n^2} \int_{-\omega_B}^{\omega_B} \Lambda_n^H(e^{j\omega})\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})\Lambda_n(e^{j\omega}) \sum_{i=0}^{n-1} \tilde{\delta}\left(\omega - \left[\frac{2\pi i}{n} - \frac{(n-1)\pi}{n}\right]\right) d\omega \\ &= \sigma_o^2\gamma \int_{-\omega_B}^{\omega_B} \sum_{i=0}^{n-1} \tilde{\delta}\left(\omega - \left[\frac{2\pi i}{n} - \frac{(n-1)\pi}{n}\right]\right) d\omega \\ &= n\sigma_o^2\gamma, \end{aligned}$$

where in the last step we have used the fact that $\omega_B > (n-1)\pi/n$. On the other hand, by Theorem 7.3.4, $r_0^{opt} \leq n\sigma_o^2\gamma$. □

7.7.2 Proof of Theorem 7.3.7

By pre- and post-multiplying (7.5) by $\Lambda_n^H(e^{j\beta})$ and $\Lambda_n(e^{j\beta})$, respectively, where $\beta \in [0,\pi]$, it must hold that

$$\Lambda_n^H(e^{j\beta}) T_n \Lambda_n(e^{j\beta}) \geq \sigma_o^2\gamma |\Lambda_n^H(e^{j\beta})\Lambda_n(e^{j\omega})|^2, \quad |\omega| \leq \omega_B, \beta \in [0,\pi]. \tag{7.14}$$

Lemma 7.9.6 (see Appendix 7.9) implies that (7.5) and (7.14) are equivalent. Now,

$$|\Lambda_n^H(e^{j\beta})\Lambda_n(e^{j\omega})|^2 = \left|\sum_{k=0}^{n-1} e^{j(\beta-\omega)k}\right|^2 = \frac{\sin^2(\frac{n}{2}[\beta-\omega])}{\sin^2(\frac{1}{2}[\beta-\omega])}$$

and

$$\Lambda_n^H(e^{j\beta}) T_n \Lambda_n(e^{j\beta}) = \sum_{m=-(n-1)}^{n-1} (n-|m|) r_k e^{-j\beta m}.$$

This implies that (7.14) is equivalent to

$$\sum_{m=-(n-1)}^{n-1} \left(1 - \frac{|m|}{n}\right) r_k e^{-j\beta m} \geq \frac{\sigma_o^2\gamma}{n} \frac{\sin^2(\frac{n}{2}[\beta-\omega])}{\sin^2(\frac{1}{2}[\beta-\omega])}, \quad |\omega| \leq \omega_B, \quad \beta \in [0,\pi]. \tag{7.15}$$

The right hand side of (7.15) is the *Fejér kernel* F_n (defined in Appendix 7.8) and, by Lemma 7.8.2 (see Appendix 7.8), the left hand side of (7.15) is the convolution of F_n and Φ_u. Thus, (7.15) is equivalent to

$$\frac{1}{2\pi}[\Phi_u * F_n](\beta) \geq \sigma_o^2\gamma F_n(\omega-\beta), \quad |\omega| \leq \omega_B, \quad \beta \in [0,\pi]. \tag{7.16}$$

This expression can be further simplified by taking the supremum over $\omega \in [-\omega_B, \omega_B]$, and using Lemma 7.8.3 (see Appendix 7.8). This implies that (7.16) is equivalent to

$$\frac{1}{2\pi}[\Phi_u * F_n](\beta) \geq \begin{cases} n\sigma_o^2\gamma, & \beta \in [0,\omega_B] \\ \sigma_o^2\gamma \sup\limits_{\beta-\omega_B<x<\beta} F_n(x), & \beta \in (\omega_B,\pi]. \end{cases} \tag{7.17}$$

Notice that, by Tonelli's Theorem (Bartle 1966), the periodicity of Φ_u, and Lemma 7.8.4 (see Appendix 7.8),

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}[\Phi_u * F_n](\beta)d\beta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Phi_u(\beta-\omega)F_n(\omega)d\omega d\beta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}F_n(\omega)d\omega\int_{-\pi}^{\pi}\Phi_u(\beta)d\beta \\ &= \int_{-\pi}^{\pi}\Phi_u(\beta)d\beta.\end{aligned}$$

Thus, if we integrate both sides of (7.17) over $[-\pi,\pi]$ using

$$\tilde{F}_n(y) := \sup_{y<x<\pi-\omega_B} F_n(x) - F_n(y) \geq 0, \quad y \in (0, \pi-\omega_B],$$

and divide by 2π, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\beta)d\beta \geq n\frac{\omega_B\sigma_o^2\gamma}{\pi} + \frac{\sigma_o^2\gamma}{\pi}\int_0^{\pi-\omega_B}F_n(\beta)d\beta + \frac{\sigma_o^2\gamma}{\pi}\int_0^{\pi-\omega_B}\tilde{F}_n(\beta)d\beta. \tag{7.18}$$

By Lemma 7.8.5 (see Appendix 7.8), there is an $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\int_{\pi-\omega_B}^{\pi}F_n(\beta)d\beta < \pi\varepsilon.$$

Therefore by Lemmas 7.8.1 and 7.8.4 (see Appendix 7.8) we have

$$\begin{aligned}\frac{\sigma_o^2\gamma}{\pi}\int_0^{\pi-\omega_B}F_n(\beta)d\beta &= \frac{\sigma_o^2\gamma}{\pi}\int_0^{\pi}F_n(\beta)d\beta - \frac{\sigma_o^2\gamma}{\pi}\int_{\pi-\omega_B}^{\pi}F_n(\beta)d\beta \\ &> \sigma_o^2\gamma(1-\varepsilon), \quad n \geq N.\end{aligned} \tag{7.19}$$

Thus, by combining (7.18) and (7.19), we obtain the lower bound

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\beta)d\beta \geq n\frac{\omega_B\sigma_o^2\gamma}{\pi} + \sigma_o^2\gamma(1-\varepsilon), \quad n \geq N.$$

This means that, for n sufficiently large, the optimal cost satisfies the asymptotic lower bound

$$r_0^{opt} \geq \left[n\frac{\omega_B}{\pi} + 1 - \varepsilon\right]\sigma_o^2\gamma.$$

□

7.7.3 Proof of Theorem 7.3.8

We proceed as in the proof of Theorem 7.3.7, to arrive at (7.17). This expression depends explicitly on n. In order to simplify it, we redefine Φ_u and β to obtain an asymptotic expression which is independent of n. To this end, divide (7.17) by $n\sigma_o^2\gamma$ and define

$$\tau := n(\beta - \omega_B).$$

To make explicit the dependence of Φ_u on n, we will use the superscript n. Then, (7.17) is rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\sigma_o^2\gamma}\frac{\Phi_u^n(\tau/n+\omega_B-\omega)}{n}\frac{1}{n}\frac{\sin^2(\frac{n}{2}\omega)}{\sin^2(\omega/2)}d\omega \geq \begin{cases} 1, & \tau\in[-n\omega_B,0] \\ \sup\limits_{\frac{\tau}{n}<x<\frac{\tau}{n}+\omega_B}\frac{1}{n^2}\frac{\sin^2(\frac{n}{2}x)}{\sin^2(x/2)}, & \tau\in(0,n(\pi-\omega_B)]. \end{cases}$$

By applying the change of variables $\omega\mapsto n\omega$, we define

$$\tilde{\Phi}_u^n(x) := \frac{1}{\sigma_o^2\gamma}\frac{\Phi_u^n(x/n+\omega_B)}{n}, \quad x\in[-n\omega_B, n(\pi-\omega_B)]. \tag{7.20}$$

To simplify the notation, we assume in the rest of the proof that $\tilde{\Phi}_u^n$ is defined in the entire real line $\mathbb{R}$. Furthermore, we assume, as will be explained later, that the integral of $\tilde{\Phi}_u^n$ in $[0,\infty)$ is finite, and that $\lim_{x\to-\infty}\tilde{\Phi}_u^n(x)=c$, for some $c>0$.

From (7.20), we notice that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u^n(x)dx = \sigma_o^2\gamma\frac{1}{\pi}\int_{-n\omega_B}^{n(\pi-\omega_B)}\tilde{\Phi}_u^n(x)dx.$$

As the limits of the integral of $\tilde{\Phi}_u^n$ vary linearly with n, to obtain an asymptotic expression for the optimal cost, we need an expression for $\tilde{\Phi}_u^n$ which is valid up to order n^{-1}. Now,

$$\frac{1}{2\pi}\int_{-n\pi}^{n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]}d\omega \geq \begin{cases} 1, & \tau\in[-n\omega_B,0] \\ \sup\limits_{\tau<x<\tau+n\omega_B}\frac{1}{n^2}\frac{\sin^2(x/2)}{\sin^2[\frac{x}{2n}]}, & \tau\in(0,n(\pi-\omega_B)]. \end{cases} \tag{7.21}$$

For a fixed $\omega\in\mathbb{R}$,

$$\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]} = \frac{\sin^2(\omega/2)}{(\omega/2)^2}+\frac{1}{3}\frac{\sin^2(\omega/2)}{n^2}+O(n^{-4}).$$

Also,

$$\begin{aligned}\frac{1}{2\pi}\int_{-n\pi}^{n\pi}&\tilde{\Phi}_u^n(\tau-\omega)\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]}d\omega \\ &= \frac{1}{2\pi}\int_{-n\pi}^{n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega \\ &\quad+\frac{1}{2\pi}\int_{0}^{n\pi}\tilde{\Phi}_u^n(\tau-\omega)\left[\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]}-\frac{\sin^2(\omega/2)}{(\omega/2)^2}\right]d\omega \\ &\quad+\frac{1}{2\pi}\int_{-n\pi}^{0}\tilde{\Phi}_u^n(\tau-\omega)\left[\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]}-\frac{\sin^2(\omega/2)}{(\omega/2)^2}\right]d\omega.\end{aligned} \tag{7.22}$$

From the assumption that the integral of $\tilde{\Phi}_u^n$ in $[0,\infty)$ is finite, the second term is $O(n^{-2})$. Also, since $\lim_{x\to-\infty}\tilde{\Phi}_u^n(x)=c>0$,

$$\begin{aligned}\lim_{n\to\infty}n\frac{1}{2\pi}\int_{-n\pi}^{0}\tilde{\Phi}_u^n(\tau-\omega)b(\omega)d\omega &= \lim_{n\to\infty}\frac{c}{2\pi}\int_0^{\pi}\sin^2(n\omega/2)\left[\frac{1}{\sin^2(\omega/2)}-\frac{1}{(\omega/2)^2}\right]d\omega \\ &= \frac{c}{\pi^2},\end{aligned}$$

where

$$b(\omega) = \frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]} - \frac{\sin^2(\omega/2)}{(\omega/2)^2}.$$

Thus, the third term of (7.22) is $c\pi^{-2}n^{-1} + O(n^{-2})$. Also,

$$\begin{aligned}\frac{1}{2\pi}\int_{-n\pi}^{n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega \\ - \frac{1}{2\pi}\int_{n\pi}^{\infty}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega \\ - \frac{1}{2\pi}\int_{-\infty}^{-n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega.\end{aligned}$$

Under the same assumptions (i.e., that the integral of $\tilde{\Phi}_u^n$ in $[0,\infty)$ is finite and $\lim_{x\to-\infty}\tilde{\Phi}_u^n(x) = c > 0$), the second term is $O(n^{-2})$, and the third term satisfies

$$\lim_{n\to\infty} n\frac{1}{2\pi}\int_{-\infty}^{-n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega = \lim_{n\to\infty}\frac{nc}{2\pi}\int_{n\pi}^{\infty}\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega = \frac{c}{\pi^2}.$$

Thus, combining all these expressions, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-n\pi}^{n\pi}\tilde{\Phi}_u^n(\tau-\omega)\frac{1}{n^2}\frac{\sin^2(\omega/2)}{\sin^2[\frac{\omega}{2n}]}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega \\ &\quad - \frac{c}{\pi^2}\frac{1}{n} + \frac{c}{\pi^2}\frac{1}{n} + O(n^{-2}) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega + O(n^{-2}).\end{aligned}$$

This means that (7.21) can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}_u^n(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega + O(n^{-2}) \geq \begin{cases}1, & \tau\in[-n\omega_B, 0] \\ \sup\limits_{\tau<x<\tau+n\omega_B}\frac{\sin^2(x/2)}{(x/2)^2}, & \tau\in(0, n(\pi-\omega_B)].\end{cases}$$

Notice that for n large the inequality is essentially independent of n, in the sense that there is a perturbation $O(n^{-2})$ which can only affect the optimal $\tilde{\Phi}_u^n$ with a term of order n^{-2}. The limits of τ in the right side also depend on n, but increasing these limits can only make the inequality more restrictive. Thus, the optimal $\tilde{\Phi}_u^{n*}$ can be written as

$$\tilde{\Phi}_u^{n*}(x) = \tilde{\Phi}_u^{0*}(x) + O(n^{-2}), \qquad x\in\mathbb{R},$$

where $\tilde{\Phi}_u^{0*}$ must be nonnegative and satisfy

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}_u^{0*}(\tau-\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}d\omega \geq \begin{cases}1, & \tau\in(-\infty, 0] \\ \sup\limits_{\tau<x<\infty}\frac{\sin^2(x/2)}{(x/2)^2}, & \tau\in(0,\infty).\end{cases} \tag{7.23}$$

Based on (7.23), we propose the following input spectrum:

$$\tilde{\Phi}_u^0(\omega) := -\frac{25}{\omega}\mu(-\omega-\pi/2) + \mu(-\omega) + \pi\delta(\omega-1), \tag{7.24}$$

where μ is the Heaviside step function.

This spectrum satisfies (7.23) and the assumptions on $\tilde{\Phi}_u^{n*}$ (i.e., $\int_0^\infty \tilde{\Phi}_u^0(x)dx < \infty$ and $\lim_{x\to-\infty} \tilde{\Phi}_u^0(x) = 1 > 0$), and has power

$$r_0 = \sigma_o^2\gamma\frac{1}{\pi}\int_{-n\omega_B}^{n(\pi-\omega_B)} \tilde{\Phi}_u^n(x)dx = \sigma_o^2\gamma\left[n\frac{\omega_B}{\pi} + \frac{25}{\pi}\ln\left(\frac{2n\omega_B}{\pi}\right) + 1\right].$$

This is the desired asymptotic upper bound for r_0^{opt}. □

7.8 Appendix: Properties of the Fejér Kernel

The *Fejér kernel* F_n is defined as

$$F_n(x) := \frac{1}{n}\frac{\sin^2(\frac{n}{2}x)}{\sin^2(\frac{1}{2}x)}, \quad x \in \mathbb{R}.$$

Some important properties of the Fejér kernel that are utilised in this chapter are included below for completeness. For the proofs, the reader is referred to standard texts on Fourier series, e.g., (Körner 1988, Vretblad 2003, Zygmund 1952).

Lemma 7.8.1 *$F_n(x) \geq 0$ and $F_n(-x) = F_n(x)$.*

Lemma 7.8.2 *If $g : \mathbb{Z} \to \mathbb{R}$ has Fourier transform G, then*

$$\sum_{m=-(n-1)}^{n-1}\left[1 - \frac{|m|}{n}\right] g_m e^{-j\omega m} = \frac{1}{2\pi}[G * F_n](\omega), \quad \omega \in [-\pi, \pi].$$

Lemma 7.8.3 *$F_n(x) \leq F_n(0) = n$ for all $x \in [-\pi, \pi]$.*

Lemma 7.8.4

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)dx = 1.$$

Lemma 7.8.5 *For every $\delta > 0$,*

$$\lim_{n\to\infty}\int_{\delta}^{\pi} F_n(x)dx = 0.$$

7.9 Appendix: Supporting Technical Lemmas

Lemma 7.9.1 *Let $\omega_B \in (0, \pi]$. Then, if $\{r_k\}_{k=0}^{n-1}$ is a solution to the input design problem* (7.5)*, it satisfies $T_n > 0$.*

Proof. Pick n different numbers $\{\omega_k\}_{k=0}^{n-1}$ from $[0,\omega_B]$. Then, from (7.5), the solution $\{r_k\}_{k=0}^{n-1}$ satisfies

$$T_n - \sigma_o^2\gamma\Lambda_n(e^{j\omega_i})\Lambda_n^H(e^{j\omega_i}) \geq 0, \quad i=0,\ldots,n-1. \tag{7.25}$$

By summing (7.25) over $i=0,\ldots,n-1$, and dividing by n, we obtain

$$T_n - \frac{\sigma_o^2\gamma}{n}UU^H \geq 0, \tag{7.26}$$

where

$$U := [\Lambda_n(e^{j\omega_0})\cdots\Lambda_n(e^{j\omega_{n-1}})] = \begin{bmatrix} 1 & \cdots & 1 \\ e^{-j\omega_0} & \ldots & e^{-j\omega_{n-1}} \\ \vdots & & \vdots \\ e^{-j(n-1)\omega_0} & \ldots & e^{-j(n-1)\omega_{n-1}} \end{bmatrix}.$$

Notice that U^H is a Vandermonde matrix (Horn and Johnson 1985), whose determinant is

$$\det(U^H) = \prod_{0\leq l<k\leq n-1}(e^{j\omega_k} - e^{j\omega_l}) \neq 0.$$

Thus, $UU^H > 0$, therefore by (7.26) we conclude that $T_n > 0$. □

Lemma 7.9.2 *Let S^n be the real linear space of Hermitian $n\times n$ continuous matrix functions on $[-\omega_B,\omega_B]$, with the inner product*[2]

$$\langle X,Y\rangle := \int_{-\omega_B}^{\omega_B}\mathrm{tr}[X(\omega)Y(\omega)]d\omega, \quad X,Y\in S^n.$$

Also, let $S_+^n \subseteq S^n$ be the closed convex cone of continuous matrix functions $X\in S^n$ such that $X(\omega)\geq 0$ for every $\omega\in[-\omega_B,\omega_B]$. Then, for every $X\in S^n$, $\langle X,Y\rangle \geq 0$ for every $Y\in S_+^n$ if and only if $X\in S_+^n$, i.e., S_+^n is self dual (Luenberger 1969).

Proof. Let $X\in S_+^n$. Then, for every $\omega\in[-\omega_B,\omega_B]$, $X(\omega) = \alpha_1 x_1(\omega)x_0^H(\omega) + \cdots + \alpha_n x_n(\omega)x_n^H(\omega)$ for some vectors $x_i(\omega)\in\mathbb{C}^n$ and real scalars $\alpha_i\geq 0$, $i=1,\ldots,n$. Thus,

$$\begin{aligned}\mathrm{tr}[X(\omega)Y(\omega)] &= \mathrm{tr}\{[\alpha_1 x_1(\omega)x_1^H(\omega)+\cdots+\alpha_n x_n(\omega)x_n^H(\omega)]Y(\omega)\} \\ &= \alpha_1 x_1^H(\omega)Y(\omega)x_1(\omega)+\cdots+\alpha_n x_n^H(\omega)Y(\omega)x_n(\omega) \\ &\geq 0,\end{aligned}$$

since $Y(\omega)\geq 0$ for every $\omega\in[-\omega_B,\omega_B]$. Thus, $\langle X,Y\rangle\geq 0$ for every $Y\in S_+^n$.

[2]Note that $\langle X,Y\rangle\in\mathbb{R}$ for all $X,Y\in S^n$, since $\langle X,Y\rangle^H = \int_{-\omega_B}^{\omega_B}\mathrm{tr}[X(\omega)Y(\omega)]^H d\omega = \int_{-\omega_B}^{\omega_B}\mathrm{tr}[Y(\omega)X(\omega)]d\omega = \langle X,Y\rangle$.

Now let $X \in S^n \setminus S^n_+$. Then, for some $\omega' \in [-\pi,\pi]$, $X(\omega')$ is not positive semidefinite, hence there exists a vector $z \in \mathbb{C}^n$ and an $\varepsilon > 0$ such that $z^H X(\omega) z < 0$ for every $\omega \in [\omega' - \varepsilon, \omega' + \varepsilon]$. Thus, if we take $Y(\omega) = zz^H f(\omega - \omega')$, where $f : [-\pi,\pi] \to \mathbb{R}^+_0$ is continuous and has nonempty support on $[\omega' - \varepsilon, \omega' + \varepsilon]$, then

$$\langle X, Y \rangle = \int_{-\omega_B}^{\omega_B} \mathrm{tr}[X(\omega) z z^H f(\omega - \omega')] d\omega = \int_{-\varepsilon}^{\varepsilon} z^H X(\omega + \omega') z f(\omega) d\omega < 0.$$

□

Lemma 7.9.3 *The Lagrangian dual (Luenberger 1969) of problem* (7.4) *is*

$$\begin{aligned} \max_{Q(\omega)=Q^H(\omega)\geq 0, |\omega|\leq\omega_B} \quad & \sigma_o^2 \gamma \int_{-\omega_B}^{\omega_B} \Lambda_n^H(e^{j\omega}) Q(\omega) \Lambda_n(e^{j\omega}) d\omega \\ s.t. \quad & \mathrm{tr}\left\{ \int_{-\omega_B}^{\omega_B} Q(\omega) d\omega \right\} = 1 \\ & \mathrm{Re} \sum_{i=1}^{n-k} \int_{-\omega_B}^{\omega_B} Q_{i,i+k}(\omega) d\omega = 0, \quad k = 1, \ldots, n-1. \end{aligned} \tag{7.27}$$

Moreover, the optimal values of (7.4) *and* (7.27) *coincide.*

Proof. Let S^n be the real linear space of Hermitian $n \times n$ continuous matrix functions on $[-\omega_B, \omega_B]$, with the inner product[3]

$$\langle X, Y \rangle := \int_{-\omega_B}^{\omega_B} \mathrm{tr}[X(\omega) Y(\omega)] d\omega, \quad X, Y \in S^n.$$

Also, let $S^n_+ \subseteq S^n$ be the closed convex cone of continuous matrix functions $X \in S^n$ such that $X(\omega) \geq 0$ for every $\omega \in [-\omega_B, \omega_B]$. According to Lemma 7.9.2, S^n_+ is self dual (Luenberger 1969). Thus, (7.4) can be written as a general convex program of the form

$$\begin{aligned} \min_{x \in \Omega} \quad & f(x) \\ \text{s.t.} \quad & G(x) \leq_{S^n_+} 0, \end{aligned}$$

where $\Omega := \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ is given by $f(x_1, \ldots, x_n) := x_1$, $G : \Omega \to S^n$ is defined as

$$[G(x)](\omega) := \sigma_o^2 \gamma \Lambda_n(e^{j\omega}) \Lambda_n^H(e^{j\omega}) - T_n,$$

and $T_n = T(\{x_i\}_{i=1}^n)$ is a linear function of x. The notation $A \leq_{S^n_+} B$ means that $B - A \in S^n_+$. Moreover, there is an $x' \in \Omega$ such that $G(x') <_{S^n_+} 0$ (e.g., take $x' = (\alpha, 0, \ldots, 0)$, with $\alpha > 0$ sufficiently large). Thus, Theorem 1 of (Luenberger 1969, section 8.6) can be applied to obtain the Lagrangian dual of (7.4), such that the optimal values of (7.4) and its dual coincide.

[3] See footnote 2.

The Lagrangian of (7.4) is

$$\begin{aligned}L(r_0,\ldots,r_{n-1},Q) &:= r_0 - \int_{-\omega_B}^{\omega_B} \mathrm{tr}\{Q(\omega)[T_n - \sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})]\}d\omega \\ &= r_0 - \int_{-\omega_B}^{\omega_B} \mathrm{tr}[Q(\omega)T_n]d\omega + \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B} \mathrm{tr}[Q(\omega)\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})]d\omega \\ &= \left(1 - \sum_{i=1}^{n}\mathbb{Q}_{i,i}\right) r_0 - \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} 2\,\mathrm{Re}(\mathbb{Q}_{i,i+k})r_k + \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B} \Lambda_n^H(e^{j\omega})Q(\omega)\Lambda_n(e^{j\omega})d\omega,\end{aligned}$$

for $Q(\omega) = Q^H(\omega) \geq 0$, $|\omega| \leq \omega_B$, where

$$\mathbb{Q} := \int_{-\omega_B}^{\omega_B} Q(\omega)d\omega.$$

Thus, the Lagrangian dual function of (7.4) is

$$\begin{aligned}g(Q) &:= \inf_{r_0,\ldots,r_{n-1}} L(r_0,\ldots,r_{n-1},Q) \\ &= \begin{cases} \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B}\Lambda_n^H(e^{j\omega})Q(\omega)\Lambda_n(e^{j\omega})d\omega, & \sum_{i=1}^{n}\mathbb{Q}_{i,i} = 1 \text{ and } \mathrm{Re}\sum_{i=1}^{n-k}\mathbb{Q}_{i,i+k} = 0, \\ & \quad k = 1,\ldots,n-1 \\ -\infty, & \text{otherwise.}\end{cases}\end{aligned}$$

This implies that the Lagrangian dual of (7.4) is (7.27). □

Lemma 7.9.4 *Let X be a real inner product space, not necessarily of finite dimension, and let K be a closed convex cone in X with nonempty interior. Then $K^{\oplus\oplus} = K$, where $Y^{\oplus} := \{x \in X : \langle x,y\rangle \geq 0 \text{ for all } y \in Y\}$ is the dual cone of a convex cone $Y \subseteq X$.*

Proof. This is a particular case of the proof of Lemma 3 in (Craven and Koliha 1977), which generalises part (iv) of (Ben-Tal and Nemirovski 2001, Theorem 2.3.1). By definition, we have that

$$K^{\oplus} = \{y \in X : \langle x,y\rangle \geq 0 \text{ for all } x \in K\}$$
$$K^{\oplus\oplus} = \{y \in X : \langle x,y\rangle \geq 0 \text{ for all } x \in K^{\oplus}\}.$$

Now, by the definition of $K^{\oplus\oplus}$, we have that $K \subseteq K^{\oplus\oplus}$. Let us assume that there is an element $a \in K^{\oplus\oplus}$ but $a \notin K$. Then, by the Separating Hyperplane Theorem (Luenberger 1969), there is an element $\bar{a} \in X$ such that

$$\langle \bar{a},a\rangle < \langle \bar{a},x\rangle, \quad x \in K. \tag{7.28}$$

In particular, we have that $\langle \bar{a},a\rangle < \langle \bar{a},0\rangle = 0$, since $0 \in K$. Also, $\langle \bar{a},x\rangle \geq 0$ for every $x \in K$, otherwise, if $\langle \bar{a},\bar{x}\rangle < 0$ for some $\bar{x} \in K$, there would be an $\alpha > 0$ such that $\langle \bar{a},\alpha\bar{x}\rangle < \langle \bar{a},a\rangle$, which is impossible as $\alpha\bar{x} \in K$, hence would contradict (7.28).

Therefore, we have found an $\bar{a} \in K^{\oplus}$ such that $\langle \bar{a},a\rangle < 0$, so $a \notin K^{\oplus\oplus}$. This contradiction proves the lemma. □

Lemma 7.9.5 *The Lagrangian dual of*

$$\begin{aligned}&\min_{r_0,\ldots,r_{n-1}} \quad r_0\\ &\quad s.t. \quad \Lambda_n^H(e^{j\beta})[T_n - \sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})]\Lambda_n(e^{j\beta}) \geq 0, \quad |\omega| \leq \omega_B, |\beta| \leq \pi\end{aligned}$$

is

$$\begin{aligned}&\max_{Q(\omega)=Q^H(\omega)\geq 0, |\omega|\leq\omega_B} \quad \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B}\Lambda_n^H(e^{j\omega})Q(\omega)\Lambda_n(e^{j\omega})d\omega\\ &\quad Q(\omega) \text{ is Toeplitz}\\ &\quad s.t. \quad \operatorname{tr}\left\{\int_{-\omega_B}^{\omega_B}Q(\omega)d\omega\right\} = 1\\ &\qquad \operatorname{Re}\sum_{i=1}^{n-k}\int_{-\omega_B}^{\omega_B}Q_{i,i+k}(\omega)d\omega = 0, \quad k = 1,\ldots,n-1.\end{aligned}$$

Proof. This essentially follows the same steps as the proof of Lemma 7.9.3. Consider T^n as the real linear space of Hermitian Toeplitz $n\times n$ continuous matrix functions on $[-\omega_B,\omega_B]$, with the inner product[4]

$$\langle X,Y\rangle := \int_{-\omega_B}^{\omega_B}\operatorname{tr}[X(\omega)Y(\omega)]d\omega, \quad X,Y\in T^n.$$

Also, let $T_+^n \subseteq T^n$ be the closed convex cone of continuous matrix functions $X\in T^n$ such that $\Lambda_n^H(e^{j\beta})X(\omega)\Lambda_n(e^{j\beta}) \geq 0$ for every $\omega\in[-\omega_B,\omega_B]$ and $\beta\in[-\pi,\pi]$. The dual of T_+^n is the convex cone T_{++}^n of Hermitian positive semidefinite Toeplitz $n\times n$ continuous matrix functions on $[-\omega_B,\omega_B]$. To see this, notice that

$$\begin{aligned}T_+^n &= \{Y\in T^n:\ \Lambda_n^H(e^{j\beta})Y(\omega)\Lambda_n(e^{j\beta}) \geq 0, \quad |\omega|\leq\omega_B, \quad |\beta|\leq\pi\}\\ &= \{Y\in T^n:\ \operatorname{tr}[\Lambda_n(e^{j\beta})\Lambda_n^H(e^{j\beta})Y(\omega)] \geq 0, \quad |\omega|\leq\omega_B, \quad |\beta|\leq\pi\}\\ &= \{Y\in T^n:\ \operatorname{tr}[X(\omega)Y(\omega)] \geq 0, |\omega|\leq\omega_B, \quad X(\omega) = \int_{-\pi}^{\pi}\Lambda_n(e^{j\beta})\Lambda_n^H(e^{j\beta})dq_\omega(\beta),\\ &\qquad\qquad \text{for some nondecreasing } q_\omega:[-\pi,\pi]\to\mathbb{R}\}\\ &= \{Y\in T^n:\ \langle X,Y\rangle \geq 0, \quad X\in T_{++}^{n+1}\},\end{aligned}$$

since $X(\omega)$ is an Hermitian Toeplitz matrix if and only if it can be written as (Grenander and Szegö 1958)

$$X(\omega) = \int_{-\pi}^{\pi}\Lambda_n(e^{j\beta})\Lambda_n^H(e^{j\beta})dq_\omega(\beta)$$

where q_ω is monotonically nondecreasing if and only if $X(\omega)\geq 0$. Thus, T_+^n is the dual cone of T_{++}^n, which is a closed convex cone, therefore by Lemma 7.9.4, T_{++}^n is the dual cone of T_+^n. The rest of the argument then follows as in the proof of Lemma 7.9.3. □

[4] See footnote 2.

Lemma 7.9.6 *The optimal value of* (7.4) *is equal to the optimal value of*

$$\begin{aligned} &\min_{r_0,\ldots,r_{n-1}} \quad r_0 \\ &\quad s.t. \quad \Lambda_n^H(e^{j\beta})[T_n - \sigma_o^2\gamma\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})]\Lambda_n(e^{j\beta}) \geq 0, \quad |\omega| \leq \omega_B, \quad |\beta| \leq \pi. \end{aligned}$$

Proof. Notice that

$$\begin{aligned} \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B}\Lambda_n^H(e^{j\omega})Q(\omega)\Lambda_n(e^{j\omega})d\omega &= \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B}\mathrm{tr}[\Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega})Q(\omega)]d\omega \\ &= \sigma_o^2\gamma\int_{-\omega_B}^{\omega_B}\left[\sum_{i=1}^{n}Q_{i,i}(\omega) + 2\,\mathrm{Re}\left\{\sum_{k=1}^{n-1}e^{j\omega k}\sum_{i=1}^{n-k}Q_{i,k+i}(\omega)\right\}\right]d\omega. \end{aligned}$$

This implies that in the Lagrangian dual of problem (7.4), according to Lemma 7.9.3, the actual decision variables are the sums of the elements in the diagonals of $Q(\omega)$, therefore this matrix can be taken as being Toeplitz. The result then follows from Lemmas 7.9.3 and 7.9.5. □

CHAPTER 8

THE COST OF COMPLEXITY IN OUTPUT ERROR SYSTEMS

8.1 Introduction

Recall that the cost of complexity is defined as the minimum amount of input power required to estimate the frequency function of a given linear time invariant system of order n with a prescribed degree of accuracy. In particular we require that the asymptotic (in the data length) variance is less or equal to γ over a prespecified frequency range $[-\omega_B, \omega_B]$.

In this chapter we extend the cost of complexity results of Chapter 7 to include more general model structures. Specifically, results are established which quantify the cost of complexity for Output Error (OE), fixed denominator and Laguerre model structures. Several properties which quantify the cost as a function of the model structure, the number of parameters n, the size of the frequency bandwidth of interest ω_B, the noise variance σ_o^2 and the required precision γ are derived. In particular, we show the monotonicity of the cost with respect to n and ω_B and determine its value when the input is restricted to be white noise and also when $\omega_B = 0$ and $\omega_B = \pi$. Furthermore, we establish an asymptotic lower bound on the cost for large values of n in the case of fixed denominator models, and study the relationship of the cost between Laguerre and FIR models.

8.2 Preliminaries

For an Output Error (OE) model structure, it is assumed that the true system is given by

$$y_t = G(z, \theta_{n_o}^o)u_t + w_t = \frac{B(z, \theta_{n_{o,b}}^o)}{A(z, \theta_{n_{o,a}}^o)}u_t + w_t,$$

where $\{u_t\}$ and $\{y_t\}$ denote the input and output signal respectively, and $\theta^o_{n_o} \in \mathbb{R}^n$. The denominator and numerator polynomials are given by

$$\begin{aligned} A(z,\theta^o_{n_{o,a}}) &:= 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n_{o,a}} z^{-n_{o,a}} \\ B(z,\theta^o_{n_{o,b}}) &:= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n_{o,b}} z^{-n_{o,b}} \\ \theta^o_{n_{o,a}} &:= [a_1\ a_2\ \cdots\ a_{n_{o,a}}]^T \\ \theta^o_{n_{o,b}} &:= [b_1\ b_2\ \cdots\ b_{n_{o,b}}]^T. \end{aligned}$$

In case of biproper systems, we can multiply $B(z,\theta^o_{n_{o,b}})$ by z. Note that $n_o = n_{o,a} + n_{o,b}$ and

$$\theta^o_{n_o} := \begin{bmatrix} \theta^o_{n_{o,a}} \\ \theta^o_{n_{o,b}} \end{bmatrix}.$$

Furthermore, $\{w_t\}$ is zero mean white noise with variance σ_o^2, and the input signal is considered to be wide-sense stationary. The model to be estimated for this system is given by

$$y_t = G(z,\theta_n)u_t + \varepsilon_t = \frac{B(z,\theta_{n_b})}{A(z,\theta_{n_a})}u_t + \varepsilon_t, \tag{8.1}$$

with the parameter vector

$$\theta_n := \begin{bmatrix} \theta_{n_a} \\ \theta_{n_b} \end{bmatrix}.$$

In this chapter it is assumed that $n_a \geq n_{o,a}$ and $n_b \geq n_{o,b}$, i.e., there is no undermodelling.

The results presented in Sections 8.4.2–8.4.3 consider a special case of the OE model structure, where the denominator of $G(z,\theta_n)$ is prespecified.

Under the assumption of no undermodelling, the asymptotic variance of the frequency function estimator is given by

$$\lim_{N\to\infty} N\ \mathrm{Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} = \sigma_o^2 \Gamma^H_{\mathrm{OE},n}(e^{j\omega},\theta^o_n) M^{-1}_{\mathrm{OE},n} \Gamma_{\mathrm{OE},n}(e^{j\omega},\theta^o_n),$$

where

$$\begin{aligned} \theta^o_n &:= [(\theta^o_{n_{o,a}})^T\ 0^T_{n_a-n_{o,a}}\ (\theta^o_{n_{o,b}})^T\ 0^T_{n_b-n_{o,b}}]^T \\ \Gamma_{\mathrm{OE},n}(z,\theta^o_n) &:= \left.\frac{\partial G(z,\theta)}{\partial\theta}\right|_{\theta=\theta^o_n} \\ M_{\mathrm{OE},n} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_{\mathrm{OE},n}(e^{j\omega},\theta^o_n)\Gamma^H_{\mathrm{OE},n}(e^{j\omega},\theta^o_n)\Phi_u(\omega)d\omega, \end{aligned}$$

and Φ_u denotes the power spectrum of $\{u_t\}$ (Ljung 1999). We note that the elements of $\Gamma_{\mathrm{OE},n}$ constitute, in general, a non-orthonormal basis. Also, for Φ_u to define a spectrum, it must satisfy the following condition,

$$\Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi. \tag{8.2}$$

The key tool utilised in this chapter to derive the cost of complexity is orthonormal basis functions. Such functions have been widely used for system identification, see e.g., (Van Den Hof et al. 1995, Wahlberg 1991). These functions have also been used to quantify the variance of different model properties (Heuberger et al. 2005, Mårtensson 2007, Ninness and Hjalmarsson 2004). For a given $\Gamma_{\mathrm{OE},n}$ it is straightforward to generate an orthonormal basis which spans the same space as the elements of $\Gamma_{\mathrm{OE},n}$ by using the Gram-Schmidt process. Hence we define

$$\Gamma_n(z,\theta_n^o) := [\mathscr{B}_0(z,\theta_n^o)\ \mathscr{B}_1(z,\theta_n^o)\ \cdots\ \mathscr{B}_{n-1}(z,\theta_n^o)]^T,$$

where the $\mathscr{B}_k(z,\theta_n^o)$ denote basis functions which are orthonormal on the unit circle $\mathbb{T}$, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{B}_k(e^{j\omega},\theta_n^o)\mathscr{B}_l^H(e^{j\omega},\theta_n^o)d\omega = \delta_{k,l},$$

and the orthonormal basis given by Γ_n spans the same model structure as the non-orthonormal basis $\Gamma_{\mathrm{OE},n}$.

The asymptotic variance of the frequency function estimator can now be equivalently written using Γ_n as

$$\lim_{N\to\infty} N \operatorname{Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} = \sigma_o^2\Gamma_n^H(e^{j\omega},\theta_n^o)M_n^{-1}\Gamma_n(e^{j\omega},\theta_n^o), \tag{8.3}$$

where

$$M_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_n(e^{j\omega},\theta_n^o)\Gamma_n^H(e^{j\omega},\theta_n^o)\Phi_u(\omega)d\omega.$$

To simplify the notation in the sequel we omit the θ_n^o dependence of Γ_n.

In this chapter we analyse the variance expression (8.3). In particular, the cost is quantified in terms of the model complexity n and the frequency range ω_B. In the derivation of these results we exploit the reproducing kernel of the space spanned by the orthonormal basis $\{\mathscr{B}_k\}_{k=0}^{n-1}$, defined by

$$K_n(\omega_1,\omega_2) := \Gamma_n^H(e^{j\omega_2})\Gamma_n(e^{j\omega_1}).$$

For a detailed treatment on the properties of the reproducing kernel we refer to (Aronszajn 1950, Ninness et al. 1998).

Remark 8.2.1 *For FIR models it holds that $\mathscr{B}_k(z) = z^{-k}$ and thus $K_n(\omega,\omega) = n$ (see Chapter 7).*

8.3 Problem Description

The input design problem we consider is the same as that in Chapter 7, namely,

$$\begin{aligned} &\min_{\Phi_u} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \\ &\text{s.t. } \Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi \\ &\qquad \lim_{N\to\infty} N \operatorname{Var}\{G(e^{j\omega}, \hat{\theta}_{N,n})\} \leq \frac{1}{\gamma}, \quad |\omega| \leq \omega_B. \end{aligned} \tag{8.4}$$

With an assumption that $M_n > 0$ in (8.3), by applying Schur complements (Boyd et al. 1994), the second constraint in (8.4) can be written as

$$M_n - \sigma_o^2 \gamma \Gamma_n(e^{j\omega})\Gamma_n^H(e^{j\omega}) \geq 0, \quad |\omega| \leq \omega_B,$$

or, equivalently as

$$M_{OE,n} - \sigma_o^2 \gamma \Gamma_{OE,n}(e^{j\omega})\Gamma_{OE,n}^H(e^{j\omega}) \geq 0, \quad |\omega| \leq \omega_B.$$

Now consider the following autocovariance representation for Φ_u:

$$\Phi_u(\omega) = L(e^{j\omega}) \sum_{k=-\infty}^{\infty} r_{|k|} e^{-j\omega k}, \tag{8.5}$$

where $L(e^{j\omega}) \geq 0$ for all $|\omega| \leq \pi$ (Jansson 2004, Jansson and Hjalmarsson 2005a). It is possible sometimes to choose $L(e^{j\omega})$ such that M_n depends only on a finite number of the parameters $\{r_k\}$, as shown in Example 8.3.1.

Example 8.3.1 *Consider the system given by*

$$G(z, \theta_n^o) = \frac{\beta z^{-1}}{1 + \alpha z^{-1}}, \quad \theta_n^o = [\alpha \ \ \beta]^T, \quad n = 2.$$

If we choose $L(e^{j\omega}) = |1 + \alpha e^{-j\omega}|^4$*, then* M_n *depends only on the parameters* r_0 *and* r_1 *of* Φ_u*, as described in* (8.5).

For fixed denominator model structures, L is equal to the square modulus of the denominator polynomial of the model, and M_n depends only on the first n parameters $\{r_k\}$ of Φ_u. This is due to $\Gamma_{\mathrm{OE},n}(z) = [A(z)]^{-1}\Lambda_n(z)$, where $\Lambda_n(z) := [z^{-1} \ \cdots \ z^{-n}]^T$ (or $\Lambda_n(z) = [1 \ z^{-1} \ \cdots \ z^{-(n-1)}]^T$, depending on whether we work with strictly proper or biproper model structures, respectively), and $A(z) = a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}$ is a polynomial in z^{-1}, hence

$$\begin{aligned} M_{OE,n} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega}) \frac{\Phi_u(\omega)}{|A(e^{j\omega})|^2} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Lambda_n(e^{j\omega})\Lambda_n^H(e^{j\omega}) \sum_{k=-\infty}^{\infty} r_{|k|} e^{-j\omega k} d\omega \\ &= T_n, \end{aligned}$$

where $T_n := T(\{r_k\}_{k=0}^{n-1})$ is a symmetric Toeplitz matrix of the vector $[r_0\ r_1\ \cdots\ r_{n-1}]$. Notice that the equality between $M_{OE,n}$ and T_n holds irrespective of which definition of Λ_n we use (i.e., whether we consider a biproper or strictly proper Λ_n).

Now, the input power can be computed as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega = A^T T_{n+1} A,$$

where $A = [a_0\ \cdots\ a_n]^T$ is the vector of coefficients of $A(z)$ (Söderström 2002). Thus, for fixed denominator structures we have to design the finite sequence $\{r_0, r_1, \ldots, r_n\}$ and ensure that there exists an extension $r_{n+1}, r_{n+2}, \ldots$ such that the nonnegativity constraint (8.2) holds. A necessary and sufficient condition for the existence of such an extension is that $T_{n+1} \geq 0$ (Byrnes et al. 2001, Grenander and Szegö 1958, Lindquist and Picci 1996). Therefore, in this particular case (8.4) can be reformulated as:

$$\begin{aligned} \min_{r_0,\ldots,r_n}\ & A^T T_{n+1} A \\ \text{s.t.}\ \ & M_n - \sigma_o^2 \gamma \Gamma_n(e^{j\omega})\Gamma_n^H(e^{j\omega}) \geq 0,\ |\omega| \leq \omega_B \\ & T_{n+1} \geq 0. \end{aligned} \tag{8.6}$$

Notice that the first constraint of (8.6) does not imply the second one. This is in contrast to the input design problem considered in Chapter 7, where the input spectrum only depends on $\{r_0, r_1, \ldots, r_{n-1}\}$ (i.e., not on r_n). Thus, the covariance extension condition becomes $T_n \geq 0$, which is actually implied by the first constraint of (8.6), hence it is redundant and can be removed from the optimisation problem (as done in Chapter 7).

We denote the optimal solution of (8.6) by $\{r_0^{\text{opt}}, r_1^{\text{opt}}, \cdots, r_n^{\text{opt}}\}$ and $f^{\text{opt}} := A^T T(\{r_k^{\text{opt}}\}_{k=0}^{n}) A$. The focus is thus to study how f^{opt} depends on the number of parameters n, the frequency bandwidth ω_B and the precision γ, by analyzing the frequency-wise LMI

$$M_n - \sigma_o^2 \gamma \Gamma_n(e^{j\omega})\Gamma_n^H(e^{j\omega}) \geq 0, \quad |\omega| \leq \omega_B. \tag{8.7}$$

Notice that the first constraint in (8.6) is infinite dimensional due to the dependence on the continuous variable ω. However, the constraint can be reformulated as a finite dimensional LMI by utilising the Generalised Kalman-Yakubovich-Popov Lemma (Iwasaki and Hara 2005), allowing the optimal input spectrum to be numerically computed by solving a semidefinite program, in a similar fashion to that described in Section 7.4 of Chapter 7.

8.4 Main Results

In this section the main contributions of the chapter are derived. In Section 8.4.1 we consider OE models of the structure (8.1). In Section 8.4.2 we consider a special case of (8.1) where the poles of $G(z,\theta_n)$ are fixed, the so called 'fixed denominator structures'. In Section 8.4.3 the results of Chapter 7 are compared to Laguerre model structures.

8.4.1 General properties of the cost

In this section we provide some general properties of the solution to the input design problem (8.4) and f^{opt}. The following theorem considers the persistence of excitation of $\{u_t\}$ and the positivity of the matrices M_n and T_n.

Lemma 8.4.1 (Persistence of excitation) *Let $\mathscr{M} := \{G(z,\theta) : \theta \in \Theta\}$ be a model structure such that $G(z,\theta)$ is an analytic function of $z \in \mathbb{T}$ for every $\theta \in \Theta$. Also let $\omega_B > 0$. Then, the optimal solution of* (8.4) *is persistently exciting of order n (Ljung 1999), and such that $M_n > 0$. Furthermore, if we consider the representation* (8.5) *of Φ_u, where $L(e^{j\omega}) > 0$ for every $\omega \in [-\pi,\pi]$, then $T_n > 0$.*

Proof. Let Φ_u^{opt} be the optimal solution of (8.4). Then, by the model quality constraint of (8.4),

$$\lim_{N\to\infty} \text{Var}\,\{G(e^{j\omega},\hat{\theta}_{N,n})\} = 0, \quad |\omega| \le \omega_B.$$

Therefore, $\lim_{N\to\infty} G(e^{j\omega},\hat{\theta}_{N,n}) = G_o(e^{j\omega})$ in mean (and thus almost surely, as well) for every $\omega \in [-\omega_B,\omega_B]$. Since $\omega_B > 0$ and $G(z,\theta)$ is an analytic function of $z \in \mathbb{T}$ for every $\theta \in \Theta$, it can be seen from analytic continuation along $\mathbb{T}$ that $\lim_{N\to\infty} G(z,\hat{\theta}_{N,n}) = G_o(z)$ almost surely for every $z \in \mathbb{T}$. This means that $\mathscr{M}$ is parameter identifiable under Φ_u^{opt} for PEM (Söderström and Stoica 1989), hence $M_n > 0$ and Φ_u^{opt} is persistently exciting of order n (Söderström and Stoica 1989). If $L(e^{j\omega}) > 0$ for every $\omega \in [-\pi,\pi]$, this also implies that Φ_u^{opt}/L is persistently exciting of order n, so T_n is nonsingular (Ljung 1999, Lemma 13.1). □

The next two results imply that the more information we require for the model, the larger the cost. In particular, Theorem 8.4.1 shows that the cost is a non-decreasing function of the model order n, and in Theorem 8.4.2 we show that the cost is a non-decreasing function with respect to ω_B.

Theorem 8.4.1 (Monotonicity of f^{opt} with respect to n) *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two model structures*

such that $\mathscr{M}_1 \subset \mathscr{M}_2$, in the sense that we can define them as

$$\mathscr{M}_1 := \{G(z,\theta):\ \theta \in \Theta,\ \theta_{n_1+1} = \cdots = \theta_{n_2} = 0\}$$
$$\mathscr{M}_2 := \{G(z,\theta):\ \theta \in \Theta\},$$

for some integers $0 < n_1 < n_2$, where $\Theta \subseteq \mathbb{R}^{n_2}$ and G is a given differentiable function of θ. Assume there exists a $\theta^* \in \mathbb{R}^{n_2}$ such that $\theta^*_{n_1+1} = \cdots = \theta^*_{n_2} = 0$ and $G_o(z) = G(z,\theta^*)$, where G_o is the true system that generates the data, i.e., $G_o \in \mathscr{M}_1 \subset \mathscr{M}_2$. Also, define $f^{opt,i}$ $(i = 0,1)$ as the solution of (8.4) *associated with $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively. Then $f^{opt,1} \leq f^{opt,2}$.*

Proof. Define $\hat{G}_i$ as the maximum likelihood estimator of G (Goodwin and Payne 1977), constrained to the model structure $\mathscr{M}_i$. Notice that, by *Gauss' approximation formula* (Ljung 1999),

$$\lim_{N\to\infty} N\text{Var}\ \hat{G}_i(e^{j\omega}) = \sigma_o^2 \Gamma^H_{OE,n}(e^{j\omega}) P_i \Gamma_{OE,n}(e^{j\omega}); \qquad i = 1,2, \tag{8.8}$$

where P_i is the covariance matrix of $\hat{\theta}_i$, the maximum likelihood estimator of θ, subject to the constraints in $\mathscr{M}_i$. Now, let $\Gamma_{OE,n}(e^{j\omega}) =: [\Gamma_1^T(e^{j\omega})\ \Gamma_2^T(e^{j\omega})]^T$, where $\Gamma_1(e^{j\omega}) \in \mathbb{C}^{n_1\times 1}$ and $\Gamma_2(e^{j\omega}) \in \mathbb{C}^{(n_2-n_1)\times 1}$, so we have that by (8.8),

$$\begin{aligned}
\lim_{N\to\infty} N\text{Var}\ \hat{G}_1(e^{j\omega}) \leq \frac{1}{\gamma} \quad &\Leftrightarrow \quad \Gamma^H_{OE,n}(e^{j\omega}) P_1 \Gamma_{OE,n}(e^{j\omega}) \leq \frac{1}{\sigma_o^2\gamma} \\
&\Leftrightarrow \quad \Gamma_1^H(e^{j\omega}) \tilde{P}_1 \Gamma_1(e^{j\omega}) \leq \frac{1}{\sigma_o^2\gamma} \\
&\Leftrightarrow \quad \frac{1}{\sigma_o^2\gamma} - \Gamma_1^H(e^{j\omega}) \tilde{P}_1 \Gamma_1(e^{j\omega}) \geq 0 \\
&\Leftrightarrow \quad \tilde{P}_1^{-1} - \sigma_o^2\gamma \Gamma_1(e^{j\omega}) \Gamma_1^H(e^{j\omega}) \geq 0,
\end{aligned}$$

where $\tilde{P}_1 \in \mathbb{R}^{n_1\times n_1}$ is the covariance matrix of the first n_1 components of $\hat{\theta}_1$. Similarly,

$$\lim_{N\to\infty} N\text{Var}\ \hat{G}_2(e^{j\omega}) \leq \frac{1}{\gamma} \quad \Leftrightarrow \quad P_2^{-1} - \sigma_o^2\gamma \Gamma_{OE,n}(e^{j\omega}) \Gamma^H_{OE,n}(e^{j\omega}) \geq 0.$$

However,

$$P_2^{-1} = \left(\text{E}\left\{\frac{\partial l(\theta)}{\partial \theta_i}\frac{\partial l(\theta)}{\partial \theta_j}\right\}\right)_{\substack{i,j=1,\ldots,n_2 \\ \theta=\theta^*}} = \left[\begin{array}{c|c} \left(\text{E}\left\{\frac{\partial l(\theta)}{\partial \theta_i}\frac{\partial l(\theta)}{\partial \theta_j}\right\}\right)_{\substack{i,j=1,\ldots,n_1 \\ \theta=\theta^*}} & * \\ \hline * & * \end{array}\right] = \left[\begin{array}{c|c} \tilde{P}_1^{-1} & * \\ \hline * & * \end{array}\right],$$

where $*$ denotes an unimportant entry and $l(\theta)$ is the log likelihood function of θ (Goodwin and Payne 1977). Thus, for every input spectrum $\Phi_u(\omega)$ and $|\omega| \leq \omega_B$,

$$\begin{aligned}
\lim_{N\to\infty} N\mathrm{Var}\hat{G}_2(e^{j\omega}) \leq \frac{1}{\gamma} \quad &\Rightarrow \quad P_2^{-1} - \sigma_o^2\gamma\Gamma_{OE,n}(e^{j\omega})\Gamma_{OE,n}^H(e^{j\omega}) \geq 0 \\
&\Rightarrow \quad \left[\begin{array}{c|c} \tilde{P}_1^{-1} - \sigma_o^2\gamma\Gamma_1(e^{j\omega})\Gamma_1^H(e^{j\omega}) & * \\ \hline * & * \end{array}\right] \geq 0 \\
&\Rightarrow \quad \tilde{P}_1^{-1} - \sigma_o^2\gamma\Gamma_1(e^{j\omega})\Gamma_1^H(e^{j\omega}) \geq 0 \\
&\Rightarrow \quad \lim_{N\to\infty} N\mathrm{Var}\hat{G}_1(e^{j\omega}) \leq \frac{1}{\gamma}.
\end{aligned}$$

This implies that $f^{\mathrm{opt},1} \leq f^{\mathrm{opt},2}$. □

Remark 8.4.1 *It is interesting to note that Theorem 8.4.1, via the duality result of Theorem 6.2.1, can be seen as a version of the Parsimony Principle (Stoica and Söderström 1982), which states that a model with less parameters gives better precision than one with more parameters.*

Theorem 8.4.2 (Monotonicity of $f^{\mathbf{opt}}$ with respect to ω_B) *Let $f^{opt,1}$ and $f^{opt,2}$ be the optimal costs of the input design problem* (8.4) *for $\omega_B = \omega_{B1}$ and $\omega_B = \omega_{B2}$, respectively, and a fixed model structure. If $0 \leq \omega_{B1} < \omega_{B2} \leq \pi$, then $f^{opt,1} \leq f^{opt,2}$.*

Proof. Follows from the fact that the set of allowable input spectra $\Phi_u(\omega)$ decreases with increasing ω_B. □

The next theorem pertains to the cost of complexity for the special case where the input spectrum is restricted to be white. This property is an extension of Theorem 7.3.4, in Chapter 7, to more general model structures, and is in fact an upper bound for the cost since the structure of the input spectrum is restricted.

Theorem 8.4.3 (White input spectrum) *For the case of white input spectra, we have $f^{opt} = f^{opt,\text{white noise}} := \sigma_o^2\gamma \sup_{|\omega|\leq\omega_B} K_n(\omega,\omega)$.*

Proof. Let $\Phi_u(\omega) = \alpha$, where $\alpha \geq 0$. Then (8.7) can be rewritten as

$$\frac{\alpha}{2\pi}\int_{-\pi}^{\pi} \Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})d\tau - \sigma_o^2\gamma\Gamma_n(e^{j\omega})\Gamma_n^H(e^{j\omega}) \geq 0, \quad |\omega| \leq \omega_B.$$

By pre- and post multiplying by $\Gamma_n^H(e^{j\beta})$ and $\Gamma_n(e^{j\beta})$ respectively it must hold that

$$\frac{\alpha}{2\pi}\int_{-\pi}^{\pi} |\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\tau})|^2 d\tau \geq \sigma_o^2\gamma|\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\omega})|^2, \quad |\omega| \leq \omega_B, \quad |\beta| \leq \pi.$$

Solving for α gives

$$\alpha \geq \sigma_o^2\gamma \frac{|\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\omega})|^2}{\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\tau})|^2 d\tau} = \sigma_o^2\gamma \frac{|K_n(\omega,\beta)|^2}{\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(\tau,\beta)|^2 d\tau} = \sigma_o^2\gamma \frac{|K_n(\omega,\beta)|^2}{K_n(\beta,\beta)},$$
$$|\omega| \leq \omega_B, \quad |\beta| \leq \pi.$$

where we have used the fact that (Ninness et al. 1998)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(\tau,\beta)|^2 d\beta = K_n(\tau,\tau), \quad \tau \in [-\pi,\pi].$$

It can then be seen that the smallest α is given by

$$\sigma_o^2\gamma \sup_{|\omega|\leq\omega_B}\sup_{|\beta|\leq\pi}\frac{|K_n(\omega,\beta)|^2}{K_n(\beta,\beta)} = \sigma_o^2\gamma \sup_{|\omega|\leq\omega_B} K_n(\omega,\omega),$$

where we have used the Cauchy-Schwarz inequality

$$|K_n(\omega,\beta)|^2 \leq K_n(\omega,\omega)K_n(\beta,\beta), \quad \omega,\beta \in [-\pi,\pi].$$

Above we have shown that $f^{\text{opt}} \geq f^{\text{opt, white noise}}$. Now, we calculate the variance of the frequency function estimator when the input is the white noise spectrum $\Phi_u^{\text{white}} := \sigma_o^2\gamma \sup_{|\omega|\leq\omega_B} K_n(\omega,\omega)$. We have

$$\begin{aligned}\lim_{N\to\infty} N \operatorname{Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} &= \sigma_o^2\Gamma_n^H(e^{j\omega})\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})\Phi_u^{\text{white}}d\tau\right]^{-1}\Gamma_n(e^{j\omega}) \\ &= \frac{1}{\gamma\sup_{|\omega|\leq\omega_B}K_n(\omega,\omega)}K_n(\omega,\omega) \leq \frac{1}{\gamma}, \quad |\omega| \leq \omega_B,\end{aligned}$$

as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})d\tau = I.$$

Thus the quality constraint is respected, i.e., $f^{\text{opt}} \leq f^{\text{opt, white noise}}$. Therefore it holds that $f^{\text{opt}} = f^{\text{opt, white noise}}$. □

The next result reveals that if $\omega_B = 0$, the cost does not depend on n.

Theorem 8.4.4 (f^{opt} **for the case** $\omega_B = 0$) *When $\omega_B = 0$, the optimal cost is given by $f^{opt} = \sigma_o^2\gamma$.*

Proof. From Lemma 8.7.2 (see Appendix 8.7), solving problem (8.22) is equivalent to solving problem (8.24). Note that (8.24) is a mass distribution problem. Hence, the optimal $\Phi_u(\tau)$ of problem (8.24) is found by concentrating all power of $\Phi_u(\tau)$ where $k(\tau)$, defined by (8.23), has its supremum. By the Cauchy-Schwarz inequality we have that $0 \leq k(\tau) \leq 1/(\sigma_o^2\gamma)$ and $\sup_{\tau\in\mathbb{R}} k(\tau) = k(0) =$

$1/(\sigma_o^2\gamma)$. Therefore, the spectrum $\Phi_u(\tau) = (2\pi\sigma_o^2\gamma)\delta(\tau)$ is an optimal solution to problem (8.22), and its power is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\sigma_o^2\gamma\delta(\tau)d\tau = \sigma_o^2\gamma.$$

Lemma 8.7.1 (see Appendix 8.7) thus implies that $f^{\text{opt}} \geq \sigma_o^2\gamma$. The next step in this proof is to show that $\Phi_u(\tau) = (2\pi\sigma_o^2\gamma)\delta(\tau)$ is a feasible input spectrum. The variance of the frequency function estimator when the input spectrum is given by $(2\pi\sigma_o^2\gamma)\delta(\tau)$ is

$$\begin{aligned}\lim_{N\to\infty} N\,\text{Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} &= \frac{1}{\gamma}\Gamma_n^H(1)\left[\int_{-\pi}^{\pi}\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})\delta(\tau)d\tau\right]^{\dagger}\Gamma_n(1)\\ &= \frac{1}{\gamma}\text{tr}\left\{\Gamma_n(1)\Gamma_n^H(1)\left[\Gamma_n(1)\Gamma_n^H(1)\right]^{\dagger}\right\}. \qquad (8.9)\end{aligned}$$

Note that since the matrix $A := \Gamma_n(1)\Gamma_n^H(1)$ has rank 1, we can introduce the decomposition

$$A = U\begin{bmatrix}\sigma_A & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \\ 0 & & & 0\end{bmatrix}U^H$$

and

$$A^{\dagger} = U\begin{bmatrix}\frac{1}{\sigma_A} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \\ 0 & & & 0\end{bmatrix}U^H,$$

where U is a unitary matrix and σ_A is the singular value of A. This implies

$$\text{tr}\left\{AA^{\dagger}\right\} = \text{tr}\left\{U\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \\ 0 & & & 0\end{bmatrix}U^H\right\} = 1.$$

Using these observations in (8.9) we obtain

$$\lim_{N\to\infty} N\,\text{Var}\{G(e^{j\omega},\hat{\theta}_{N,n})\} = \frac{1}{\gamma},$$

which means that the spectrum $(2\pi\sigma_o^2\gamma)\delta(\tau)$ is indeed a feasible input spectrum. Thus $f^{\text{opt}} \leq \sigma_o^2\gamma$. At the beginning of this proof we showed that $f^{\text{opt}} \geq \sigma_o^2\gamma$, so we conclude that $f^{\text{opt}} = \sigma_o^2\gamma$. □

This result implies that if we 'hide' all properties of the system except for the static properties, the cost is low irrespective of the complexity of the system.

Remark 8.4.2 *The case of $\omega_B = 0$ will not give a positive definite M_n. This has been treated in Chapter 7.*

The next theorem considers the case $\omega_B = \pi$, where a lower bound for the cost is derived.

Theorem 8.4.5 (Lower bound for f^{opt} as $\omega_B = \pi$) *When $\omega_B = \pi$, we have $f^{opt} \geq n\sigma_o^2\gamma$.*

Proof. The constraint is

$$\sigma_o^2\gamma\Gamma_n^H(e^{j\omega})\Psi_n^{-1}\Gamma_n(e^{j\omega}) \leq 1,$$

where

$$\Psi_n := \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\tau)\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})d\tau.$$

We first multiply this constraint by $\frac{\Phi_u(\omega)}{2\pi}$ and then integrate with respect to ω from $-\pi$ to π. This gives

$$\sigma_o^2\gamma\text{tr}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)\Gamma_n^H(e^{j\omega})\Psi_n^{-1}\Gamma_n(e^{j\omega})d\omega\right\} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega$$

which is equivalent to

$$\sigma_o^2\gamma\text{tr}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)\Gamma_n(e^{j\omega})\Gamma_n^H(e^{j\omega})d\omega\Psi_n^{-1}\right\} = \sigma_o^2\gamma\text{tr}\{\Psi_n\Psi_n^{-1}\} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega = f^{\text{opt}}.$$

Thus we have $n\sigma_o^2\gamma \leq f^{\text{opt}}$. □

8.4.2 Fixed denominator structure: lower bound for the cost

In this section we consider fixed denominator structures. This terminology refers to structures (8.1) in which the poles of $G(z,\theta_n)$ are prespecified, i.e., $A(z,\theta_{n_{o,a}}^o) = A(z)$. The orthonormal basis corresponding to such model structures is given by

$$\begin{aligned}\mathscr{B}_0(z) &= \frac{\sqrt{1-|\xi_1|^2}}{z-\xi_1}\\ \mathscr{B}_k(z) &= \frac{\sqrt{1-|\xi_{k+1}|^2}}{z-\xi_{k+1}}\prod_{l=1}^{k}\frac{1-\bar{\xi}_l z}{z-\xi_l}, \quad k \geq 1,\end{aligned} \tag{8.10}$$

where $\xi_k \in \mathbb{D}$, $k \in \mathbb{N}$, denote the zeros of $A(z)$ (Ninness and Hjalmarsson 2004, Ninness et al. 1998).

Before exploiting the basis (8.10), we recall Observation 7.3.1 in Chapter 7, which suggests that f^{opt} is asymptotically proportional to the model complexity n, to the accuracy γ and the bandwidth ω_B. In Theorem 8.4.6 we derive a more refined version (asymptotic in n) of the lower bound for f^{opt} provided in Observation 7.3.1.

Theorem 8.4.6 (Lower bound for the asymptotic cost of fixed denominator structures) *Consider the orthonormal basis given by* (8.10)*. Assume that* $\omega_B \in (0,\pi]$*. Then, there exists an* $n_{b,as} \in \mathbb{N}$*, depending on* σ_o^2*,* γ *and* ω_B*, such that, for all* $n_b \geq n_{b,as}$*,*

$$f^{opt} \geq \sigma_o^2 \gamma \frac{\omega_B}{\pi} \frac{\left(\sum_{i=1}^{n} \frac{1-|\xi_i|}{1+|\xi_i|}\right)^2}{\left(\sum_{i=1}^{n} \frac{1+|\xi_i|}{1-|\xi_i|}\right)}.$$

Furthermore, if there exists a $\delta > 0$ *such that* $|\xi_i| \leq \delta$ *for every* $i = 1,\ldots,n$*, then*

$$f^{opt} \geq n\sigma_o^2 \gamma \frac{\omega_B}{\pi} \left(\frac{1-\delta}{1+\delta}\right)^3.$$

Proof. By pre- and post-multiplying (8.7) by $\Gamma_n^H(e^{j\beta})$ and $\Gamma_n(e^{j\beta})$, respectively, where $\beta \in [-\pi,\pi]$, it must hold that

$$\Gamma_n^H(e^{j\beta}) M_n \Gamma_n(e^{j\beta}) \geq \sigma_o^2 \gamma |\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\omega})|^2, \quad |\omega| \leq \omega_B, \quad |\beta| \leq \pi. \tag{8.11}$$

Now,

$$|\Gamma_n^H(e^{j\beta})\Gamma_n(e^{j\omega})|^2 = |K_n(\omega,\beta)|^2$$

and

$$\Gamma_n^H(e^{j\beta}) M_n \Gamma_n(e^{j\beta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau)|K_n(\tau,\beta)|^2 d\tau.$$

This implies that (8.11) is equivalent to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau)|K_n(\tau,\beta)|^2 d\tau \geq \sigma_o^2 \gamma |K_n(\omega,\beta)|^2, \quad |\omega| \leq \omega_B, \quad |\beta| \leq \pi,$$

or, equivalently, to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau)|K_n(\tau,\beta)|^2 d\tau \geq \sigma_o^2 \gamma \sup_{|\omega|\leq\omega_B} |K_n(\omega,\beta)|^2, \quad |\beta| \leq \pi.$$

Now, if we integrate this expression with respect to β in $[-\pi,\pi]$, and use Tonelli's Theorem (Bartle 1966), we obtain

$$\int_{-\pi}^{\pi} \Phi_u(\tau)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(\tau,\beta)|^2 d\beta\right] d\tau \geq \sigma_o^2 \gamma \int_{-\pi}^{\pi} \sup_{|\omega|\leq\omega_B} |K_n(\omega,\beta)|^2 d\beta. \tag{8.12}$$

By the defining property of the reproducing kernel (Ninness et al. 1998) we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(\tau,\beta)|^2 d\beta = K_n(\tau,\tau)$$

and

$$\sum_{i=1}^{n} \frac{1-|\xi_i|}{1+|\xi_i|} \leq K_n(\omega,\omega) \leq \sum_{i=1}^{n} \frac{1+|\xi_i|}{1-|\xi_i|},$$

(see the *Christoffel-Darboux Formula* (Ninness et al. 1998, Theorem 3.1)). This implies that

$$\sup_{|\omega|\leq\omega_B} |K_n(\omega,\beta)|^2 \geq |K_n(\beta,\beta)|^2 \geq \left[\sum_{i=1}^{n} \frac{1-|\xi_i|}{1+|\xi_i|}\right]^2, \quad |\beta| \leq \omega_B$$

and

$$\int_{-\pi}^{\pi} \Phi_u(\tau) \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(\tau,\beta)|^2 d\beta\right] d\tau \leq \left[\sum_{i=1}^{n} \frac{1+|\xi_i|}{1-|\xi_i|}\right] \int_{-\pi}^{\pi} \Phi_u(\tau) d\tau.$$

Thus, by subtituting these expressions into (8.12) we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau) d\tau \geq \sigma_o^2 \gamma \frac{\omega_B}{\pi} \frac{\left(\sum_{i=1}^{n} \frac{1-|\xi_i|}{1+|\xi_i|}\right)^2}{\left(\sum_{i=1}^{n} \frac{1+|\xi_i|}{1-|\xi_i|}\right)}.$$

In the case of $|\xi_i| \leq \delta$ for every $i = 1,\ldots,n$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau) d\tau \geq n\sigma_o^2 \gamma \frac{\omega_B}{\pi} \left(\frac{1-\delta}{1+\delta}\right)^3.$$

This concludes the proof. □

8.4.3 Comparison between the Laguerre and FIR case

In this section we quantify the relationship between the cost of complexity for the FIR (see Chapter 7) and Laguerre model structures. The Laguerre basis (Wahlberg 1991) is a special case of (8.10) where $\xi_k = \xi \in \mathbb{R}$. However, in order to make the comparison in the cost of complexity for the FIR and Laguerre basis, we need to modify the standard Laguerre basis by adding a constant filter. Thus we consider the Laguerre basis to be defined by

$$\begin{aligned} \mathscr{B}_0(z) &= 1 \\ \mathscr{B}_k(z) &= \frac{\sqrt{1-\xi^2}}{z-\xi}\left[\frac{1-\xi z}{z-\xi}\right]^{k-1}, \quad k \geq 1, \end{aligned} \tag{8.13}$$

where $\xi \in [-1,1]$ is the pole of the Laguerre basis. Note that the FIR basis is a special case of the Laguerre basis, corresponding to $\xi = 0$, and that for the FIR case, f^{opt} corresponds to r_0^{opt}.

We make one further definition before developing the main result:

$$\tilde{\omega}_B := \frac{1}{j}\ln\left(\frac{e^{j\omega_B}-\xi}{1-\xi e^{j\omega_B}}\right). \tag{8.14}$$

The following theorem states the relationship between the Laguerre case (as defined in (8.13)) and the FIR case.

Theorem 8.4.7 (Relatioship between the costs for FIR and Laguerre basis) *The cost for the Laguerre and FIR basis functions are related by*[1]

$$(f^{opt})^{\omega_B}_{Lag} = (f^{opt})^{\tilde{\omega}_B}_{FIR}$$
$$\Phi_u^{opt,\,Lag}(\omega) = \frac{1-\xi^2}{|e^{j\omega}-\xi|^2}\Phi_u^{opt,\,FIR}\left(\frac{1}{j}\ln\left[\frac{e^{j\omega}-\xi}{1-\xi e^{j\omega}}\right]\right)$$

where 'Lag' denotes the expressions related to the Laguerre problem and 'FIR' is related to the associated FIR problem.

Proof. Notice that $\{\mathscr{B}_k\}_{k=0}^{n-1}$ spans the model structure composed of all $G \in \mathscr{H}_2$ of the form[2]

$$G(z,\theta) = \frac{\theta_0 + \theta_1 z^{-1} + \cdots + \theta_{n-1} z^{-(n-1)}}{(1-\xi z^{-1})^{n-1}} \tag{8.15}$$

(see e.g. (Heuberger et al. 2005, Section 4.4.2)). This implies that $\{\mathscr{B}_k\}_{k=0}^{n-1}$ spans the same model structure as the (non-orthogonal) basis $\{\tilde{\mathscr{B}}_k\}_{k=0}^{n-1}$ given by

$$\tilde{\mathscr{B}}_k(z) = \left(\frac{1-\xi z}{z-\xi}\right)^k; \quad k = 0,\ldots,n-1,$$

since $\{\tilde{\mathscr{B}}_k\}_{k=0}^{n-1}$ consists of n linearly independent biproper rational functions with at most $n-1$ poles at $z=\xi$, and it also spans all $G \in \mathscr{H}_2$ of the form (8.15). On the other hand, the input design problem (8.4) can be written as:

$$\begin{aligned}
&\min_{\Phi_u} \frac{1}{2\pi j}\oint_{\mathbb{T}} \Phi_u(z)\frac{dz}{z} \\
&\text{s.t. } \Phi_u(z) \geq 0, \quad z \in \mathbb{T} \\
&\qquad \sigma_o^2 \Gamma_n^H(e^{j\omega})\left[\frac{1}{2\pi j}\oint_{\mathbb{T}} \Gamma_n(z)\Gamma_n^H(z)\Phi_u(z)\frac{dz}{z}\right]^{-1}\Gamma_n(e^{j\omega}) \leq \frac{1}{\gamma}, \quad |\omega| \leq \omega_B.
\end{aligned} \tag{8.16}$$

Notice that, with some abuse of notation, we have changed the argument in Φ_u and Γ_n from ω to z, in order to rewrite the integrals of (8.4) as contour integrals.

Since $\text{span}\{\mathscr{B}_k\}_{k=0}^{n-1} = \text{span}\{\tilde{\mathscr{B}}_k\}_{k=0}^{n-1}$, there is a non-singular (constant) matrix $T \in \mathbb{R}^{n\times n}$ such that $\Gamma_n(z) = T\tilde{\Gamma}_n(z)$, where $\tilde{\Gamma}_n(z) := [\tilde{\mathscr{B}}_0(z) \;\cdots\; \tilde{\mathscr{B}}_{n-1}(z)]^T$. Thus,

$$\begin{aligned}
&\sigma_o^2 \Gamma_n^H(e^{j\omega})\left[\frac{1}{2\pi j}\oint_{\mathbb{T}} \Gamma_n(z)\Gamma_n^H(z)\Phi_u(z)\frac{dz}{z}\right]^{-1}\Gamma_n(e^{j\omega}) \\
&\qquad = \sigma_o^2 \tilde{\Gamma}_n^H(e^{j\omega})\left[\frac{1}{2\pi j}\oint_{\mathbb{T}} \tilde{\Gamma}_n(z)\tilde{\Gamma}_n^H(z)\Phi_u(z)\frac{dz}{z}\right]^{-1}\tilde{\Gamma}_n(e^{j\omega}).
\end{aligned}$$

[1] ln is considered to be the principal branch of the logarithm, i.e., $\text{Im}\{\ln(z)\} \in (-\pi,\pi]$ for all $z \in \mathbb{C}$.

[2] $\mathscr{H}_2$ denotes the Hardy space of all functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic in $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ and such that $\lim_{r\to 1_+}\int_{-\pi}^{\pi} \|f(re^{j\omega})\|_2^2 d\omega < \infty$ (Duren 1970, Koosis 1998).

This means that (8.16) can be written in terms of $\tilde{\Gamma}_n$ instead of Γ_n.

Consider now the bilinear transformation

$$\lambda^{-1} = \frac{1-\xi z}{z-\xi}; \qquad z = \frac{\lambda+\xi}{1+\xi\lambda}. \tag{8.17}$$

For this transformation we have that

$$\frac{dz}{z} = \frac{1-\xi^2}{(\lambda+\xi)(\lambda^{-1}+\xi)}\frac{d\lambda}{\lambda}, \tag{8.18}$$

where

$$\frac{1-\xi^2}{(\lambda+\xi)(\lambda^{-1}+\xi)} = \frac{1-\xi^2}{|\lambda+\xi|^2} > 0, \quad \lambda \in \mathbb{T}.$$

Also, $z \in \mathbb{D} \Leftrightarrow \lambda \in \mathbb{D}$, and $z \in \mathbb{T} \Leftrightarrow \lambda \in \mathbb{T}$. Therefore, applying (8.17) and (8.18) to (8.16) gives

$$\begin{aligned}
&\min_{\Phi_u} \frac{1}{2\pi j}\oint_{\mathbb{T}} \Phi_u\left(\frac{\lambda+\xi}{1+\xi\lambda}\right)\frac{1-\xi^2}{(\lambda+\xi)(\lambda^{-1}+\xi)}\frac{d\lambda}{\lambda} \\
&\text{s.t. } \Phi_u\left(\frac{\lambda+\xi}{1+\xi\lambda}\right)\frac{1-\xi^2}{(\lambda+\xi)(\lambda^{-1}+\xi)} \geq 0, \quad \lambda \in \mathbb{T} \\
&\qquad \sigma_o^2\Lambda_n^H(e^{j\omega})\left[\frac{1}{2\pi j}\oint_{\mathbb{T}} \Lambda_n(\lambda)\Lambda_n^H(\lambda)\Phi_u\left(\frac{\lambda+\xi}{1+\xi\lambda}\right)\frac{1-\xi^2}{(\lambda+\xi)(\lambda^{-1}+\xi)}\frac{d\lambda}{\lambda}\right]^{-1}\Lambda_n(e^{j\omega}) \leq \frac{1}{\gamma}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\omega| \leq \tilde{\omega}_B,
\end{aligned} \tag{8.19}$$

where $\Lambda_n(\lambda) := [1 \ \lambda \ \cdots \ \lambda^{-(n-1)}]^T = \tilde{\Gamma}_n((\lambda+\xi)/(1+\xi\lambda))$.

It can be seen from (8.19), that by defining

$$\tilde{\Phi}_u(z) := \frac{1-\xi^2}{(z+\xi)(z^{-1}+\xi)}\Phi_u\left(\frac{z+\xi}{1+\xi z}\right), \quad z \in \mathbb{T}, \tag{8.20}$$

we have turned the original input design problem (8.16) into an FIR input design problem. Replacing Φ_u by Φ_u^{FIR} and $\tilde{\Phi}_u$ by Φ_u^{Lag} in (8.20) concludes the proof. □

This theorem can be interpreted as saying that: the Laguerre input design problem is simply a 'frequency-warped' version of the FIR case. This has several interesting consequences:

1. From (8.14) we have that $\omega_B = \pi$ gives $\tilde{\omega}_B = \pi$. This means that if we are interested in obtaining a good model for the entire frequency range, the pole ξ of the Laguerre basis does not have any influence on the input power required to obtain such a model.

2. By applying the duality result of Theorem 6.2.1 (see Chapter 6), the previous observation can be stated as a fundamental limitation in the estimation of Laguerre models:

$$\min_{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega)d\omega=1} \max_{-\pi\leq\omega\leq\pi} \text{Var}\left\{G(e^{j\omega}, \hat{\theta}_{N,n})\right\} = \frac{n\sigma_o^2}{N}, \tag{8.21}$$

for $N \gg 1$, independently of the location of the pole ξ. Clearly, the solutions to problem (8.4) and (8.21) are simply proportional to each other. Note that the optimal Φ_u for $\omega_B = \pi$ has power $n\sigma_o^2\gamma$, hence it has to be scaled by $1/(n\sigma_o^2\gamma)$ in (8.21). This implies that, with this new input spectrum, the maximum variance of $G(e^{j\omega}, \hat{\theta}_{N,n})$ changes from $1/\gamma$ to $n\sigma_o^2$, which is similar to the so-called *water-bed effect in spectral estimation* (Stoica et al. 2004). Notice that the effect is similar as the optimal maximum variance given in (8.21) is independent of the frequency ω since the entire frequency range $|\omega| \leq \pi$ is considered. Again the pole ξ of the Laguerre basis does not have any influence on the maximum value of the variance curve.

3. If $0 < \xi < 1$, then $\tilde{\omega}_B > \omega_B$ (if $\omega_B < \pi$). Thus, due to the monotonicity of f^{opt} with respect to ω_B for the FIR case (see Chapter 7), for a given ω_B the optimal cost f^{opt} will be greater if we consider a Laguerre model structure with $0 < \xi < 1$ than if an FIR model is considered. Conversely, f^{opt} will be smaller if we consider a Laguerre model structure with $-1 < \xi < 0$ than in the case of an FIR model.

4. Basically all the results of Chapter 7 can be translated to the Laguerre case by using the transformation developed in this section. For example, the asymptotic bounds on f^{opt} can be rewritten for the Laguerre model, giving

$$\left(\frac{n}{\pi j}\ln\left[\frac{e^{j\omega_B}-\xi}{1-\xi e^{j\omega_B}}\right]+1\right)\sigma_o^2\gamma \leq f^{\text{opt}} \leq \left(\frac{n}{\pi j}\ln\left[\frac{e^{j\omega_B}-\xi}{1-\xi e^{j\omega_B}}\right]+1+\frac{25}{\pi}\ln\left[\frac{2n}{\pi j}\ln\left(\frac{e^{j\omega_B}-\xi}{1-\xi e^{j\omega_B}}\right)\right]\right)\sigma_o^2\gamma,$$

which holds when n is sufficiently large.

8.5 Numerical Example

The following example illustrates some aspects of the results presented in Sections 8.4.2 and 8.4.3. Here we consider Laguerre models where the order of the denominator polynomial $A(z)$ is the same as the numerator polynomial $B(z, \theta_{n_b})$ and $n = n_b$. First we illustrate the result of Theorem 8.4.7. Let $\sigma_o^2 = \gamma = 1$. In Figure 8.1 the optimal solution f^{opt} is plotted for a Laguerre model with $\xi = 0.2$ using $\omega_B = 0.8\pi$. From (8.14) it is concluded that this frequency corresponds to the 'warped' frequency 0.86π (≈ 2.72). Therefore, we have also plotted the cost associated with an FIR model (i.e., $\xi = 0$) where $\omega_B = 0.86\pi$. It can be seen that the cost is the same for these two cases.

Next the dependence on the location of the pole ξ in the Laguerre model is shown. In Figure 8.1 we have plotted f^{opt} using $\omega_B = 0.1\pi$ for three cases:

1. A Laguerre model with $\xi = 0.2$.

2. A Laguerre model with $\xi = -0.2$.

3. An FIR model.

We see that the location of the pole affects f^{opt} for the Laguerre model as predicted, i.e. for the Laguerre model with $\xi = 0.2 > 0$ the cost is higher than for the FIR model, and the cost is lower for the Laguerre model with $\xi = -0.2 < 0$.

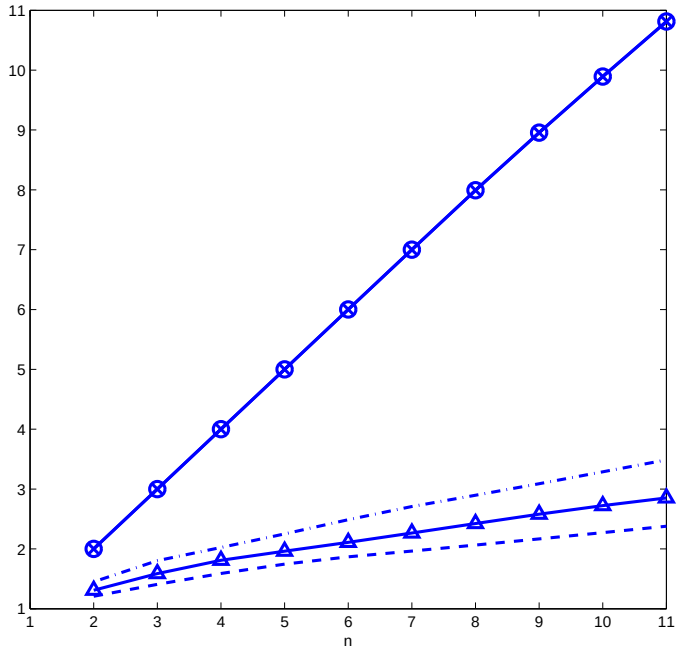

Figure 8.1. The cost f^{opt} versus model order n. Laguerre model with $\omega_B = 0.8\pi$ and $\xi = 0.2$ (-○-). FIR model with $\omega_B = 0.86\pi \approx 2.72$ (-×-). Laguerre model with $\omega_B = 0.1\pi$ and $\xi = 0.2$ (dash-dotted line). FIR model with $\omega_B = 0.1\pi$ (-△-). Laguerre model with $\omega_B = 0.1\pi$ and $\xi = -0.2$ (dashed line).

8.6 Summary

This chapter extends the FIR model based results in Chapter 7 to more general model structures. In particular we have considered OE models, fixed denominator models and Laguerre models. The input design problem examined here is to minimise the input power required to estimate a model of order n of an LTI system with a prescribed precision γ over the frequency range $[-\omega_B, \omega_B]$. The optimal input power is denoted as the cost of complexity. For simplicity we assumed no undermodelling. The

results quantify the cost associated with different model structures and overmodelling as well as the cost of extracting more information about the system. For example, if the input is restricted to white spectra in the identification of OE models, the cost is given by $\sigma_o^2\gamma \sup_{|\omega|\leq\omega_B} K_n(\omega,\omega)$, where the reproducing kernel $K_n(\omega,\omega)$ accounts for model structure. If we hide all frequencies except the static properties (i.e., $\omega_B = 0$), the cost is given by $\sigma_o^2\gamma$ which means that the cost is independent of both n and the model structure. Furthermore, for fixed denominator structures a lower bound for the cost is given by $\sigma_o^2\gamma\omega_B/\pi \left(\sum_{i=1}^n \frac{1-|\xi_i|}{1+|\xi_i|}\right)^2 / \left(\sum_{i=1}^n \frac{1+|\xi_i|}{1-|\xi_i|}\right)$, where $\{\xi_i\}$ denote the poles of the system transfer function. Finally, a result which states the relationship between FIR models and Laguerre models is presented.

8.7 Appendix: Technical lemmas

Lemma 8.7.1 *For the case when $\omega_B = 0$, a lower bound for f^{opt} is obtained by solving*

$$\begin{aligned} \min_{\Phi_u} \ & \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)d\omega \\ s.t. \ & \Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi \\ & \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\tau)\Phi_u(\tau)d\tau \geq 1, \end{aligned} \tag{8.22}$$

where

$$k(\tau) := \frac{1}{\sigma_o^2\gamma}\frac{|\Gamma_n^H(1)\Gamma_n(e^{j\tau})|^2}{|\Gamma_n^H(1)\Gamma_n(1)|^2}. \tag{8.23}$$

Proof. First note that for the case $\omega_B = 0$, the second constraint in (8.4) can be written as

$$\lim_{N\to\infty} N \text{ Var } \{G(1,\hat{\theta}_{N,n})\} = \sigma_o^2\Gamma_n^H(1)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})\Phi_u(\tau)d\tau\right]^{-1}\Gamma_n(1) \leq \frac{1}{\gamma},$$

which by the Schur complement is equal to

$$\frac{1}{2\pi\sigma_o^2\gamma}\int_{-\pi}^{\pi}\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})\Phi_u(\tau)d\tau \geq \Gamma_n(1)\Gamma_n^H(1).$$

By pre- and post-multiplying with $\Gamma_n^H(1)$ and $\Gamma_n(1)$ respectively, it must hold that

$$\frac{1}{2\pi\sigma_o^2\gamma}\int_{-\pi}^{\pi}\Gamma_n^H(1)\Gamma_n(e^{j\tau})\Gamma_n^H(e^{j\tau})\Gamma_n(1)\Phi_u(\tau)d\tau \geq \Gamma_n^H(1)\Gamma_n(1)\Gamma_n^H(1)\Gamma_n(1),$$

which can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} k(\tau)\Phi_u(\tau)d\tau \geq 1.$$

□

Lemma 8.7.2 (A mass distribution problem) *The minimisation problem* (8.22) *is equivalent to the following optimisation problem:*

$$\begin{aligned} &\max_{\Phi_u} \int_{-\pi}^{\pi} k(\tau)\Phi_u(\tau)d\tau \\ &s.t.\ \ \Phi_u(\omega) \geq 0, \quad |\omega| \leq \pi \\ &\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\tau)d\tau \leq c_{const}, \end{aligned} \tag{8.24}$$

where c_{const} is a constant and k is defined in (8.23)*, in the sense that the solutions of both problems are proportional to each other.*

Proof. The proof follows from Theorem 6.2.1 of Chapter 6. □

CHAPTER 9

AN ALGORITHM TO GENERATE BINARY SIGNALS

9.1 Introduction

As evident from the literature and the results in previous chapters of the Thesis, optimal test signals are frequently specified in terms of their second order properties, e.g. autocovariance or spectrum. However, to utilise these signals in practice, one needs to be able to produce realisations whose second order properties closely approximate the prescribed properties. Of particular interest are binary waveforms since they have the highest form-factor in the sense that they achieve maximal energy for a given amplitude. In this chapter we utilise ideas from model predictive control to generate a binary waveform whose sampled autocovariance is as close as possible to some prescribed autocovariance.

We develop a relatively simple procedure to generate a binary waveform, based on the use of the Receding Horizon concept commonly employed in Model Predictive Control (Goodwin et al. 2001). Heuristically speaking the idea is to solve, for each time instant, a finite horizon optimisation problem to find the optimal set of the next, say, T values of the sequence such that the sampled autocovariance sequence so obtained is as close as possible (by some measure) to the desired autocovariance. One then takes the first term of this optimal set for the sequence, advances time by one step and repeats the procedure.

Note that in order to find the true optimal binary sequence of length N, we would have to compute the sample autocovariance function of all sequences in $\{0,1\}^N$ and then choose the sequence whose autocovariance is closest to the desired one according to a specified norm. This procedure, however, would be computationally intractable as it involves 2^N comparisons, a truly large number in general.

Several kinds of measures can be used to compare the sampled autocovariance of the generated signal with the desired autocovariance, including the Euclidean or the infinity norm of their difference. However, we have verified via simulations that the Euclidean norm produces very good results when compared to other norms. Furthermore, by Theorem 9.4.1 of Section 9.4, the algorithm is shown to converge for a special case when the Euclidean norm is used.

The algorithm is fast and easy to implement when compared with the existing methods, and can also be run in realtime. To demonstrate the application of the algorithm, two examples, motivated by experiment design, are provided.

9.2 Description of the Receding Horizon Algorithm

In this section we formulate and develop the receding horizon algorithm that generates a binary signal with a prescribed autocovariance. This is done in two parts. First we convert the problem to an equivalent one that allows us to simplify the computation and to force the generated signal to have zero mean. We then develop the algorithm as a series of steps and finally present it as Matlab® code.

Let $\{r_k^d\}_{k=0}^{\infty}$ be a given desired autocovariance sequence. Also, let N be the length of the signal to be generated, n the number of lags of $\{r_k^d\}_{k=0}^{\infty}$ to be compared to the corresponding lags of the sampled autocovariance sequence of the designed signal, and m be the length of the receding horizon over which we apply the optimisation algorithm.

Note that in order for $\{r_k^d\}_{k=0}^{\infty}$ to be a valid autocovariance sequence, it must be positive definite (Papoulis 1991), i.e.

$$\sum_{1\le i\le j\le M} a_i a_j^* r_{i-j} \ge 0$$

for every $M \in \mathbb{N}$ and $\{a_i\}_{k=1}^{M} \subseteq \mathbb{C}^M$.

For simplicity, we force the designed signal to have zero mean and restrict its values to $\{-1,1\}$. This implies that r_0^d must be equal to 1.

The two parts of the proposed algorithm are described below:

(A) Conversion to an Equivalent Problem

First we convert the desired autocovariance sequence $\{r_k^d\}_{k=0}^{\infty}$ into the non-central autocovariance of a $\{0,1\}$ sequence. That is, define

$$\tilde{r}_k^d := \frac{1}{4}(r_k^d + 1), \qquad k = 0,\dots,n. \tag{9.1}$$

Remark 9.2.1 *The idea is that the algorithm will generate a sequence $\{\tilde{y}_i\}_{i=1}^{N}$ taking only the values*

$\{0,1\}$ *such that*

$$\frac{1}{N}\sum_{i=k+1}^{N}\tilde{y}_i\tilde{y}_{i-k} \approx \tilde{r}_k^d, \qquad k=0,\ldots,n, \tag{9.2}$$

where the left side corresponds to the sampled non-central autocovariance of the signal evaluated at lag k. The approximation criterion will be the Euclidean norm, as shown in step 5 below.

Note that since $\tilde{y}_i \in \{0,1\}$ *for every i, we see that equation* (9.2) *for* $k=0$ *is equivalent to*

$$\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_i \approx \tilde{r}_0^d = \frac{1}{2}, \qquad k=0,\ldots,n.$$

This implies that equation (9.2) *is actually forcing* $\{\tilde{y}_i\}_{i=1}^N$ *to have sampled mean* $1/2$*, or equivalently, forcing the designed signal to have zero sampled mean.*

(B) The Main Loop

In this part, the user should provide 3 variables: the number N of points to be generated, the horizon length m and the number of lags n to be considered. For the ease of explanation we provide an outline of the algorithm as a series of steps:

1) Set $t=1$.

2) Set $(y'_t,\ldots y'_{t+m-1}) = 0_{1,m} \in \{0,1\}^m$, where $0_{1,m}$ denotes a zero matrix of dimension $1\times m$.

3) Compute the first n lags of the sampled non-central autocovariance of $(\tilde{y}_1,\ldots,\tilde{y}_{t-1},y'_t,y'_{t+1},\ldots y'_{t+m-1})$ (or of $(y'_1,\ldots y'_m)$, if $t=1$) via

$$r'_k := \frac{1}{t+m-1}\sum_{i=k+1}^{t+m-1} y'_i y'_{i-k}, \qquad k=0,\ldots,n,$$

where we are considering $y'_i = \tilde{y}_i$ for $i=1,\ldots,t-1$.

4) Generate a new m-tuple $(y'_{t+1},\ldots,y'_{t+m}) \in \{0,1\}^m$ and repeat step 3 until all m-tuples have been tested.

5) Let $\tilde{y}_t = y'_t$ for the m-tuple $(y'_t,\ldots y'_{t+m-1}) \in \{0,1\}^m$ for which $\|\{r'_i\}_{i=0}^n - \{\tilde{r}_i^d\}_{i=0}^n\|_2$ is minimum. If this norm is equal for both values of y'_t, take $\tilde{y}_t = 0$.

6) If $t<N$, let $t=t+1$ and go to step 2.

7) Convert the $\{0,1\}$ N-tuple $(\tilde{y}_1,\ldots,\tilde{y}_N)$ into a $\{-1,1\}$ N-tuple $(y_1,\ldots,y_N)$ via

$$y_t := 2\tilde{y}_t - 1, \qquad t=1,\ldots,N. \tag{9.3}$$

It is straightforward to extend the method to more general cases. For example, to generate signals with non zero mean $\bar{y}$ and/or taking values in $\{a,b\}$, it is necessary to alter equations (9.1) and (9.3), and to let $r_0^d = \bar{y}^2$.

To provide further insight into the implementation of the algorithm we add the following remarks:

Remark 9.2.2 *The computation of the sampled autocovariance at step* 3 *can be done in a recursive manner (with respect to t), which reduces the execution time of the algorithm.*

Remark 9.2.3 *The execution time of the algorithm depends exponentially on m. However, it can be empirically verified that* $m = 1$ *gives very good results. In fact, we show in Section 9.4 that the algorithm converges successfully for* $m = 1$ *in a particular case. Thus, for ease of reference, we present below an optimised version of the algorithm for* $m = 1$ *in Matlab® code:*

```
% Initialisation
y  = zeros(N+n, 1);
r0 = zeros(n+1, 1);
r1 = zeros(n+1, 1);

% Conversion of the autocovariance
% sequence to the equivalent problem
rd = 0.25*(rd + 1);

for i = 1:N,

   % Calculation of the next
   % autocovariance sequence, if we
   % add "0" or "1" to the output
   % signal, respectively
   r0 = r0 - rd;
   r1 = r0 + [1; y(n+i-1:-1:i)];

   % Comparison of the resulting
   % autocovariance sequences
   if norm(r0) > norm(r1),
      y(n+i) = 1;
```

```
      r0      = r1;
   end

end

% Conversion to the original problem
% to obtain the desired sequence
y = 2*y(n+1:end) - 1;
```

Here, `y` *is the generated input signal;* `r0` *and* `r1` *are the non-central autocovariances of the signal plus an additional* 0 *or* 1*, respectively; and* `rd` *is the desired autocovariance sequence (to be specified by the user).*

As with $\{r_k^d\}_{k=0}^{\infty}$, the user of the algorithm also has to choose three other variables: N, n and m. The choice of N depends on the external circumstances which arise in the particular context where the user needs to generate a binary signal. For example, in experiment design, N usually depends on a number of factors such as the sampling period, the total time the system will be available for experimentation and the required precision of the parameter estimates. The variable n depends on the characteristics of the prescribed autocovariance sequence $\{r_k^d\}_{k=0}^{\infty}$. In particular, it is convenient to choose n such that $r_k^d \approx 0$ for all $k > n$. The choice of m is a tradeoff between precision and execution time, as will be seen in the examples provided in the next section, however, for most practical applications it seems that $m = 1$ provides very good performance.

9.3 Input Signal Design Examples

In this section we present two examples. The first example deals with the problem of generating pseudo random signals (i.e. pseudo white noise). The second example involves the generation of bandlimited '$1/f$' noise, which has been shown to possess important robustness properties in experiment design (see chapters 2 and 3).

9.3.1 Pseudo White Noise

The tuning parameters of the algorithm are N, the number of data points to be generated; n, the number of lags to be considered for the comparison of the desired and sampled autocovariance sequences; and

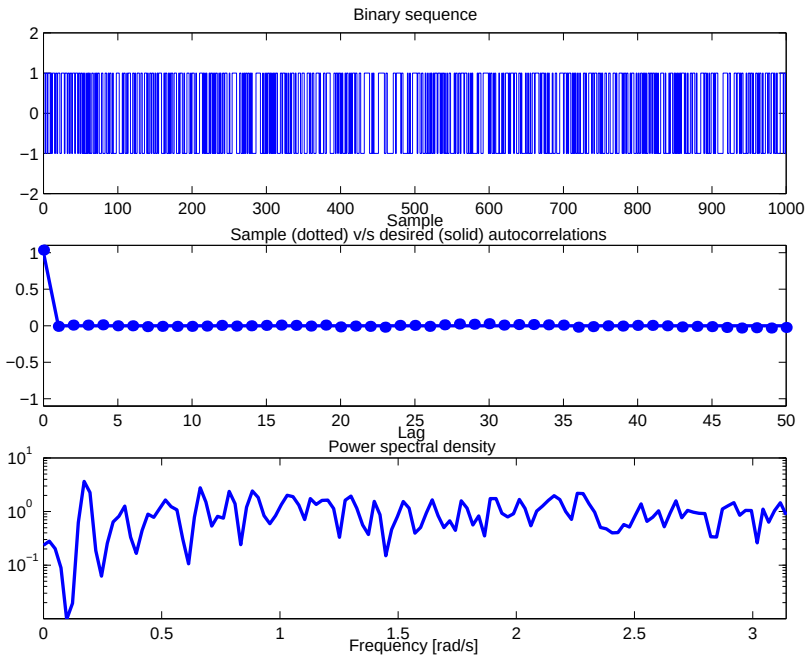

Figure 9.1. Characteristics of the generated pseudo white noise signal for $m = 1$, $N = 1000$ and $n = 50$.

m, the length of the receding horizon.

For $m = 1$, $N = 1000$ and $n = 50$, we obtain the results presented in Figure 9.1. From this figure, we see that both the autocovariance and spectrum of the generated signal are similar to those of white noise. If we increase N to 10^6, we obtain Figure 9.2, which shows that the algorithm has remarkably good asymptotic properties. With respect to the execution time, we find that the algorithm requires only a small amount of time to run, e.g. on a PC with a Pentium III 871 Mhz CPU and 512 Mb of RAM it takes less than 42 sec to generate 10^6 points! A plot of the dependence of the cost function on N is given in Figure 9.3. Note that the cost is on a logarithmic scale. From this figure it can be seen that the convergence rate of the algorithm appears to be $O(1/N)$. However, the proof of convergence given in Section 9.4 establishes a convergence rate of $O(1/\sqrt{N})$, as it is based on a conservative upper bound for the cost function $\|\{r_i'\}_{i=0}^n - \{\tilde{r}_i^d\}_{i=0}^n\|_2$.

In Figure 9.4 the dependence of the cost function on the horizon length m is shown (for $N = 10^4$ and $n = 50$). As expected, it can be seen that the cost function decreases with m. Note however that the computational complexity of the algorithm depends exponentially on m, thus a tradeoff needs to be considered between accuracy and execution time.

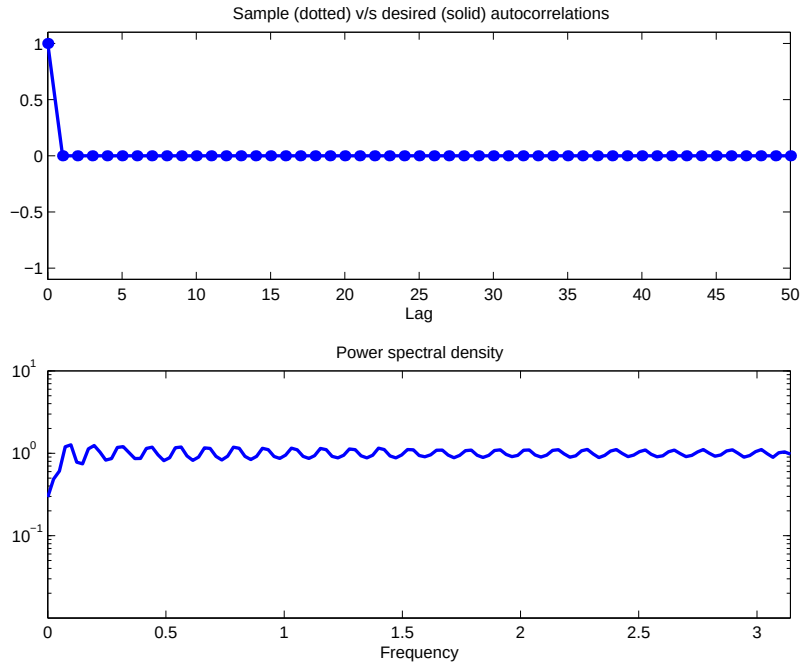

Figure 9.2. Characteristics of the generated pseudo white noise signal for $m = 1$, $N = 10^6$ and $n = 50$.

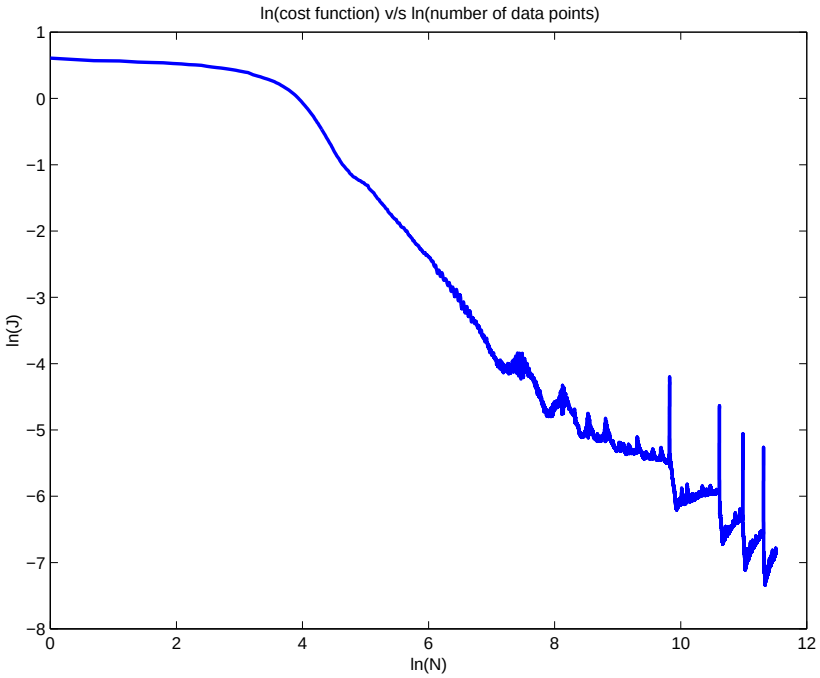

Figure 9.3. Dependence of the cost function $J := \|\{r'_i\}_{i=0}^n - \{\tilde{r}_i^d\}_{i=0}^n\|_2$ on N, for $m = 1$ and $n = 50$, when generating pseudo white noise.

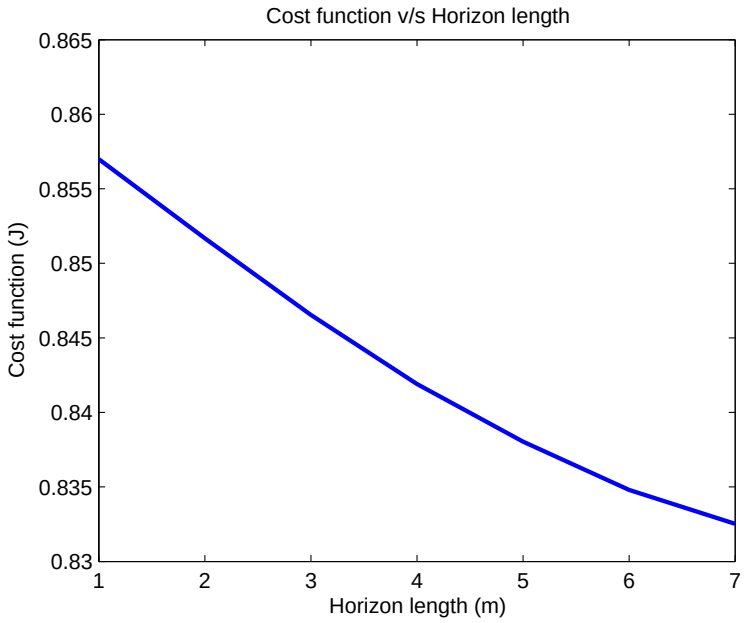

Figure 9.4. Dependence of the cost function $J := \|\{r_i'\}_{i=0}^n - \{\tilde{r}_i^d\}_{i=0}^n\|_2$ on the horizon length, for $N = 1000$ and $n = 50$, when generating pseudo white noise.

9.3.2 Bandlimited '$1/f$' Noise

Bandlimited '$1/f$' noise is defined by the following spectrum:

$$\Phi^{1/f}(\omega) := \begin{cases} \dfrac{1/\omega}{\ln\overline{\omega} - \ln\underline{\omega}}, & \omega \in [\underline{\omega}, \overline{\omega}], \\ 0, & \text{otherwise}, \end{cases}$$

where $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$ ($\underline{\omega} < \overline{\omega}$). The autocovariance sequence of this signal is given by

$$r_k^{1/f} := \frac{1}{\ln\overline{\omega} - \ln\underline{\omega}} \int_{\underline{\omega}}^{\overline{\omega}} \frac{\cos kx}{x} dx, \quad k \in \mathbb{N}_0.$$

Figure 9.5 shows the ideal spectral density of bandlimited '$1/f$' noise for $\underline{\omega} = 1$, $\overline{\omega} = 2$. In Figure 9.6 we present the results obtained from the receding horizon algorithm for $\underline{\omega} = 1$, $\overline{\omega} = 2$, $m = 1$, $N = 10^6$ and $n = 50$. Figure 9.6 verifies the ability of the algorithm to generate a binary non-white noise signal. The discrepancies between the desired and the achieved autocovariances seem to be due to the impossibility of generating a binary signal with a true bandlimited '$1/f$' spectrum, as the results do not appear to improve significantly by increasing m and n.

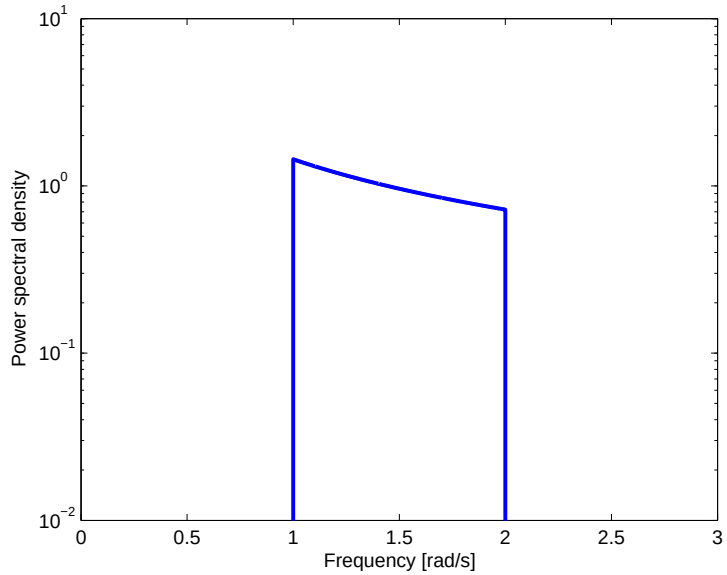

Figure 9.5. Power spectral density of bandlimited '$1/f$' noise signal for $\underline{\omega} = 1$ and $\overline{\omega} = 2$.

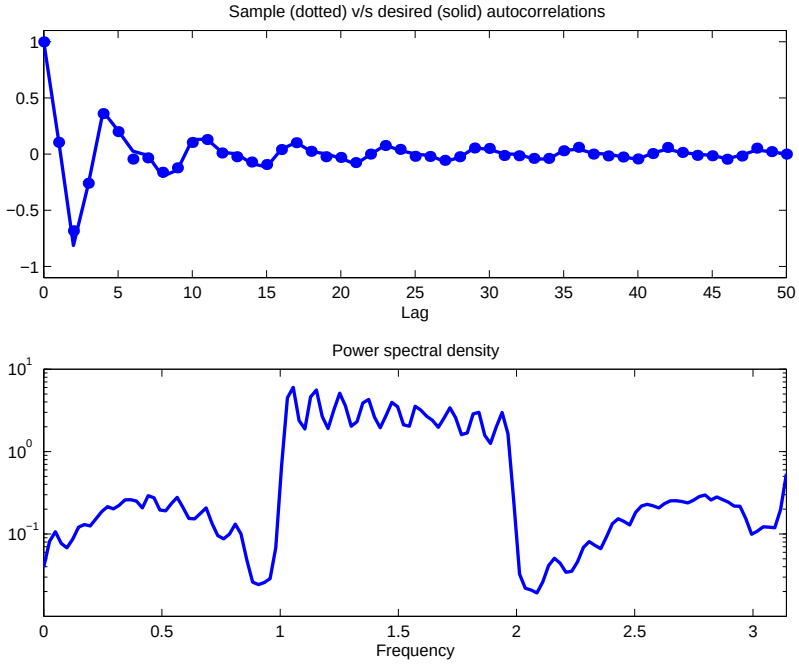

Figure 9.6. Characteristics of the generated pseudo bandlimited '$1/f$' noise signal for $m = 1$, $N = 10^6$ and $n = 50$.

9.4 Convergence of the Receding Horizon Algorithm

In this section we study the convergence of the receding horizon algorithm for the special case of generating 'pseudo' white noise, i.e., when the desired autocovariance sequence is a Kronecker delta ($r_0^d = 1$ and $r_k^d = 0$ for $k \neq 0$). We proceed by describing the algorithm as a switched linear system and then apply a simple geometric inequality to establish its convergence.

9.4.1 A Switched Linear System Representation of the Algorithm

To aid the development of the switched linear system to represent the algorithm, let

$$\begin{aligned}
\mathbf{x}_t &:= [\mathbf{y}_t^T \;\; \mathbf{r}_t^T]^T, \\
\mathbf{y}_t &:= [y_{t-n} \;\; \cdots \;\; y_{t-1}]^T \in \mathbb{R}^{n\times 1}, \\
\mathbf{r}_t &:= [r_{n,t} \;\; \cdots \;\; r_{1,t}]^T \in \mathbb{R}^{n\times 1},
\end{aligned}$$

where

$$r_{k,t} := \sum_{i=k+1}^{t-1} (y_i y_{i-k} - r_k^d), \quad 1 \leq k \leq \min(t-2, n) \tag{9.4}$$

and

$$\mathbf{r}^d := [r_n^d \;\; \cdots \;\; r_1^d]^T \in \mathbb{R}^{n\times 1}.$$

Now,

$$r_{k,t+1} = r_{k,t} + y_t y_{t-k} - r_k^d,$$

and the dynamics of $\mathbf{r}_t$ are given by

$$\mathbf{r}_{t+1} = \mathbf{r}_t + y_t \mathbf{y}_t - \mathbf{r}^d$$

with the initial condition

$$\mathbf{r}_0 = 0_{n,1}. \tag{9.5}$$

The dynamics of $\mathbf{y}_t$ are then

$$\mathbf{y}_{t+1} = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{bmatrix} \mathbf{y}_t + \begin{bmatrix} 0_{n-1,1} \\ 1 \end{bmatrix} y_t,$$

and the initial condition

$$\mathbf{y}_0 = 0_{n,1}.$$

For the generation of pseudo white noise, we have that $\mathbf{r}^d = 0_{n,1}$. This simplifies the expressions, and allows the algorithm to be written as the following switched linear system:

$$\mathbf{x}_{t+1} = \begin{cases} \left[\begin{array}{cc|c} 0_{n-1,1} & I_{n-1} & \\ 0 & 0_{1,n-1} & 0_{n,n+1} \\ \hline -I_n & & I_n \end{array}\right] \mathbf{x}_t + \left[\begin{array}{c} 0_{n-1,1} \\ -1 \\ \hline 0_{n,1} \end{array}\right], & y_t = -1, \\ \left[\begin{array}{cc|c} 0_{n-1,1} & I_{n-1} & \\ 0 & 0_{1,n-1} & 0_{n,n+1} \\ \hline I_n & & I_n \end{array}\right] \mathbf{x}_t + \left[\begin{array}{c} 0_{n-1,1} \\ 1 \\ \hline 0_{n,1} \end{array}\right], & y_t = 1. \end{cases} \tag{9.6}$$

Notice that from (9.6) we have that

$$\mathbf{r}_{t+1} = \mathbf{r}_t \pm \mathbf{y}_t, \tag{9.7}$$

where the $\pm$ sign is chosen to make $\|\mathbf{r}_{t+1}\|_2$ as small as possible.

9.4.2 Proof of Convergence

The basic idea for the proof of convergence is to establish a worst case bound for $\|\mathbf{r}_t\|_2$, and to check that according to this bound, $\|\mathbf{r}_t/t\|_2 \to 0$ as $t \to \infty$. Thus, to proceed, we require the following result:

Lemma 9.4.1 (An inequality in an inner product space) *Let x, y be elements of an inner product space $(X, \langle,\rangle)$. Then*

$$\min\{\|x+y\|^2, \|x-y\|^2\} \leq \|x\|^2 + \|y\|^2,$$

where $\|z\| := \sqrt{\langle z,z\rangle}$ for every $z \in X$.

Proof. Notice that

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\,\mathrm{Re}\langle x,y\rangle,$$

hence

$$\min\{\|x+y\|^2, \|x-y\|^2\} = \|x\|^2 + \|y\|^2 - 2|\,\mathrm{Re}\langle x,y\rangle| \leq \|x\|^2 + \|y\|^2.$$

□

The convergence of the algorithm for the special case of generating pseudo white noise is established in the following theorem.

Theorem 9.4.1 (Convergence of the algorithm for white noise) *For the algorithm described by* (9.6)*, where* $\{y_t\}_{t=1}^{\infty} \subseteq \{-1,1\}^{\mathbb{N}}$ *is chosen such that*

$$\|\mathbf{r}_t + y_t\mathbf{y}_t\|_2 = \min\{\|\mathbf{r}_t - \mathbf{y}_t\|_2, \|\mathbf{r}_t + \mathbf{y}_t\|_2\}, \quad t \in \mathbb{N},$$

it holds that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=k+1}^{t} y_i y_{i-k} = r_k^d, \quad k = 1,\ldots,n.$$

Proof. First note that

$$\|\mathbf{y}_t\|_2^2 = n, \quad t > n.$$

Hence, by Lemma 9.4.1 and (9.7) we have that

$$\|\mathbf{r}_{t+1}\|_2^2 = \min\{\|\mathbf{r}_t + \mathbf{y}_t\|_2^2, \|\mathbf{r}_t - \mathbf{y}_t\|_2^2\} \leq \|\mathbf{r}_t\|_2^2 + \|\mathbf{y}_t\|_2^2 = \|\mathbf{r}_t\|_2^2 + n, \quad t > n. \tag{9.8}$$

Since $\mathbf{r}_0 = 0_{n,1}$ (see (9.5)), we can iterate (9.8) over $t \in \mathbb{N}$, giving

$$\|\mathbf{r}_t\|_2^2 \leq nt + c, \quad t > n, \tag{9.9}$$

where $c \in \mathbb{R}^+$ is an upper bound on $\sum_{t=1}^{n} \|\mathbf{r}_t\|_2^2$. For example, by applying the Cauchy-Schwarz inequality to (9.4) and using the fact that $|y_t| = 1$ and $r_k^d = 0$ we have that $\|\mathbf{r}_t\|_2^2 \leq n(t-1)$. Therefore,

$$\sum_{t=1}^{n} \|\mathbf{r}_t\|_2^2 \leq \sum_{t=1}^{n} n(t-1) = \frac{n^3}{2},$$

hence we can take $c = n^3/2$. Now, if we divide (9.9) by t^2 and recall the definition of $\mathbf{r}_t$ (see (9.4)), we obtain

$$\sum_{k=1}^{n} \left[\sum_{i=k+1}^{t-1} \frac{y_i y_{i-k} - r_k^d}{t} \right]^2 \leq \frac{n}{t} + \frac{c}{t^2}, \quad t \geq n+2.$$

Therefore,

$$\left| \frac{1}{t} \sum_{i=k+1}^{t-1} (y_i y_{i-k} - r_k^d) \right| \leq \sqrt{\frac{n}{t} + \frac{c}{t^2}}, \quad t \geq n+2. \tag{9.10}$$

Since the right side of (9.10) tends to 0 as $t \to \infty$, we conclude that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=k+1}^{t} y_i y_{i-k} = \lim_{t\to\infty} \frac{t+1}{t} \frac{1}{t+1} \sum_{i=k+1}^{t} y_i y_{i-k} = r_k^d, \quad k = 1,\ldots,n.$$

□

Theorem 9.4.1 establishes that the algorithm generates a binary signal whose sampled autocovariance converges, as t goes to ∞, to the autocovariance of white noise.

9.5 Summary

We have developed a novel method for generating binary signals with a specified autocovariance. The algorithm is based on ideas from model predictive control, hence utilises a receding horizon algorithm. The algorithm is simple and straightforward to implement, and exhibits fast convergence as verified by simulation studies. We have shown empirically that the algorithm has good asymptotic properties. We have also established global convergence of the algorithm for the case of generating pseudo white noise.

CHAPTER 10

CONCLUSIONS AND FUTURE WORK

The focus of this Thesis has been on robust experiment design. Robust experiment design is fundamental in overcoming the paradox in optimal experiment design by removing the dependence on having exact knowledge of the system under test. In these conclusions, we summarise the main results as described in this Thesis, provide some discussion, and suggest some future research directions.

10.1 Conclusions

A min-max approach to robust optimal experiment design for dynamic system identification has been described and analysed. Several properties of the robust optimal input for one-parameter systems were established, namely, existence and uniqueness, and the fact that the optimal spectra have finite support. We evaluated and compared several different design criteria: one based on the minimum eigenvalue of a scaled version of the information matrix, another based on relative frequency domain errors, and one related to the ν gap. A somewhat astonishing observation was obtained when we showed that bandlimited '$1/f$' noise performs, in general, better than pseudo white noise inputs such as PRBS.

Diffuse prior information presents another problem for robust experiment design involving the need of a general measure of 'goodness'. To this end, we proposed and analysed a general class of criteria for measuring how 'good' an experiment is. We developed a specific measure, based on an asymptotic (in model order) variance expression, that provides a system independent optimal experiment design, suitable for the case when one only has a vague idea about the system to be identified. It was also shown that bandlimited '$1/f$' noise is optimal according to this cost function, which reinforces the observation in Chapter 2 regarding the use of bandlimited '$1/f$' noise as an input signal for system identification.

In order for the results in Chapter 3 to consider non-asymptotic (in model order) variance expressions, we digressed slightly and established fundamental limitations on the variance of estimated parametric models, for both open and closed loop identification. These limitations were shown to give rise to a 'water-bed' effect in system identification. In the closed loop case, we presented results for both direct and indirect identification methods. These results have been utilised to show that in the case

of multisine inputs, the well-known asymptotic (in model order) variance expressions provide upper bounds on the actual variance of the estimated model for finite orders.

Based on the fundamental limitations derived in Chapter 4, several robust experiment design problems were next studied. First, we derived a closed form expression for the input spectrum which minimises the maximum value of an integrated weighted variance of the frequency response estimator, over all model structures with a given number of parameters. With this expression, the problem studied in Chapter 3, i.e. finding a class of cost functions that give an optimal input being independent of the true system and the noise dynamics, has been revisited. The result verifies that obtained in Chapter 3, namely that bandlimited '$1/f$' noise is optimal, also for finite order models. Finally, we analysed Yuan & Ljung's unprejudiced optimal input design approach. In this case, an unprejudiced optimal input which is valid for finite order models has been developed, which solves the apparent paradox present in Yuan & Ljung's original result. It is important to note that, as in Yuan & Ljung's approach, the variance expressions utilised here do not consider undermodelling and both approaches assume the noise dynamics as known.

Robustness is also of importance in experiment design for control. We have shown that a recently developed paradigm, called 'least costly experiment design for control', can be formulated in a more traditional setting, via a duality result, by stating it as an $\mathscr{H}_\infty$ or minimax optimisation problem depending on the particular constraints being taken into account. Specifically, we considered 4 problems from the least costly framework. In the open loop case, equivalence to the traditional experiment design problems was established using high order and finite order approximations of the covariance, where both independently and non-independently parameterised system and noise models are considered. For systems operating in closed loop, equivalence has been shown using covariance expressions which are non asymptotic in the model order. The duality result that we have established between least costly and traditional experiment design provides new insights into both frameworks. It also offers practical advantages e.g. by allowing the computational tools developed for each problem to be used in either framework. Furthermore, we believe the results provide a better understanding of the link between identification and robust control.

We have considered robustness from many directions in the Thesis. In Chapter 7 we have analysed the robustness of the system identification methodology against model complexity, using an experiment design perspective. In particular, we determined the minimum amount of input power, r_0^{opt}, required to estimate a Finite Impulse Response (FIR) model with prescribed precision γ over the frequency range $[-\omega_B, \omega_B]$, as a function of the model order n. Using the assumption of no undermodelling, we

first established that if n is large, r_0^{opt} is asymptotically proportional to n, ω_B and γ. This was deduced in a heuristic fashion from an asymptotic variance expression, and was justified in a rigorous manner by establishing asymptotic bounds on r_0^{opt}. A loose upper bound for r_0^{opt} is given by white noise input spectra, and it was shown that if ω_B is sufficiently close to (but not necessarily equal to) π, then r_0^{opt} equals the white noise solution. Furthermore a loose lower bound for r_0^{opt} was obtained by considering the solution for $\omega_B = 0$. Results have also been developed which provide tighter asymptotic lower and upper bounds for r_0^{opt}. Essentially these bounds quantify the cost of extracting more information about the system and overmodelling. They allow us to conclude that, asymptotically in n, the amount of system information to be extracted (as measured by ω_B), the accuracy (as measured by γ) and the noise power are all on an equal weighting with respect to the model complexity n in terms of the 'cost of complexity'. Hence, with respect to the capability of identifying complex systems, using a limited input power budget and limited time, we see that to identify highly complex systems within a certain accuracy, a more limited bandwidth is required than for a less complex system. It is important to note that the excitation needs to be carefully designed, as the cost of complexity is the minimum required input power to meet the model quality specifications. In particular, broadband excitation may not be suitable when a limited frequency range is of interest. The results obtained illustrate that the amount of information that we ask to be extracted from a system will determine how sensitive the cost of the identification experiment is with respect to the system (and model) complexity.

Chapter 8 extends the FIR model based results in Chapter 7 to more general model structures. In particular, we have considered Output Error (OE), fixed denominator and Laguerre models. The results quantify the cost associated with different model structures and overmodelling as well as the cost of extracting more information about the system. Many of the properties derived for FIR models have been extended to these more general model structures. Furthermore, a result stating the relationship between FIR models and Laguerre models was presented. This result shows that the cost of complexity problem for Laguerre models is essentially a frequency warped version of the FIR case.

Finally, we have developed a novel method for generating binary signals with a specified autocovariance. The method is based on ideas from model predictive control. In particular, it utilises a receding horizon algorithm. The algorithm is simple and straightforward to implement, and has been shown to exhibit fast convergence as verified by simulation studies. It has also been shown empirically that the algorithm has good asymptotic properties. Global convergence has been established for the particular case of generating pseudo white noise.

10.2 Suggestions for Further Research

In this section we present some suggestions for future research directions, based on the outcomes of the research presented in this Thesis.

- Generalise known properties of the optimal robust solution for the one-parameter case to the multi-parameter (SISO) case.
- Modify the min-max approach by embedding a prior distribution on the bounds of the parameter set.
- Use methods for determining Bayesian prior distributions (for example, maximum entropy, Jeffreys' or Barnard's methods (Berger 1985)) to generate robust solutions via Bayesian experiment design techniques.
- Extend the robust min-max approach to closed loop optimal robust experiments.
- Consider oversampling issues in robust experiment design.
- Extend the robust approach to deal with multi-input multi-output and non-linear systems.
- Explicitly consider the undermodelling problems in experiment design.
- Analyse the possibility of using the information from the step response or relay experiments to design a better experiment.
- Study the effect of undermodelling in the variance expression which appear in experiment design.
- Further extend the results of Chapters 7 and 8 to more general model structures, such as Box-Jenkins models.
- Extend the convergence result of the MPC algorithm for generating binary signals to more general cases.

APPENDIX A

NOTATION

Symbols

$\mathbb{N}$	Set of the natural numbers
$\mathbb{N}_0$	Set of the natural numbers including 0
$\mathbb{Z}$	Set of the integers
$\mathbb{R}$	Real line
$\mathbb{C}$	Complex plane (including ∞)
$\mathbb{T}$	Unit circle (i.e., $\{z \in \mathbb{C} : \ \lvert z\rvert = 1\}$)
$\mathbb{D}$	Open unit disk (i.e., $\{z \in \mathbb{C} : \ \lvert z\rvert < 1\}$)
$\mathbb{E}$	Exterior of the closed unit disk (i.e., $\{z \in \mathbb{C} : \ \lvert z\rvert > 1\}$)
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{R}_0^+$	Set of nonnegative real numbers
$\mathbb{K}^{n\times m}$	Set of the matrices of dimension $n \times m$ with elements in $\mathbb{K}$
$\mathbb{K}^n$	Set of the vectors of dimension n with elements in $\mathbb{K}$
$\mathbb{K}^{\mathbb{N}}$	Set of the infinite sequences with elements in $\mathbb{K}$
$\mathscr{M}_n$	Set of all stable model structures with n parameters (i.e., $\mathscr{M}_n := \{G : \ \mathbb{C} \times \Theta \to \mathbb{C} :$ $G(z,\cdot)$ is differentiable in the connected open set $\Theta \subseteq \mathbb{R}^n$ for all $z \in \mathbb{T}$, and $G(\cdot,\theta) \in \mathscr{H}_2$ for all $\theta \in \Theta\}$)
$[a,b]$	$\{x \in \mathbb{R} : \ a \leq x \leq b\}$
(a,b)	$\{x \in \mathbb{R} : \ a < x < b\}$
$(a,b]$	$\{x \in \mathbb{R} : \ a < x \leq b\}$
$[a,b)$	$\{x \in \mathbb{R} : \ a \leq x < b\}$
$A \subseteq B$	A is a subset of B
$C^1(X,Y)$	Space of the continuously differentiable functions from X to Y
$C(\mathbb{T},\mathbb{R}_0^+)$	Space of the continuous functions $f : \ \mathbb{T} \to \mathbb{R}_0^+$ such that $f(z^*) = [f(z)]^*$ for every $z \in \mathbb{T}$
$\mathscr{H}_2^n$	Hardy space of all functions $f : \mathbb{C} \to \mathbb{C}^n$ which are analytic in $\mathbb{E}$ and such that $\lim_{r\to 1_+} \int_{-\pi}^{\pi} \lVert f(re^{j\omega})\rVert^2 d\omega < \infty$
$\mathscr{S}(X)$	Set of all generalised functions f on $\mathbb{R}^n$ such that f is the derivative of some probability distribution function on $\mathbb{R}^n$, and $\operatorname{supp} f \subset X$

$\overline{A}$	Topological closure of X (with respect to a metric space $X \supseteq A$)
$\overline{\lim}_{n\to\infty} A_n$	Outer limit of the sequence of subsets of a metric space X, $\{A_n\}$
	(i.e., $\cap_{m\in\mathbb{N}}\overline{\cup_{n\geq m}A_n}$)
$\underline{\lim}_{n\to\infty} A_n$	Inner limit of the sequence of subsets of a metric space X, $\{A_n\}$
	(i.e., $\cap_{\{n_i\}\in K}\overline{\lim}_{i\to\infty}A_{n_i}$, where K is the set of all infinite subsequences $\{n_i\}$ in $\mathbb{N}$)
$\mathscr{X}_A$	Indicator function of the set A (i.e., $\mathscr{X}_A(x) = 1$ if $x \in A$, and $\mathscr{X}_A(x) = 0$ otherwise)
$A \setminus B$	Set theoretic difference of A and B (i.e., $A \setminus B = \{x \in A : x \notin B\}$)
$\text{span}\{\mathscr{B}_k\}_{k=1}^n$	Linear span of the elements $\mathscr{B}_k$ of a given vector space X
I_n (or I)	Identity matrix of dimension $n \times n$ (or of appropriate dimension)
$0_{n,m}$ (or 0)	Zero matrix of dimension $n \times m$ (or of appropriate dimension)
A^T	Transpose of A
A^H	Complex conjugate transpose of A
$\det A$	Determinant of A
$\text{tr} A$	Trace of A
$\lambda_{\min}(A)$	Minimum eigenvalue of A
$\lambda_{\max}(A)$	Maximum eigenvalue (or spectral radius) of A
$\sigma_{\max}(A)$	Maximum singular value of A
$\|x\|_2$	Euclidean norm of the vector x
$\text{perm}(A)$	Permanent of A
$\|A\|$	Absolute value of A
$A^\dagger$	Moore-Penrose generalised inverse of A
$[A]_+$	A Cholesky Factor of $(1/2)V^T(D+\|D\|)V$, where V is such that $A = V^T DV$,
	with D a diagonal matrix (see Definition 6.4.1)
$\mathscr{N}(A)$	Null space of A
$\mathscr{R}(A)$	Range space of A
$\text{diag}[a_1,\ldots,a_n]$	Diagonal matrix with diagonal elements $\{a_1,\ldots,a_n\}$
$\text{diag}(A,B)$	Block diagonal matrix with A and B as its diagonal blocks
$A > B$	$A - B$ is positive definite
$A \geq B$	$A - B$ is positive semi-definite
$A \otimes B$	Kronecker product of A and B
$\langle X, Y\rangle$	Inner product between X and Y
$A \leq_S B$	$B - A$ belongs to the cone S
$A <_S B$	$B - A$ belongs to the interior of the cone S
$K^\oplus$	Dual cone of the convex cone $K \subseteq X$, i.e., $\{x \in X : \langle x, y\rangle \geq 0 \text{ for all } y \in K\}$

s	Heaviside operator (i.e., derivation operator in the time domain, and Laplace variable in the frequency domain)
z	Forward shift operator
$\mathscr{F}\{f\}$	Fourier transform of the function f
$\|G\|_\infty$	$\mathscr{H}_\infty$-norm of the transfer function G (i.e., $\|G\|_\infty := \lim_{r\to 1_+} \sup_{z\in\mathbb{T}} \sigma_{\max}[G(rz)]$, or $\|G\|_\infty := \operatorname{ess\,sup}_{\omega\in[-\pi,\pi]} \bar{\sigma}[G(\omega)]$ if $G:[-\pi,\pi]\to\mathbb{C}^{n\times n}$ is is (essentially) bounded)
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$C(e^{j\omega}I-A)^{-1}B+D$
$G*H$	Convolution between G and H
$N(m,P)$	Gaussian distribution of mean m and covariance P
$\xrightarrow{d}$	Convergence in distribution
$\hat{\theta}$	Estimator of the quantity θ
$\mathrm{E}\{X\}$	Expected value of the random variable (or vector) X
$\bar{E}\{f_t\}$	$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E\{f_t\}$
$\mathrm{Var}\,X$	Variance of the random variable X
$\mathrm{Rel.\,Var}\{X\}$	Relative variance of the random variable X (i.e., $\mathrm{Rel.\,Var}\{X\} := \mathrm{Var}\{X\}/\|\mathrm{E}\{X\}\|^2$)
$\mathrm{Cov}\,\hat{\theta}$	Covariance matrix of $\hat{\theta}$
Φ_x	Spectrum of the stochastic process x
j	Imaginary number (i.e., $j := \sqrt{-1}$)
$\mathrm{Re}\,z$	Real part of z
$\mathrm{Im}\,z$	Imaginary part of z
z^*	Complex conjugate of z
$\mathrm{sgn}(x)$	Sign function (i.e. $\mathrm{sgn}(x)=1$ if $x>0$, $\mathrm{sgn}\,x=-1$ if $x<0$ and $\mathrm{sgn}\,x=0$ if $x=0$)
$\delta(x)$	Dirac Delta function
$\delta_{i,k}$ (or δ_k)	Kronecker Delta function ($\delta_k := \delta_{k,0}$)
$\mu(x)$	Heaviside step function
$\mathrm{Si}(x)$	Sine Integral (i.e., $\mathrm{Si}(x) = \int_0^x [\sin(t)/t]dt$)
$F_n(x)$	Fejér Kernel (i.e., $F_n(x) = \sin^2((n+1)x/2)/[(n+1)\sin^2(x/2)]$)
$O(f(n))$	Ordo $f(n)$, i.e. a function tending to zero at least at the same rate as $f(n)$ as $n\to\infty$
$\min_x f(x)$	Minimum of $f(x)$
$\max_x f(x)$	Maximum of $f(x)$
$\inf_x f(x)$	Infimum of $f(x)$
$\sup_x f(x)$	Supremum of $f(x)$
$\operatorname{ess\,sup}_x f(x)$	Essential supremum of $f(x)$
$\mathrm{supp}\,f$	Support of f

$\arg\min f(x)$	Argument which minimises $f(x)$
$\arg\max f(x)$	Argument which maximises $f(x)$

Abbreviations and Acronyms

ARMAX	AutoRegressive Moving Average model with eXogeneous input
ARX	AutoRegressive model with eXogeneous input
BJ	Box-Jenkins model
FIR	Finite Impulse Response model
LMI	Linear Matrix Inequality
LP	Linear Programming
ML	Maximum Likelihood
MPC	Model Predictive Control
OE	Output Error Model
PEM	Prediction Error Method
SISO	Single-Input Single-Output system
s.t.	Subject To

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