Generation of Amplitude Constrained Signals with a Prescribed Spectrum

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Abstract

In this paper a real-time method for generating signals of constrained amplitude and a given (arbitrary) spectrum is presented. This technique is based on the concatenation of sinusoidal signals of suitably chosen frequencies in order to obtain a signal with the desired sample autocovariance sequence as the number of samples tends to infinity. The effectiveness of the method is demonstrated theoretically and via simulations.

1 Introduction

The problem of generating a waveform having specified second order properties arises in many fields, see for example Cule and Torquato (1999); Gujar and Kavanagh (1968); Koutsourelakis and Deodatis (2005); Liu and Munson (1982); Sheehan and Torquato (2001); Yeong and Torquato (1998a,b). For example, in experiment design (Goodwin and Payne; 1977; Ljung; 1999) one typically obtains an optimal test signal specified in terms of its spectral properties. This leads to the problem of implementing a real signal with a specified spectrum, or spectral density. Moreover, it is usual that the input should also be constrained in its amplitude, i.e. the amplitude must lie in an interval $[a, b] \subset \mathbb{R}$. In general, frequency domain techniques do not work properly with this kind of constraint, and as such are translated into an 'equivalent' power constraint under which the input is designed to satisfy the conditions.

In many applications it is important to implement an input signal which, within the constraints of its amplitude, has maximum power. This is the case, for example, in experiment design, where the quality of the estimation typically increases with the signal to noise ratio. The signal to noise ratio is obviously improved by choosing an input with high power. Binary signals have precisely this desirable property: their power is maximum for a given amplitude constraint (Tan and Godfrey; 2001).

Several techniques have been proposed to design a binary signal with a given autocovariance (see e.g. van den Bos and Krol (1979); Boufounos (2007); Cule and Torquato (1999); Gujar and Kavanagh (1968); Koutsourelakis and Deodatis (2005); Liu and Munson (1982); Rojas, Welsh and Goodwin (2007); Sheehan and Torquato (2001); Yeong and Torquato (1998a,b) and the references therein). For example, in Rojas, Welsh and Goodwin (2007) a technique based on Model Predictive Control (Goodwin et al.; 2001) is developed, where, for each time instant, a finite horizon optimisation problem to find the optimal set of the next, say, T values of the sequence such that the sampled autocovariance sequence so obtained is as close as possible (in a prescribed sense) to the desired autocovariance. One then takes the first term of this optimal set for the sequence, advances time by one step and repeats the procedure.

It is known, however, that binary processes cannot have an arbitrary autocovariance sequence (De Carvalho and Clark; 1983; Karakostas and Wynn; 1993; Masry; 1972; McMillan; 1955). Therefore, in this paper we relax the binary constraint and concentrate on the problem of generating a sequence with bounded amplitude and prescribed autocovariance. The algorithm proposed here provides a quasi-stationary sequence whose sample autocovariance sequence has guaranteed convergence for arbitrarily prescribed spectral densities. The algorithm is very fast and easy to implement, requiring from the user only the ability to generate independent random variables with a given distribution (for which several algorithms are available (Devroye; 1986)). The resulting signal has a crest factor (the quotient between its squared amplitude and its power (Ljung; 1999)) of approximately 2, while a binary signal has a crest factor of 1. In addition, the algorithm works in real time, i.e., it is not necessary to specify a priori the number of samples to be generated, and the method can be extended so that the de-

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sired spectrum can be modified during the execution of the algorithm. In this case the generated signal would have a spectrum equal to the last one being prescribed, if this spectrum is kept fixed from some time on. This is advantageous in some applications, such as adaptive experiment design (Gerencsér and Hjalmarsson; 2005), where the spectrum of the input is designed in real time, based on a recursively estimated model.

To demonstrate the application of the algorithm, two examples, motivated by experiment design, are provided. A typical input signal used in system identification is bandlimited white noise (Ljung; 1999, Section 13.3). In this paper we show how the proposed algorithm can be used to generate this type of signal and also provide the obtained spectral density to highlight how closely it approximates the desired spectral density. The second example is inspired by recent work on experiment design where it was shown that a more robust input for a particular class of systems is in fact one with a bandlimited '$1/f$' spectrum (Rojas, Welsh, Goodwin and Feuer; 2007; Goodwin et al.; 2006). We again provide the spectral density generated by the proposed algorithm as well as that of the prescribed signal, for the purpose of comparison.

The paper is structured as follows. In Section 2 we present the algorithm and provide a detailed explanation. In Section 3 we prove convergence of the sample autocovariance coefficients of the signal generated by the algorithm to their desired values. Section 4 shows the results of some numerical examples that illustrate the quality of the signals generated by the algorithm. We present conclusions in Section 5.

2 The Proposed Method

In this section we introduce the proposed method for generating signals of constrained amplitude and prespecified spectral density. The idea of the method comes from the following simple observation (inspired by Example 10-4 of Papoulis (1991)):

Lemma 2.1 *Let $\Phi \in \mathcal{L}_1([-\pi,\pi],\mathbb{R}_0^+)$ be such that $(2\pi)^{-1}\int_{-\pi}^{\pi}\Phi(\omega)d\omega = 1$. Let ω and ϕ be independent random variables, where ω has density $\Phi/2\pi$ and ϕ is uniformly distributed in $[-\pi,\pi]$. Then, $\{y_t\}_{t\in\mathbb{N}}$, where $y_t := \sqrt{2}\cos(\omega t+\phi)$, is a sequence of random variables of zero mean and spectral density Φ.*

PROOF. By direct calculation we have

$$\mathrm{E}\, y_t = \sqrt{2}\,\mathrm{E}\cos\omega t\,\mathrm{E}\cos\phi - \sqrt{2}\,\mathrm{E}\sin\omega t\,\mathrm{E}\sin\phi = 0,$$

since $\mathrm{E}\cos\phi = (2\pi)^{-1}\int_{-\pi}^{\pi}\cos\phi d\phi = 0$ and similarly $\mathrm{E}\sin\phi = 0$. In addition,

$$\begin{aligned}\mathrm{E}\, y_t y_s &= \mathrm{E}\cos(\omega[t+s])\,\mathrm{E}\cos 2\phi \\ &\quad - \mathrm{E}\sin(\omega[t+s])\,\mathrm{E}\sin 2\phi + \mathrm{E}\cos(\omega[t-s]) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega[t-s])\Phi(\omega)d\omega,\end{aligned}$$

which shows that $\{y_t\}_{t\in\mathbb{N}}$ has spectral density Φ. □

Lemma 2.1 shows how to construct a signal having a given spectral density and range $[-\sqrt{2},\sqrt{2}]$. However, the resulting stochastic process is not ergodic, since

$$\frac{1}{n}\sum_{t=1}^{n} y_t = \frac{\sqrt{2}}{n}\sum_{t=1}^{n}\cos(\omega t+\phi) \xrightarrow[n\to\infty]{} 0$$

but, as $n\to\infty$,

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n} y_t y_{t+m} &= \frac{1}{n}\sum_{t=1}^{n}[\cos(2\omega t+\omega m+2\phi)+\cos\omega m] \\ &\to \cos\omega m \neq \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega m)\Phi(\omega)d\omega.\end{aligned}$$

In fact, $\{y_t\}_{t\in\mathbb{N}}$ is a *purely predictable process* (Papoulis; 1991). One way to overcome this problem is to split the time into intervals, and to use different (independent) random variables ω,ϕ for each interval. This suggests the following algorithm to generate $\{y_t\}_{t\in\mathbb{N}}$ having a desired spectral density $\Phi \in \mathcal{L}_1([-\pi,\pi],\mathbb{R}_0^+)$ of total power 1:

(1) Choose integers $1 = n_0 < n_1 < \cdots$.
(2) Generate i.i.d. (independent and identically distributed) random variables $\omega_1,\dots,\omega_L$ with density $\Phi/2\pi$, and i.i.d. variables $\phi_1,\dots,\phi_L$ with uniform distribution on $[-\pi,\pi]$ (independent of $\omega_1,\dots,\omega_L$).
(3) For every $t\in\mathbb{N}$, define $y_t = \sqrt{2}\cos(\omega_q t+\phi_q)$, where q is the unique integer such that $n_{q-1} < t \le n_q$.

The performance of this method depends on the choice of the integers $\{n_q\}_{q\in\mathbb{N}_0}$. We show below that a sufficient condition for convergence of the algorithm is that the strictly increasing sequence $\{n_q\}_{q\in\mathbb{N}_0}\subseteq\mathbb{N}$ (with $n_0 = 1$) satisfy the following conditions:

$$\sum_{q=1}^{\infty}\left(\frac{n_q-n_{q-1}}{q}\right)^2 < \infty. \tag{1}$$

$$\lim_{q\to\infty} n_q - n_{q-1} = \infty. \tag{2}$$

An example of such a sequence is

$$n_q = \left\lfloor 1+\sum_{k=1}^{q} k^{\gamma}\right\rfloor, \quad \gamma\in(0,1/2). \tag{3}$$

Remark 2.1 *The signal generated by the proposed algorithm is evidently nonstationary. However, it is* quasi-stationary *(Ljung; 1999), since its expected value is bounded by* $\sqrt{2}$ *and, as established in the next section (c.f. Theorem 3.1) its sample autocovariance coefficients converge to the prescribed ones with probability* 1*. This implies that its spectral density is well defined, in the quasi-stationary sense described in Ljung (1999), and equals the prescribed* Φ*.*

It is important to remark that quasi-stationarity (and its associated concept of spectral density) is a relaxation of the concept of stationarity. Quasi-stationarity provides a suitable framework for most applications, particularly in relation to system identification and experiment design. For more details, the reader is referred to Ljung (1999).

Remark 2.2 *The algorithm presented in this section has close connections to, so-called, Schroeder's method (Schroeder; 1970), which provides a simple rule for generating multi-sinusoidal signals with low peak factor. Indeed, (Schroeder; 1970) considered a multi-sinusoid as an approximation of a frequency modulated signal, whose spectrum (according to Woodward's theorem (Blachman and McAlpine; 1969)) is approximately proportional to the distribution of its instantaneous frequency. The algorithm proposed here is based on a similar idea, since it constructs a signal from segments of sinusoids whose frequencies are distributed according to the spectral density* Φ*. This suggests the possibility of extending the proposed technique to generate continuous time 'chirp' signals with a prescribed spectrum. However, for reasons of space we will not pursue this idea in the present paper.*

Remark 2.3 *There exist several techniques in the literature for generating signals with limited amplitude and a prescribed spectrum. In addition to the references cited in the Introduction, many researchers have proposed crest factor minimization techniques (see, e.g., Schroeder (1970); Guillaume et al. (1991) and the references therein) which produce multi-sinusoidal signals with low crest factor and prescribed amplitudes. While the method proposed in this paper has some connections with these techniques (c.f. Remark 2.2), it is not restricted to line spectra (i.e., multi-sines). This can be seen as an advantage in areas such as experiment design, because signals with a continuous spectrum excite the entire frequency range, thus providing some degree of robustness against the lack of knowledge of the true system. The interested reader is referred to (Ljung; 1999), where further comments on the use of periodic and non-periodic signals in system identification is provided.*

3 Analysis of Convergence

In this section we study the convergence of the sample covariance sequence of the signal generated by the algorithm presented in Section 2, i.e., we establish that

$$R_m^N := \frac{1}{N}\sum_{t=m+1}^{N} y_t y_{t-m}, \quad N \in \mathbb{N}, \quad m \in \mathbb{N}_0$$

converges almost surely to $r_m := \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega m)\Phi(\omega)d\omega$. To this end, let us first define (for $q \in \mathbb{N}$)

$$S_{q,m} := \sum_{t=n_q+1}^{n_q+m} y_t y_{t-m},\ W_{q,m} := \sum_{t=n_q+m+1}^{n_{q+1}} y_t y_{t-m}. \quad (4)$$

We then have the following result:

Theorem 3.1 *Consider the algorithm of Section 2, where* $\{n_q\}_{q\in\mathbb{N}_0} \subseteq \mathbb{N}$ *is a strictly increasing sequence (with* $n_0 = 1$*) satisfying* (1) *and* (2)*. Then,* $\lim_{N\to\infty} R_m^N = r_m$ *almost surely for every* m*.*

PROOF. By (2) there is an $M \in \mathbb{N}$ such that $n_{q+1} - n_q > m$ for every $q > M$. Consider then the decomposition

$$R_m^N = \frac{1}{N}\sum_{t=m+1}^{n_{M+1}} y_t y_{t-m} + \frac{1}{N}\sum_{t=n_{M+1}+1}^{n_T} y_t y_{t-m} \quad (5)$$
$$+ \frac{1}{N}\sum_{t=n_T+1}^{N} y_t y_{t-m},$$

where $T \in \mathbb{N}$ is such that $n_T + 1 \leq N < n_{T+1}$. The first term in the right side of (5) converges to 0 as $N \to \infty$. For the third term we also have that

$$\left|\frac{1}{N}\sum_{t=n_T+1}^{N} y_t y_{t-m}\right| < \frac{1}{n_T}\sum_{t=n_T+1}^{N} |y_t y_{t-m}|$$
$$< 2\frac{n_{T+1} - n_T}{n_T}.$$

However, by part 2 of Lemma A.1 (see the Appendix), $(n_{T+1} - n_T)/n_T \to 0$. Therefore, the third term in (5) also vanishes as $N \to \infty$. Hence we need only focus on the second term of the right-hand side of (5). Now (recall the definitions in (4)),

$$\begin{aligned}
&\frac{1}{N}\sum_{t=n_{M+1}+1}^{n_T} y_t y_{t-m} \\
&\quad = \frac{1}{N}\sum_{q=M+1}^{T-1}\left[\sum_{t=n_q+1}^{n_q+m} y_t y_{t-m} + \sum_{t=n_q+m+1}^{n_{q+1}} y_t y_{t-m}\right] \\
&\quad = \frac{1}{N}\sum_{q=M+1}^{T-1}(S_{q,m} + W_{q,m}). \qquad (6)
\end{aligned}$$

By construction, $\{S_{q,m}\}_{q=M+1}^{T-1}$ and $\{W_{q,m}\}_{q=M+1}^{T-1}$ are sets of independent random variables, respectively. For $S_{q,m}$, part 1 of Lemma A.2 implies that $\sum_{q=M+1}^{\infty} \mathrm{Var}\{S_{q,m}\}/q^2 \leq \sum_{q=M+1}^{\infty} m^2/q^2 < \infty$, while for $W_{q,m}$, part 2 of Lemma A.2 and condition (1) implies

$$\begin{aligned}
&\sum_{q=M+1}^{\infty} \mathrm{Var}\{W_{q,m}\}/q^2 \qquad (7)\\
&\quad \leq (1 + r_{2m}/2 - r_m^2) \sum_{q=M+1}^{\infty} (n_q - n_{q-1} - m)^2/q^2\\
&\quad < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{q=M+1}^{T-1} (S_{q,m} + W_{q,m} - (n_{q+1} - n_q - m)r_m)\\
&\quad = \lim_{N\to\infty} \frac{T-M-1}{N} \frac{1}{T-M-1}\\
&\qquad \cdot \sum_{q=M+1}^{T-1} (S_{q,m} + W_{q,m} - (n_{q+1} - n_q - m)r_m)\\
&\quad = \lim_{N\to\infty} \frac{T-M-1}{N} \lim_{T\to\infty} \frac{1}{T-M-1} \cdot \qquad (8)\\
&\qquad \sum_{q=M+1}^{T-1} (S_{q,m} + [W_{q,m} - (n_{q+1} - n_q - m)r_m]).
\end{aligned}$$

The first limit in (8) is zero, because for $T > M+1$, $|(T-M-1)/N| < T/(n_T+1)$, which tends to 0 as $T\to\infty$ by part 1 of Lemma A.1. On the other hand, the second limit in (8) is also zero, by Kolmogorov's strong law of large numbers (Chung; 2001, Theorem 5.4.1), Lemma A.2 (see the Appendix) and (7) (together with the analogous result obtained for $\mathrm{Var}\{S_{q,m}\}$). Therefore, expression (8) is equal to zero. Now,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{q=M+1}^{T-1} (n_{q+1} - n_q - m)r_m\\
&\quad = r_m \lim_{N\to\infty} \frac{1}{N} \sum_{q=M+1}^{T-1} (n_{q+1} - n_q - m) \qquad (9)\\
&\quad = r_m \lim_{N\to\infty} \left[\frac{n_T - n_{M+1}}{N} - m\frac{T-M-1}{N} \right].
\end{aligned}$$

The first term in brackets tends to 1, since

$$\frac{n_T - n_{M+1}}{n_{T+1}} < \frac{n_T - n_{M+1}}{N} \leq \frac{n_T - n_{M+1}}{n_T + 1},$$

and both sides of this inequality tend to 1 as $N\to\infty$ (by part 2 of Lemma A.1). The second term in brackets in (9) tends to 0 by part 1 of Lemma A.1. This means that (9) is equal to r_m. Finally, combining (6), (8) and (9) gives

$$\begin{aligned}
&\frac{1}{N} \sum_{t=n_{M+1}+1}^{n_T} y_t y_{t-m}\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{q=M+1}^{T-1} (S_{q,m} + W_{q,m} - (n_{q+1} - n_q - m)r_m)\\
&\quad + \lim_{N\to\infty} \frac{1}{N} \sum_{q=M+1}^{T-1} (n_{q+1} - n_q - m)r_m\\
&= r_m,
\end{aligned}$$

which concludes the proof. □

4 Numerical Examples

In this section we present two examples. The first example deals with the problem of generating pseudo random signals (i.e. pseudo white noise). The second example relates to the generation of bandlimited '$1/f$' noise. Such signals have recently been shown to possess important robustness properties in experiment design (Rojas, Welsh, Goodwin and Feuer; 2007).

4.1 Pseudo White Noise

Consider the problem of generating a signal of amplitude $\sqrt{2}$ and flat spectral density, i.e., $\Phi(\omega) = 1$. In order to execute the algorithm of Section 2, we need to generate independent samples $\omega_1, \omega_2, \ldots$ with density $1/2\pi$ in $[-\pi, \pi]$. This can be easily done in most programming languages. For example, in Matlab one can use the command `2*pi*(rand(1)-0.5)`.

Figure 1 shows the sample spectral density and autocorrelation of a signal generated with the proposed algorithm (considering the sequence (3) with $\gamma = 0.49$, and 100000 samples). Figure 2 shows part of the signal generated (in the time domain).

Figure 3 presents a histogram of the distribution of the values of the generated signal. The distribution of such values is consistent with the fact that the algorithm delivers blocks (of increasing length) of sinusoids of random frequency and phase. Indeed, the empirical distribution of the values of a sinusoid $\sqrt{2}\cos(\omega t + \phi)$ ($t \in \mathbb{N}$), where ω is irrational, converges, by Weyl's Equidistribution Theorem (Körner; 1988), to the distribution of $A\cos x$, where x is uniformly distributed in $[-\pi, \pi]$, whose density is $1/\pi\sqrt{2-x^2}$ (for $|x| < \sqrt{2}$) (Papoulis; 1991, page 97).

Finally, Figure 4 exhibits the evolution of a typical realization of the algorithm, by presenting the maximum of the absolute values of the sample autocorrelation sequence R_m^t for lags $1 \leq m \leq 50$, as a function of the time

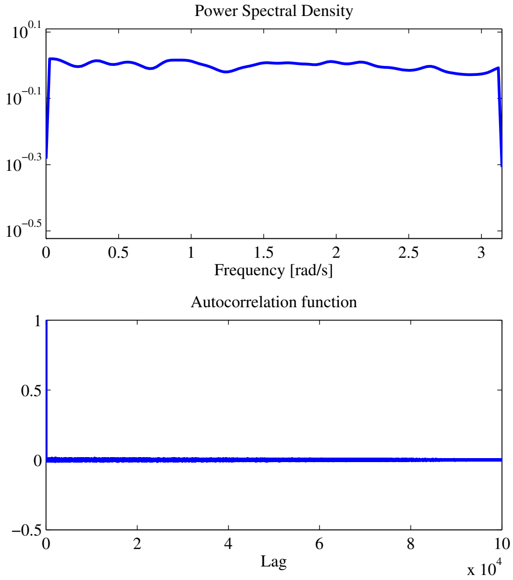

Fig. 1. Sample spectral density and autocorrelation function of a pseudo white noise signal generated by the proposed algorithm.

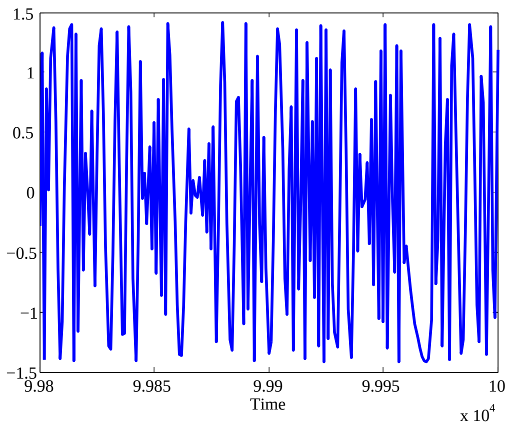

Fig. 2. Last 200 samples of a pseudo white noise signal generated by the proposed algorithm.

t. According to Theorem 3.1, this quantity should decay to zero almost surely as $t \to \infty$, in agreement with the plot of Figure 4.

4.2 Bandlimited '1/f' Noise

Bandlimited '$1/f$' noise is defined by the following (unilateral) spectral density:

$$\Phi^{1/f}(\omega) := \begin{cases} \dfrac{\pi/\omega}{\ln \overline{\omega} - \ln \underline{\omega}}, & \omega \in [\underline{\omega}, \overline{\omega}] \\ 0, & \text{otherwise,} \end{cases}$$

where $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$ $(\underline{\omega} < \overline{\omega})$. In this case, the proposed algorithm can be implemented by generating the variables

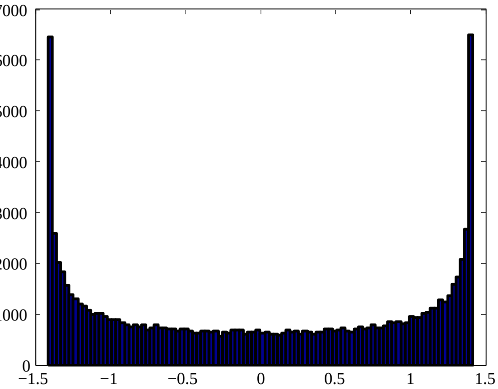

Fig. 3. Histogram of 100000 samples of a pseudo white noise signal generated by the proposed algorithm.

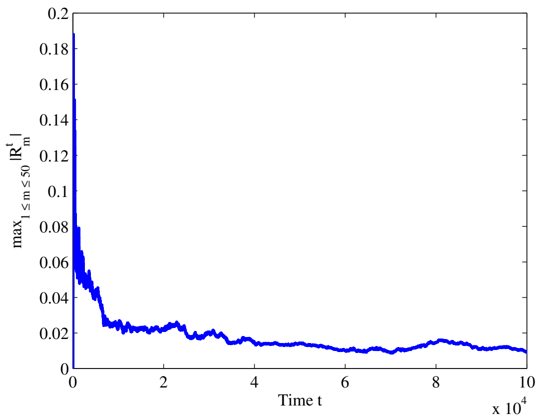

Fig. 4. Evolution of the maximum of the absolute values of the sample autocorrelation sequence R_m^t for lags $1 \le m \le 50$, as a function of the time t, for a typical realization.

ω_q as $\omega_q = \underline{\omega}(\overline{\omega}/\underline{\omega})^x$, where x is uniformly distributed in $[0, 1]$ (Devroye; 1986).

Figure 5 shows the ideal spectral density of bandlimited '$1/f$' noise for $\underline{\omega} = 1$, $\overline{\omega} = 2$, and the results obtained with the proposed algorithm (for $\gamma = 0.499$ and 10000 samples). This figure verifies the ability of the algorithm to generate an amplitude constrained non-white noise signal.

5 Conclusions

In this paper we have presented a novel method for generating signals of constrained amplitude with a specified spectral density. The algorithm is based on a blocking technique from probability. The algorithm is simple and straightforward to implement. It generates signals whose sample spectral density exhibits fast convergence as verified by simulation studies. In addition, we have established the convergence of the sample autocovariance sequence of the signal generated by the algorithm for arbitrary spectral densities.

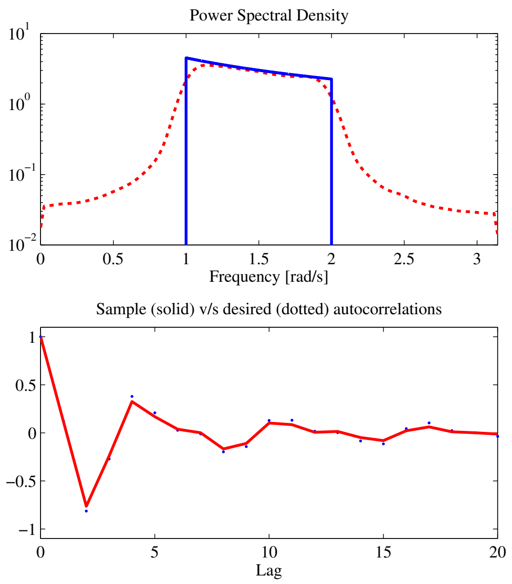

Fig. 5. Sample spectral density (red dotted line) and autocorrelation function of a bandlimited '$1/f$' noise signal generated by the proposed algorithm. The ideal bandlimited '$1/f$' noise spectrum for $\underline{\omega} = 1$ and $\overline{\omega} = 2$ is also shown (solid blue line).

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A Appendix: Technical Lemmas

Lemma A.1 *Let $\{n_q\}_{q\in\mathbb{N}_0}$ be a strictly increasing sequence in $\mathbb{N}$ satisfying* (1) *and* (2)*. Then,*

1) $\lim_{q\to\infty} q/n_q = 0$.

2) $\lim_{q\to\infty}(n_q - n_{q-1})/n_{q-1} = 0$ *(or, equivalently,* $\lim_{q\to\infty} n_q/n_{q-1} = 1$*).*

PROOF. 1) Let $\epsilon > 0$. By (2), there is an $M \in \mathbb{N}$ such that $n_q - n_{q-1} > 2/\epsilon$ for every $q \geq M$. Then,

$$\left|\frac{q}{n_q}\right| = \left|\frac{q}{n_0 + \sum_{k=1}^{Q-1}(n_k - n_{k-1}) + \sum_{k=Q}^{q}(n_k - n_{k-1})}\right|$$

$$< \left|\frac{q}{1 + \sum_{k=1}^{Q-1}(n_k - n_{k-1}) + 2(q - Q + 1)/\epsilon}\right|$$

$$= \epsilon\left|\frac{q}{2(q - Q + 1) + \epsilon + \epsilon\sum_{k=1}^{Q-1}(n_k - n_{k-1})}\right|.$$

The last expression can be made less than ϵ for all q sufficiently large. This establishes 1).

2) By (1), it must hold that $\lim_{q\to\infty}(n_q - n_{q-1})/q = 0$. Combining this equation with 1) gives

$$\lim_{q\to\infty}\frac{n_q - n_{q-1}}{n_{q-1}} = \lim_{q\to\infty}\frac{n_q - n_{q-1}}{q}\frac{q}{q-1}\frac{q-1}{n_{q-1}} = 0.$$

□

Lemma A.2 *Let $N \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then (c.f.* (4)*),*

1) $\mathrm{E}\, S_{q,m} = 0$ *and* $\mathrm{Var}\,\{S_{q,m}\} = \sum_{t,s=n_q+1}^{n_q+m} r^2_{|t-s|} \leq m^2$ *for all* $q \in \mathbb{N}$.

2) If $m < n_q - n_{q-1}$ *we have that* $\mathrm{E}\, W_{q,m} = (n_{q+1} - n_q - m)r_m$ *and* $\mathrm{Var}\,\{W_{q,m}\} \leq (n_q - n_{q-1} - m)^2(1 + r_{2m}/2 - r_m^2)$.

PROOF. 1) Notice that $S_{q,m}$ is a sum of products of independent random variables (y_t and y_{t-m}). Therefore, by Lemma 2.1,

$$\mathrm{E}\, S_{q,m} = \sum_{t=n_q+1}^{n_q+m} \mathrm{E}\, y_t y_{t-m} = \sum_{t=n_q+1}^{n_q+m} \mathrm{E}\, y_t\, \mathrm{E}\, y_{t-m} = 0$$

and

$$\begin{aligned}\mathrm{Var}\,\{S_{q,m}\} &= \sum_{t=n_q+1}^{n_q+m}\sum_{s=n_q+1}^{n_q+m} \mathrm{E}\, y_t y_{t-m} y_s y_{s-m} \\ &= \sum_{t=n_q+1}^{n_q+m}\sum_{s=n_q+1}^{n_q+m} \mathrm{E}\, y_t y_s\, \mathrm{E}\, y_{t-m} y_{s-m} \\ &= \sum_{t=n_q+1}^{n_q+m}\sum_{s=n_q+1}^{n_q+m} r_{t-s}^2.\end{aligned}$$

The proof is concluded by noting that $r_\tau = r_{-\tau}$ and $|r_\tau| \leq r_0 = (2\pi)^{-1}\int_{-\pi}^{\pi}\Phi(\omega)d\omega = 1$ for all $\tau \in \mathbb{N}_0$.

2) If $m < n_q - n_{q-1}$, then the products $y_t y_{t-m}$ in the definition of $W_{q,m}$ are related to the same random variables ω_{q+1}, ϕ_{q+1}. Hence, by Lemma 2.1,

$$\mathrm{E}\, W_{q,m} = \sum_{t=n_q+m+1}^{n_{q+1}} \mathrm{E}\, y_t y_{t-m} = (n_{q+1} - n_q - m) r_m.$$

Now,

$$\mathrm{E}\, W_{q,m}^2 = \sum_{t=n_q+m+1}^{n_{q+1}}\sum_{s=n_q+m+1}^{n_{q+1}} \mathrm{E}\, y_t y_{t-m} y_s y_{s-m}$$

where, after some algebra,

$$\begin{aligned}&\mathrm{E}\, y_t y_{t-m} y_s y_{s-m} \\ &\quad= 4\,\mathrm{E}\cos(\omega_{q+1}t + \phi_{q+1})\cos(\omega_{q+1}[t-m] + \phi_{q+1}) \\ &\qquad\cdot\cos(\omega_{q+1}s + \phi_{q+1})\cos(\omega_{q+1}[s-m] + \phi_{q+1}) \\ &\quad= \frac{1}{2}r_{2(t-s)} + \frac{1}{2}r_{2m} + \frac{1}{2}.\end{aligned}$$

Therefore, using the fact that $|r_\tau| \leq 1$ for all $\tau \in \mathbb{N}_0$,

$$\begin{aligned}\mathrm{Var}\,\{W_{q,m}\} &= \mathrm{E}\, W_{q,m}^2 - [\mathrm{E}\, W_{q,m}]^2 \\ &= \frac{1}{2}\sum_{t,s=n_q+m+1}^{n_{q+1}}[r_{2(t-s)} + r_{2m} + 1] \\ &\qquad - (n_{q+1} - n_q - m)^2 r_m^2 \\ &\leq (n_{q+1} - n_q - m)^2\left(1 + \frac{1}{2}r_{2m} - r_m^2\right).\end{aligned}$$

□