Finite Model Order Optimal Input Design for Minimum Variance Control

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***Abstract*— It is well known that if we intend to use a minimum variance controller to stabilize a minimum phase plant (with exactly one time delay), which is designed based on a model obtained from an identification experiment, the best experiment which can be performed on the system to determine such a model (subject to output power constraints, or for some specific model structures) is to use the true minimum variance controller. This result has been derived under several circumstances, first using asymptotic (in model order) variance expressions but also more recently for ARMAX models of finite order. In this paper we re-approach this problem by using a recently developed geometric approach to variance analysis, which is non asymptotic in model order, with which we generalize some of the previous results established in the literature. We also believe the new derivations to be more transparent than earlier contributions.**

I. Introduction

Research in experiment design has been substantial both in the statistical literature [3, 19, 20] and in engineering [7, 11, 17, 21]. In particular, in the field of system identification it has been noted since a long time [6] the importance of focusing the problem of designing the experiment on the particular application for which the model will be used, e.g., prediction, control or simulation [13]. This gave rise to the area of system identification for control [5, 9].

In case we were interested in designing a minimum variance (MV) controller for a linear time invariant (LTI) system, it has been established in the literature that under several conditions, the model to be used for the MV design should be identified in closed loop, using the MV controller of the true system during the estimation stage [6, 10, 4, 8]. Specifically, this has been established under an output power constraint, and for models of large order in [6, 4], and for ARMAX models of finite order (subject to some degree and factorization conditions) under general input-output power constraints in [8].

Notice that the optimality of the MV controller does not hold for all situations. For instance, it has been established in [1] that for Box-Jenkins models, the best experiment to be applied for identification purposes, under an input power constraint, is an open loop experiment for a very general class of cost functions (which includes the one involved in the problem of designing a MV controller).

In this paper we reapproach the problem of designing an experiment for the purpose of constructing a MV controller. To this end, we utilize a recently developed geometric approach to variance analysis (see [14]). These techniques allow us to work with models of finite order in a simplified manner, and with them we will rederive and generalize some results from [8] in a transparent way.

The geometric approach developed in [14] can be explained as follows. Let us assume that unknown system parameters $\theta = [\theta_1 \ \cdots \ \theta_n] \in \mathbb{R}^{1\times n}$ (vectors will be taken as row vectors; we denote the true value by θ^o) are estimated using a data set consisting of measured inputs and outputs resulting in the parameter estimate $\hat{\theta}_N \in \mathbb{R}^{1\times n}$ which has the property that the (normalized) model error $\sqrt{N}(\hat{\theta}_N - \theta^o)$ becomes normal distributed as the sample size N of the data set grows to infinity

$$\sqrt{N}\left(\hat{\theta}_N - \theta^o\right) \in \mathrm{As}\mathcal{N}\left(0, \mathrm{AsCov}\,\hat{\theta}_N\right) \tag{1}$$

The asymptotic covariance matrix $\mathrm{AsCov}\,\hat{\theta}_N$ of the limit distribution is a measure of the model accuracy. This is reinforced by that, under mild conditions [13],

$$\lim_{N\to\infty} N \cdot \mathbf{E}\left[(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)^{\mathrm{T}}(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)\right] = \mathrm{AsCov}\,\hat{\theta}_N$$

In prediction error identification, which is the identification method we will consider,

$$\mathrm{AsCov}\,\hat{\theta}_N = \left[\langle \Psi, \Psi\rangle\right]^{-1} \tag{2}$$

where $\Psi : \mathbb{C} \to \mathbb{C}^{n\times 2}$ is the gradient with respect to the estimated parameters of the one-step ahead predictor, normalized by the inverse of the noise standard deviation, and where $\langle \Psi, \Psi\rangle$ denotes the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(\mathrm{e}^{\mathrm{j}\omega})\Psi^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega$ (superscript * denotes complex conjugate transpose). Furthermore, our interest will not be the model parameters θ themselves but some "system theoretic" quantity. We will let such a quantity be represented by a differentiable function $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times q}$. Given an estimate $\hat{\theta}_N$ of θ^o, a natural estimate of $J(\theta^o)$ is $J(\hat{\theta}_N)$. Using Gauss' approximation formula and (2), it can be shown [13], that the asymptotic covariance of $J(\hat{\theta}_N)$ is given by

$$\mathrm{AsCov}\ J(\hat{\theta}_N) = \Lambda^{\mathrm{T}}\left[\langle \Psi, \Psi\rangle\right]^{-1}\overline{\Lambda} \tag{3}$$

where Λ is the derivative $\Lambda \triangleq J'(\theta^o) \in \mathbb{C}^{n\times q}$.

As shown in [14, 16], (3) can be given the geometric interpretation that it is the projection of a given function onto the space spanned by the elements of the predictor gradient Ψ. This key insight is the core of the geometric approach we will exploit in the current paper.

This paper is structured as follows. In Section II we present the mathematical preliminaries of the geometrical

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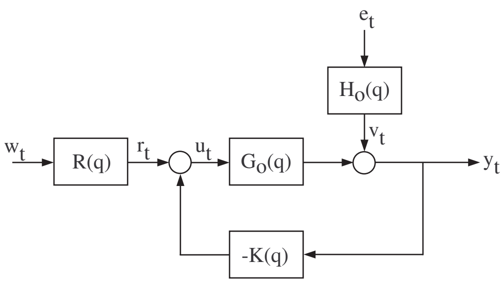

Fig. 1. Block diagram of SISO LTI system with output feedback

approach to variance analysis, developed in [14, 16, 15], which will be used in the sequel, together with the standing assumptions which will be considered in the remainder of the paper. Section III develops the core results of the paper, related to the problem of experiment design for the design of MV controllers. Finally, Section IV gives conclusions.

NOTATION

We will consider vector valued complex functions as row vectors and the inner product of two such functions $f, g : \mathbb{C} \to \mathbb{C}^{1\times m}$ is defined as $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{j}\omega})g^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega$ where g^* denotes the complex conjugate transpose of g. Likewise, $\overline{g}$ denotes the complex conjugate of g. When f and g are matrix-valued functions, we will still use the notation $\langle f, g\rangle$ to denote $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{j}\omega})g^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega$ whenever the dimensions of f and g are compatible. The $\mathcal{L}_2$-norm of $f : \mathbb{C} \to \mathbb{C}^{1\times m}$ is given by $\|f\| = \sqrt{\langle f, f\rangle}$, and its W-weighted $\mathcal{L}_2$-norm is given by $\|f\|_W = \sqrt{\langle Wf, f\rangle}$, where $W : \mathbb{C} \to \mathbb{R}_0^+$. The space $\mathcal{L}_2^{n\times m}$ consists of all functions $f : \mathbb{C} \to \mathbb{C}^{n\times m}$ such that all elements of f have bounded $\mathcal{L}_2$-norm and if $f \in \mathcal{L}_2^{n\times m}$, f is said to be an $\mathcal{L}_2^{n\times m}$-function; when $n = 1$, the notation is simplified to $\mathcal{L}_2^m$. For $f : \mathbb{C} \to \mathbb{C}^{n\times m}$, $f_i : \mathbb{C} \to \mathbb{C}^{1\times m}$ denotes the ith row of f. If $f \in \mathcal{L}_2^m$ and $\mathcal{S}$ is a closed subspace of $\mathcal{L}_2^m$, $\mathrm{Proj}_{\mathcal{S}}\{f\}$ denotes the orthogonal projection of f on $\mathcal{S}$. If $\Psi \in \mathcal{L}_2^{n\times m}$ for some positive integers n and m, then $\mathcal{S}_\Psi$ denotes the subspace to $\mathcal{L}_2^m$ generated by the span of the rows of Ψ. For a finite dimensional subspace $\mathcal{S}$ to $\mathcal{L}_2^m$, $\dim \mathcal{S}$ is the dimension of $\mathcal{S}$.

For a differentiable function $f : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times q}$, $f'(\bar{x})$ is a $n \times q$ matrix with $\frac{\mathrm{d}f_j(x)}{\mathrm{d}x_i}\big|_{x=\bar{x}}$ as ijth entry, the partial derivative $\frac{\partial f(\bar{x})}{\partial x_i}$ is defined analogously.

For two matrices A and B, $A > B$ ($A \geq B$) denotes that $A - B$ is positive definite (semidefinite).

II. PRELIMINARIES

In this section we present the assumptions and mathematical preliminaries from [14] which are necessary for developing the results of Section III.

A. System and model

Assumption 2.1: The true system is given by the single-input single-output (SISO) LTI system $G_o(\mathrm{q})$ (q is the forward shift operator) depicted in Figure 1 where u_t and y_t represent the measured input and output, respectively, where e_t and w_t are zero mean white noise sequences with variance λ_o and 1, respectively, and bounded moments of order $4+\delta$ for some $\delta > 0$. The LTI filter R represents the stable minimum phase spectral factor of the reference signal r_t, and H_o is an inversely stable LTI filter that is normalized to be monic, i.e., $\lim_{z\to\infty} H_o(z) = 1$. The system G_o includes at least one unit time delay, and we assume the entire system to be internally stabilized by the LTI controller K. ■

Next, we introduce a quite general family of model structures that will be covered.

Assumption 2.2: The system is modelled by

$$y_t = T(\mathrm{q}, \theta)\chi_t \tag{4}$$

where $T(\mathrm{q}, \theta) = [G(\mathrm{q}, \theta) \;\; H(\mathrm{q}, \theta)]$ is an LTI model of the system and the noise dynamics, parameterized by the vector $\theta \in \mathbb{R}^{1\times n}$, and where $\chi_t = [u_t, \;\; e_t]^{\mathrm{T}}$. The noise model may also be independently parameterized by a separate vector η, and then we write $H(\mathrm{q}, \eta)$. This distinction is only used when it has important implications and for the general treatment we can consider the noise model $H(\mathrm{q}, \theta)$.

The model parametrization is such that the *true* system is in the model set, that is, there is a parameter θ^o such that

$$G_o(\mathrm{q}) \triangleq G(\mathrm{q}, \theta^o), \quad H_o(\mathrm{q}) \triangleq H(\mathrm{q}, \theta^o)$$

Finally, following the definitions in [13], the model structure is uniformly stable and globally identifiable at θ^o. ■

The type of model described above includes all standard black-box model structures such as ARMAX, output error and Box-Jenkins.

B. Geometric Approach to Variance Analysis

The main result in [14, 16] is the linking of (3) to orthogonal projection which is embodied in the following theorem.

Theorem 2.1 (Theorem A.2.5 in [14]): Suppose that $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times q}$ is differentiable and let the asymptotic covariance matrix $\mathrm{AsCov}\ J(\hat{\theta}_N)$ be defined by (3) where $\Psi \in \mathcal{L}_2^{n\times m}$. Suppose that $\gamma \in \mathcal{L}_2^{q\times m}$ is such that

$$\Lambda = \langle \Psi, \gamma\rangle \tag{5}$$

then

$$\mathrm{AsCov}\ J(\hat{\theta}_N) = \big\langle \mathrm{Proj}_{\mathcal{S}_\Psi}\{\gamma\}, \mathrm{Proj}_{\mathcal{S}_\Psi}\{\gamma\}\big\rangle^{\mathrm{T}} \tag{6}$$

where $\mathcal{S}_\Psi$ is the space spanned by the rows of Ψ.

In particular, when J is scalar,

$$\mathrm{AsVar}\, J(\hat{\theta}_N) = \|\mathrm{Proj}_{\mathcal{S}_\Psi}\{\gamma\}\|^2 \tag{7}$$

The applicability of Theorem 2.1 for rewriting the asymptotic covariance matrix as (6) hinges on whether there exists an $\mathcal{L}_2$-function γ such that (5) holds. Lemma A.2.6 in [14] completely characterizes the family of such functions.

C. Asymptotic covariance matrix for the parameters

Under Assumptions 2.1 and 2.2, the asymptotic covariance matrix $\mathrm{AsCov}\,\hat{\theta}_N$ obtained in prediction error identification can be written as (2) with Ψ being the prediction error gradient [13]. By expressing the signal pair $\chi_t = [u_t \;\; e_t]^{\mathrm{T}}$

in terms of $\xi_t = [w_t \;\; e_t]^{\mathrm{T}}$, which has a stable spectral factor given by

$$R_\chi := \begin{bmatrix} S_o R & -K S_o H_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_o} \end{bmatrix} \tag{8}$$

where $S_o(\mathrm{q}) = 1/(1 + K(\mathrm{q})G_o(\mathrm{q}))$ is the closed loop sensitivity function, Ψ in (2) (which is closely related to the prediction error gradient) is given by

$$\Psi(z) = T'(z, \theta^o) R_{\mathrm{SNR}}(z) \tag{9}$$

where $T'(z,\theta) = \begin{bmatrix} \frac{\partial G(z,\theta)}{\partial \theta} & \frac{\partial H(z,\theta)}{\partial \theta} \end{bmatrix}$, $R_{\mathrm{SNR}}(z) = R_\chi(z) R_v^{-1}(z)$ is a signal-to-noise spectral factor, and $R_v(z) = \sqrt{\lambda_o} H_o(z)$.

D. Asymptotic covariance of LTI system properties

In this section we will derive an expression for the asymptotic covariance (3) of the estimate $J(\hat{\theta}_N)$ of an arbitrary differentiable quantity $J : \mathbb{R}^{1\times n} \to \mathbb{C}^{1\times q}$ when Ψ in (3) is given by (9). While this can be done on a case by case basis for different model structures using Theorem 2.1, we will instead use (a generalization of) impulse response coefficients as an intermediate parametrization in order to obtain an expression that is valid regardless of the model structure.

Take $\{\mathcal{G}_k(z)\}_{k=1}^{\infty}$ and $\{\mathcal{H}_k(z)\}_{k=1}^{\infty}$ to be two orthonormal bases for $\mathcal{H}_2$ and for $k = 1, 2, \ldots$ define the orthonormal functions

$$\mathcal{T}_{2k-1}(z) = [\mathcal{G}_k(z) \;\; 0],\; \mathcal{T}_{2k}(z) = [0 \;\; \mathcal{H}_k(z)] \tag{10}$$

With $\tau = [\tau_1 \;\; \tau_2 \;\; \cdots]$, any transfer function $T = [G \;\; H]$ satisfying Assumption 2.1 can be parameterized as

$$T(z) = [G(z) \;\; H(z)] = \sum_{k=1}^{\infty} \tau_k \, \mathcal{T}_k(z) \tag{11}$$

With $\mathcal{G}_k(z) = \mathcal{H}_k(z) = z^{-k}$, this corresponds to the usual impulse response representation.

Also the original model (4), which is parameterized by the vector θ, can be expressed through the parametrization (11):

$$T(z,\theta) = \sum_{k=1}^{\infty} \tau_k(\theta) \mathcal{T}_k(z) \tag{12}$$

or

$$G(z,\theta) = \sum_{k=1}^{\infty} g_k(\theta) \mathcal{G}_k(z), \quad H(z,\theta) = \sum_{k=1}^{\infty} h_k(\theta) \mathcal{H}_k(z)$$

where $g_k = \tau_{2k-1}, \; h_k = \tau_{2k}$.

Theorem 2.2: Let the true system be given by $T_o(z) = [G_o(z) \; H_o(z)]$, defined from a τ_o in the parametrization (11), and suppose that the quantity $J_\tau(\tau_o) \in \mathbb{C}^{1\times q}$ is estimated by $J(\hat{\theta}_N) = J_\tau(\tau(\hat{\theta}_N))$ where $\tau(\cdot)$ is the map from model parameters θ to the system parameters τ.

Suppose that J_τ and $\tau(\theta)$ are differentiable. Define $\mathcal{S}_\Psi \subset \mathcal{L}_2^2$ to be the subspace spanned by the rows of (9), and define

$$\nabla J_\tau(z) \triangleq \sum_{k=1}^{\infty} \left(\frac{\partial J_\tau(\tau)}{\partial \tau_k} \right)^* \mathcal{T}_k(z) \Bigg|_{\tau=\tau(\theta^o)} \tag{13}$$

where R_{SNR} is defined as in (8). Assume that $\nabla J_\tau R_{\mathrm{SNR}}^{-*} \in \mathcal{L}_2^{q\times 2}$, and that J_τ and τ are such that the chain rule applies:

$$J'(\theta) = \sum_{k=1}^{n_\tau} \tau_k'(\theta) \frac{\partial J_\tau(\tau(\theta))}{\partial \tau_k} \tag{14}$$

Then the asymptotic covariance (3) of $J(\hat{\theta}_N)$ can be expressed as

$$\begin{aligned} &\mathrm{AsCov}\, J(\hat{\theta}_N) \\ &\quad = \left\langle \mathrm{Proj}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau R_{\mathrm{SNR}}^{-*} \right\}, \mathrm{Proj}_{\mathcal{S}_\Psi} \left\{ \nabla J_\tau R_{\mathrm{SNR}}^{-*} \right\} \right\rangle^{\mathrm{T}} \end{aligned} \tag{15}$$

Notice that ∇J_τ is weighted by $R_{\mathrm{SNR}}^{-*}(z^{-*})$ which is a spectral factor of the ratio $\Phi_v(z)\Phi_\chi^{-1}(z)$. This ratio is known from the expression

$$\lim_{m\to\infty} \frac{1}{m} \mathrm{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \Phi_v(\mathrm{e}^{\mathrm{j}\omega}) \Phi_\chi^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega}) \tag{16}$$

derived in [12] and can be interpreted as the frequency-wise noise to signal ratio.

E. An explicit expression of the asymptotic variance

The most complicated step of evaluating the variance expression (15) is the projection onto the space $\mathcal{S}_\Psi$. If an orthonormal basis $\{\mathcal{B}_k(z)\}_{k=1}^{n}$ of $\mathcal{S}_\Psi$ is known, then the projection can be computed from

$$\mathrm{Proj}_{\mathcal{S}_\Psi}\{f\} \triangleq \sum_{k=1}^{n} \langle f, \mathcal{B}_k \rangle \mathcal{B}_k$$

For some properties J, the asymptotic variance (15) can be expressed directly as a function of an orthonormal basis for the space $\mathcal{S}_\Psi$, and this result is presented in the following theorem.

Theorem 2.3: Consider the conditions of Theorem 2.2. If, in addition, the condition

$$\frac{\partial J_\tau(\tau)}{\partial \tau_k} \Bigg|_{\tau=\tau(\theta^o)} = \mathcal{T}_k(z_o)\alpha \tag{17}$$

holds for some $\alpha \in \mathbb{C}^{2\times q}$ and $z_o \in \mathbb{C}$ such that $\sum_{k=1}^{\infty} \mathcal{T}_k^*(z_o)\mathcal{T}_k \in \mathcal{H}_2$, then the asymptotic covariance can be expressed as

$$\begin{aligned} &\mathrm{AsCov}\, J(\hat{\theta}_N) \\ &\quad = \alpha^{\mathrm{T}} R_{\mathrm{SNR}}^{-\mathrm{T}}(z_o) \sum_{k=1}^{n} \mathcal{B}_k^{\mathrm{T}}(z_o) \overline{\mathcal{B}_k(z_o)}\; \overline{R_{\mathrm{SNR}}^{-1}(z_o)}\; \overline{\alpha} \end{aligned} \tag{18}$$

where $\{\mathcal{B}_k\}_{k=1}^{n}$ is any orthonormal basis for the space $\mathcal{S}_\Psi$.

F. Frequency response

For the covariance of the frequency response estimate, i.e. $J(\theta) = T(\mathrm{e}^{\mathrm{j}\omega}, \theta)$, we obtain

$$\Lambda = T'(\mathrm{e}^{\mathrm{j}\omega}, \theta^o) = \Psi(\mathrm{e}^{\mathrm{j}\omega}) R_{\mathrm{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega})$$

where Ψ is given by (9). Theorem 2.3 can be applied to obtain the covariance expression

$$\begin{aligned} &\mathrm{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) \\ &\quad = R_{\mathrm{SNR}}^{-\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega}) \sum_{k=1}^{n} \mathcal{B}_k^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega}) \overline{\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega})}\, \overline{R_{\mathrm{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega})} \end{aligned} \tag{19}$$

where $\{\mathcal{B}_k\}_{k=1}^n$ is any orthonormal basis for the space $\mathcal{S}_\Psi$. This expression has also been derived in [18] using the theory of reproducing kernels.

III. Minimum variance control

When the true system G_o is minimum phase with exactly one pure time delay, the minimum variance (MV) controller for the configuration in Figure 1 is given by

$$K_{\text{MV}}(G_o, H_o) = \frac{H_o - 1}{G_o}$$

and this controller produces the output $y_t = e_t$ with variance λ_o. The performance of any controller K can be measured by the variance of the output as compared to the optimal white output e_t:

$$V(K) \triangleq \mathbf{E}\left[(y_t - e_t)^2\right]$$

Replacing the true system with an estimated model gives the certainty equivalence controller $K_{\text{MV}}(G(\text{q}, \hat{\theta}_N), H(\text{q}, \hat{\theta}_N))$. We will now analyze

$$J(\hat{\theta}_N) \triangleq V(K_{\text{MV}}(G(\text{q}, \hat{\theta}_N), H(\text{q}, \hat{\theta}_N)))$$

In [6, 4] it is shown that

$$\delta J \triangleq \lim_{N\to\infty} N \cdot \mathbf{E}J(\hat{\theta}_N) = \|X_o\|^2_{\text{AsCov}\, T(\cdot, \hat{\theta}_N)} \tag{20}$$

where $X_o = \frac{\sqrt{\lambda_o}}{H_o}\begin{bmatrix}\frac{H_o - 1}{G_o} & -1\end{bmatrix}$ and where the expectation is taken over the experimental conditions in the identification experiment. In [6, 4], $\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$ is approximated with the result (16) (asymptotic in model order), yielding

$$\delta J \approx \delta J_{\text{ho}} \triangleq m\|X_o\|^2_{\Phi_v \Phi_\chi^{-\mathrm{T}}} \tag{21}$$

where m is the model order (which may differ from the number of estimated parameters). This δJ_{ho} is subsequently used for optimizing the experimental conditions, with the result that the MV control is the optimal experiment, yielding $\delta J_{\text{ho}} = \lambda_o m$, provided that this experiment does not violate any constraint on the experiment.

We will now re-examine (20) and instead use the exact expression (19) for $\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$.

We will assume that the model structure is identifiable, that the identification experiment is stable and persistently exciting of sufficient order so that the prediction error gradient Ψ, defined in (9), is in $\mathcal{L}_2^{n\times m}$ (for some integers n and m) and so that

$$\langle \Psi, \Psi \rangle > 0 \tag{22}$$

The condition (22) implies that the dimension of $\mathcal{S}_\Psi$, the space spanned by the rows of Ψ, equals the number of rows n of Ψ (see Lemma A.2.1 in [14]).

Now, take $\{\mathcal{B}_k\}_{k=1}^n$ to be an orthonormal basis for the subspace spanned by the rows of the predictor gradient (9) and denote the elements of $\mathcal{B}_k$ by $\mathcal{B}_k^G$ and $\mathcal{B}_k^H$ so that $\mathcal{B}_k = [\mathcal{B}_k^G\ \mathcal{B}_k^H]$. Notice that

$$1 = \|\mathcal{B}_k\|^2 = \|\mathcal{B}_k^G\|^2 + \|\mathcal{B}_k^H\|^2 \tag{23}$$

Now, replacing $\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$ in (20) by (19) yields

$$\delta J = \lambda_o \sum_{k=1}^n \left\| Z_o \mathcal{B}_k^G + \mathcal{B}_k^H \right\|^2 \tag{24}$$

where

$$Z_o = \frac{\sqrt{\lambda_o} H_o}{R}\left[\frac{H_o - 1}{G_o} - K\right] \tag{25}$$

and *K is the controller used in the identification stage*. To see this, notice that when using an MV controller, $S_o = 1/H_o$, and, by (19),

$$\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \frac{\lambda_o |H_o|^4}{|R|^2}\begin{bmatrix} 1 & 0 \\ K & \frac{R_o}{\sqrt{\lambda_o} H_o}\end{bmatrix} \cdot \left(\sum_{k=1}^n \mathcal{B}_k^{\mathrm{T}}(z_o)\overline{\mathcal{B}_k(z_o)}\right)\begin{bmatrix} 1 & \overline{K} \\ 0 & \overline{\frac{R_o}{\sqrt{\lambda_o} H_o}}\end{bmatrix}$$

Substituting this expression into (20) and using the decomposition of $\mathcal{B}_k$ gives (24).

We will now analyze (24) by studying the structure of the row space of the predictor gradient (9).

1) A global lower bound: Let n_H be the number of elements in $\theta = [\theta_1\ \cdots\ \theta_n]$ that only appear in the noise model H, and not in G, and let θ_k be such an element. Then the kth row in T' will have the form $[0\ X]$ for some transfer function X. Due to the structure of R_χ (see (8)), it then follows that also the kth row of Ψ will have the structure $[0\ \tilde{X}]$ for some transfer function $\tilde{X}$, and hence there will be n_H rows in Ψ with this structure. This, together with the linear independence of the rows of Ψ, by (22), in turn implies that n_H basis functions in $\{\mathcal{B}_k\}$ can be taken of the form $\mathcal{B}_k = [0\ \mathcal{B}_k^H]$ where, due to (23), $\|\mathcal{B}_k^H\| = 1$. Without lack of generality, we may assume these to be the first n_H functions, which gives

$$\begin{aligned}\delta J &= \lambda_o \sum_{k=1}^n \left\| Z_o \mathcal{B}_k^G + \mathcal{B}_k^H \right\|^2 \\ &= \lambda_o \sum_{k=1}^{n_H} \left\| \mathcal{B}_k^H \right\|^2 + \lambda_o \sum_{k=n_H+1}^{n} \left\| Z_o \mathcal{B}_k^G + \mathcal{B}_k^H \right\|^2 \\ &\geq \lambda_o \sum_{k=1}^{n_H} \left\| \mathcal{B}_k^H \right\|^2 = \lambda_o n_H \end{aligned} \tag{26}$$

Thus it is not possible to reduce the loss in performance below the number of parameters that only appear in the noise model, scaled by the noise variance. This is consistent with intuition - these parameters are identified using the driving noise only and therefore there is a lower bound for the accuracy with which these parameters can be identified.

2) An upper bound for the optimal performance: Let us denote by $\delta J|_{MV}$ the performance measure (20) when the MV control is used in the identification experiment. In this case Z_o collapses to $Z_o = 0$ and hence we have the following upper bound for the minimum cost

$$\min_K \delta J \leq \delta J|_{MV} = \lambda_o \sum_{k=1}^n \left\| \mathcal{B}_k^H \right\|^2 \leq \lambda_o n \tag{27}$$

where the inequality follows from (23). Thus (27) provides an upper bound for the minimum achievable δJ for *any* model structure.

3) ARMAX models: Let us now specialize to ARMAX models

$$A(\mathrm{q})y_t = B(\mathrm{q})u_t + C(\mathrm{q})e_t \tag{28}$$

where

$$\begin{aligned} A(\mathrm{q}) &= 1 + a_1\mathrm{q}^{-1} + \cdots + a_{n_a}\mathrm{q}^{-n_a} \\ B(\mathrm{q}) &= b_1\mathrm{q}^{-1} + \cdots + b_{n_b}\mathrm{q}^{-n_b} \\ C(\mathrm{q}) &= 1 + c_1\mathrm{q}^{-1} + \cdots + c_{n_c}\mathrm{q}^{-n_c} \end{aligned}$$

a case studied in [8]. We order the parameters as $\theta = [c_1 \;\cdots\; c_{n_c} \; a_1 \;\cdots\; a_{n_a} \; b_1 \;\cdots\; b_{n_b}]^T$. With the controller given by $K(\mathrm{q}) = K_N(\mathrm{q})/K_D(\mathrm{q})$ for some polynomials K_N and K_D and introducing the closed loop characteristic polynomial $A_c = A_oK_D + B_oK_N$ (where A_o and B_o denote the true A and B polynomials) it is straightforward to show that the predictor gradient Ψ is given by

$$\Psi = \frac{1}{\sqrt{\lambda_o}} \begin{bmatrix} 0 & \frac{\sqrt{\lambda_o}}{C_o}\Gamma_{n_c} \\ -\frac{RB_oK_D}{C_oA_c}\Gamma_{n_a} & -\frac{\sqrt{\lambda_o}K_D}{A_c}\Gamma_{n_a} \\ \frac{R\,A_oK_D}{C_oA_c}\Gamma_{n_b} & -\frac{\sqrt{\lambda_o}K_N}{A_c}\Gamma_{n_b} \end{bmatrix} \tag{29}$$

where $\Gamma_n(\mathrm{q}) = [\mathrm{q}^{-1} \;\cdots\; \mathrm{q}^{-n}]^T$.

Let us further assume MV control $K = (H_o - 1)/G_o = (C_o - A_o)/B_o$ which gives $A_c = B_oC_o$. Then

$$\Psi = \frac{1}{\sqrt{\lambda_o}} \begin{bmatrix} 0 & \frac{\sqrt{\lambda_o}}{C_o}\Gamma_{n_c} \\ -\frac{RB_o}{C_o^2}\Gamma_{n_a} & -\frac{\sqrt{\lambda_o}}{C_o}\Gamma_{n_a} \\ \frac{R\,A_o}{C_o^2}\Gamma_{n_b} & -\frac{\sqrt{\lambda_o}(C_o-A_o)}{B_oC_o}\Gamma_{n_b} \end{bmatrix} \tag{30}$$

Using Gram-Schmidt orthonormalization on the first n_c rows, we have that the first n_c functions of an orthonormal basis can be taken to have the structure

$$\mathcal{B}_k = [0 \; \mathcal{B}_k^H], \quad k = 1, \ldots, n_c$$

for some orthonormal $\mathcal{B}_k^H$. Furthermore, the first $\min(n_a, n_c)$ elements of $-\frac{\sqrt{\lambda_o}}{C_o}\Gamma_{n_a}$ are spanned by the elements of $\frac{\sqrt{\lambda_o}}{C_o}\Gamma_{n_c}$. Thus, using Gram-Schmidt, one can append $\min(n_a, n_c)$ functions

$$\mathcal{B}_k = [\mathcal{B}_k^G, \; 0], \quad k = n_c + 1, \ldots, n_c + \min(n_a, n_c) \tag{31}$$

to obtain an orthonormal basis for the span of the first $n_c + \min(n_a, n_c)$ rows of Ψ. By applying Gram-Schmidt to the ensuing rows of Ψ one finally obtain an orthonormal basis $\{\mathcal{B}_k\}_{k=1}^n$ for the row space of Ψ.

As already mentioned, when MV control is used in the identification experiment, Z_o collapses to $Z_o = 0$ and hence (24) is in this case given by

$$\delta J|_{MV} = \lambda_o \sum_{k=1}^{n} \left\|\mathcal{B}_k^H\right\|^2 \le \lambda_o(n - \min(n_a, n_c)) \tag{32}$$

where the inequality follows from (23) and (31). Combining (32) with (26), and using that $n = n_a + n_b + n_c$ and that $n_H = n_c$ for ARMAX models, gives

$$\lambda_o n_c \le \delta J|_{MV} \le \lambda_o(n_a + n_b + n_c - \min(n_a, n_c)) \tag{33}$$

Let us now further specialize to the case where

$$n_c \ge n_a, \quad \text{and} \quad C_o - A_o = B_oX \tag{34}$$

for some polynomial X of finite order. Consider again (30). Since the elements of $-\frac{(C_o-A_o)}{B_oC_o}\Gamma_{n_b}$ are spanned by the elements of $\frac{1}{C_o}\Gamma_{n_c}$ it follows that the last n_b functions of the orthonormal basis $\{\mathcal{B}_k\}$ have the same structure as (31). But then the upper bound (32) can be refined to

$$\delta J|_{MV} \le \lambda_o(n_a + n_b + n_c - n_a - n_b) = \lambda_o n_c$$

which when combined with the lower bound in (33), which holds independent of the experimental conditions, gives that MV control is optimal for identification of ARMAX systems subject to (34).

We summarize our results in the following theorem.

Theorem 3.1: Consider the performance degradation criterion (20) for certainty equivalence MV control. Suppose that the system and experimental conditions are given by Section II-A with the restrictions that the true system is minimum phase and contains exactly one pure time delay. Furthermore, assume that (22) holds. Then

i) regardless of the experimental conditions,

$$\delta J \ge \lambda_o \, n_H \tag{35}$$

where n_H is the number of parameters that only appear in the noise model.

ii) using MV control in the identification experiment, results in a cost

$$\delta J|_{MV} \le \lambda_o \, n$$

where n is the number of estimated parameters.

Furthermore, for ARMAX models (28),

i') using MV control in the identification experiment results in a cost (20) that satisfies the bounds

$$\lambda_o n_c \le \delta J|_{MV} \le \lambda_o(n_a + n_b + n_c - \min(n_a, n_c)) \tag{36}$$

ii') MV control is optimal and yields the minimum achievable cost $\delta J|_{MV} = \lambda_o \, n_c$ when

$$n_c \ge n_a, \quad \text{and} \quad C_o - A_o = B_oX$$

for some polynomial X of finite order.

Remarks 3.1:

i) We stress that the bound (35) is applicable to *any* model structure. The bound is quite natural. The accuracy of parameters that only appear in the noise model cannot be influenced by the input signal and thus limits the modeling accuracy. This bound has previously been established for the case of ARMAX models in Eqn. (7) of [8] through lengthy calculations.

ii) Result ii') in the theorem is a generalization of Theorem 1 of [8] which covers the case B_o constant and $n_c \geq n_a$.

iii) This theorem only holds in general when there are no input/output power constraints, since the MV controller might give an input which does not satisfy these constraints. In fact, for Box-Jenkins models, it has been shown in [1] that under input power constraints, the best experiment should be performed in open loop. However, if there is only an output power constraint, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(e^{j\omega})d\omega \leq c$$

where[1] $c > \lambda_o$, then the MV controller is feasible (and thus optimal), since in this case

$$\Phi_y = \left|\frac{G_o}{H_o}\right|^2 \Phi_r + \lambda_o$$

and the performance loss, $\delta J|_{MV}$, does not depend on the reference spectrum, hence the output power constraint can be satisfied by making R small enough.

IV. Conclusions

In this paper we have utilized a recently developed geometric approach to variance analysis [18, 16, 15, 14], to derive upper and lower bounds for the performance degradation in minimum variance control due to the model error, extending some recent results in [8], which at the same time are generalization of previous results from [6, 4] to models of finite order. We believe that the geometric approach to variance analysis complements algebraic derivations such those in [8], allowing new insights.

References

[1] Agüero, J. C. and Goodwin, G. C. [2007]. Choosing between open- and closed-loop experiments in linear system identification, *IEEE Transactions on Automatic Control* **52(8)**: 1475–1480.

[2] Åström, K. J. [1970]. *Introduction to Stochastic Control Theory*, Academic Press, New York.

[3] Fedorov, V. V. [1972]. *Theory of Optimal Experiments*, Academic Press, New York.

[4] Forssell, U. and Ljung, L. [2000]. Some results on optimal experiment design, *Automatica* **36**(5): 749–756.

[5] Gevers, M. [2005]. Identification for control: From the early achievements to the revival of experiment design, *European Journal of Control* **11**: 1–18.

[6] Gevers, M. and Ljung, L. [1986]. Optimal experiment designs with respect to the intended model application, *Automatica* **22**(5): 543–554.

[7] Goodwin, G. C. and Payne, R. L. [1977]. *Dynamic System Identification: Experiment Design and Data Analysis*, Academic Press, New York.

[1] Notice that the condition that $c > \lambda_o$ is not restrictive, since from the MV control theory [2] it is well known that the fact that u can only depend causally on y implies that λ_o is a lower bound on the output power.

[8] Hildebrand, R. and Solari, G. [2005]. Identification for control: optimal input intended to identify a minimum variance controller, *Automatica* **43**: 758–767.

[9] Hjalmarsson, H. [2005]. From experiment design to closed-loop control, *Automatica* **41**(3): 393–438.

[10] Hjalmarsson, H., Gevers, M. and De Bruyne, F. [1996]. For model-based control design, closed loop identification gives better performance, *Automatica* **32(12)**: 1659–1673.

[11] Jansson, H. [2004]. *Experiment design with applications in identification for control*, PhD thesis, Royal Institute of Technology (KTH). TRITA-S3-REG-0404.

[12] Ljung, L. [1985]. Asymptotic variance expressions for identified black-box transfer function models, *IEEE Transactions on Automatic Control* **30(9)**: 834–844.

[13] Ljung, L. [1999]. *System Identification: Theory for the User, 2nd Edition*, Prentice Hall, Upper Saddle River, New Jersey.

[14] Mårtensson, J. [2007]. *Geometric Analysis of Stochastic Model Errors in System Identification*, PhD thesis, Royal Institute of Technology (KTH). TRITA-EE 2007:061.

[15] Mårtensson, J. and Hjalmarsson, H. [2007a]. A geometric approach to variance analysis in system identification: Linear time-invariant systems, *46th IEEE Conference on Decision and Control*, New Orleans, USA.

[16] Mårtensson, J. and Hjalmarsson, H. [2007b]. A geometric approach to variance analysis in system identification: Theory and nonlinear systems, *46th IEEE Conference on Decision and Control*, New Orleans, USA.

[17] Mehra, R. K. [1974]. Optimal input signals for parameter estimation in dynamic systems – survey and new results, *IEEE Transactions on Automatic Control* **19(6)**: 753–768.

[18] Ninness, B. and Hjalmarsson, H. [2004]. Variance error quantifications that are exact for finite-model order, *IEEE Transactions on Automatic Control* **49(8)**: 1275–1291.

[19] Pukelsheim, F. [1993]. *Optimal Design of Experiments*, John Wiley, New York.

[20] Silvey, S. D. [1980]. *Optimal Design: An Introduction to the Theory for Parameter Estimation*, Chapman and Hall, London.

[21] Zarrop, M. [1979]. *Optimal Experiment Design for Dynamic System Identification*, Vol. 21 of *Lecture Notes in Control and Information*, Springer, Berlin, New York.