EL3370 Mathematical Methods in Signals, Systems and Control

Topic 5: Orthogonal Expansions

Cristian R. Rojas

Division of Decision and Control Systems
KTH Royal Institute of Technology

Outline

Outline

Orthonormal Sets

The notion of basis is very important, since it allows to define "coordinates" in a space, thus allowing explicit computations in Hilbert spaces.

Definition

In an inner product space V, a family $(e_\alpha)_{\alpha \in I}$ in $V \setminus \{0\}$ is an *orthogonal set* if $e_\alpha \perp e_\beta$ for $\alpha \neq \beta$. If also $\|e_\alpha\| = 1$ for all $\alpha \in I$, $(e_\alpha)_{\alpha \in I}$ is an *orthonormal set*. In case I is finite, $\mathbb{N}$ or $\mathbb{Z}$, (e_α) is an *orthogonal*/*orthonormal sequence*.

Examples of orthonormal sets

1. In $\mathbb{C}^n$, take the standard basis vectors.
2. In ℓ_2, take $(e_n)_{n \in \mathbb{N}}$ with $e_n = (0, \dots, 0, 1, 0, \dots)$. (The 1 is in the n-th position.)
3. In $L_2[-\pi, \pi]$, take $(e_n)_{n \in \mathbb{Z}}$, with $e_n(t) = (2\pi)^{1/2} e^{int}$ for $n \in \mathbb{Z}$. (*Fourier basis*)

Orthonormal Sets (cont.)

Definition
If (e_n) is an orthonormal sequence in a Hilbert space H, then, for every $x \in H$, (x, e_n) is the *n-th Fourier coefficient of x w.r.t.* (e_n), and $\sum_{n=1}^{\infty} (x, e_n) e_n$ is the *Fourier series w.r.t.* (e_n).

Lemma
Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space V; $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x \in V$. Then, $\left\| x - \sum_{k=1}^{n} \lambda_k e_k \right\|^2 = \|x\|^2 + \sum_{k=1}^{n} |\lambda_k - c_k|^2 - \sum_{k=1}^{n} |c_k|^2$, where $c_k := (x, e_k)$.
(*Exercise!*)

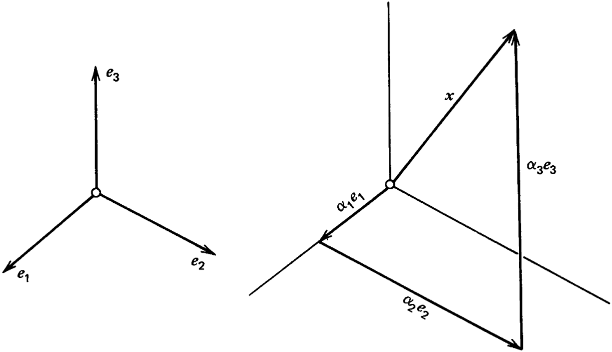
Since $\{e_1, \ldots, e_n\}$ span $\text{lin}\{e_1, \ldots, e_n\}$, we have

Theorem
Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space V. The closest point y of $\text{lin}\{e_1, \ldots, e_n\}$ to a point $x \in V$ is $y = \sum_{k=1}^{n} (x, e_k) e_k$, and $\|x - y\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, e_k)|^2$.

Orthonormal Sets (cont.)

Corollary

If $x \in \operatorname{lin}\{e_1, \ldots, e_n\}$, then $x = \sum_{k=1}^{n} (x, e_k) e_k$, and $\|x\|^2 = \sum_{k=1}^{n} |(x, e_k)|^2$.



$\alpha_k = (x, e_k)$

Outline

Bessel Inequality

Theorem (Bessel Inequality)
If (e_n) is an orthonormal sequence in an inner product space V, and $x \in V$, then

$$\sum_{n=1}^{\infty} |(x, e_n)|^2 \leqslant \|x\|^2.$$

Proof. For $N \in \mathbb{N}$, $\sum_{k=1}^{N} |(x, e_k)|^2 = \|x\|^2 - \left\|x - \sum_{k=1}^{N} (x, e_k) e_k\right\|^2 \leqslant \|x\|^2$. Take $N \to \infty$. $\square$

We want to study the meaning of $\sum_{k=1}^{\infty} (x, e_k) e_k$.

Definition (Infinite sum in a normed space)
Let (x_n) be a sequence in a normed space V. We say that $\sum_{n=1}^{\infty} x_n$ *converges* and has *sum* x (*i.e.*, $\sum_{n=1}^{\infty} x_n = x$) if $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$, *i.e.*, $\left\|x - \sum_{n=1}^{N} x_n\right\| \to 0$ as $N \to \infty$.

Theorem

Let (e_n) is an orthonormal sequence in a Hilbert space H, and let (λ_n) be a sequence in $\mathbb{C}$. Then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in H iff $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

Proof

($\Rightarrow$) Let $x = \sum_{n=1}^{\infty} \lambda_n e_n$ and $x_N = \sum_{n=1}^{N} \lambda_n e_n$. Then, $(x_N, e_n) = \lambda_n$ for $n \leqslant N$, and taking $N \to \infty$ gives $(x, e_n) = \lambda_n$. Then, by Bessel inequality: $\sum_{n=1}^{\infty} |\lambda_n|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 \leqslant \|x\|^2 < \infty$.

($\Leftarrow$) Assume that $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$, and let $x_N = \sum_{n=1}^{N} \lambda_n e_n$. Then,

$$\|x_{N+P} - x_N\|^2 = \left\| \sum_{n=N+1}^{N+P} \lambda_n e_n \right\|^2 = \sum_{n=N+1}^{N+P} \|\lambda_n e_n\|^2 = \sum_{n=N+1}^{N+P} |\lambda_n|^2 \to 0 \quad \text{as } N \to \infty.$$

Therefore, (x_n) is Cauchy, and it converges in H. $\square$

Observation

If $H = L_2[a, b]$, then the above convergence is *in norm* (or L_2 convergence).
A *different* type is *point-wise convergence*: $\sum_{n=1}^{\infty} x_n(t) = x(t)$ for all $t \in [a, b]$.

Outline

Total Orthonormal Sequences

Goal: When does $\sum_{n=1}^{\infty}(x,e_n)e_n = x$? $\quad\Rightarrow\quad$ We need conditions on (e_n).

Observation

The approximation error $y = x - \sum_{n=1}^{\infty}(x,e_n)e_n$ satisfies $(y,e_k) = 0$ for all k. Thus, we could ensure that $\sum_{n=1}^{\infty}(x,e_n)e_n = x$ if $y = 0$ is the only vector s.t. $(y,e_k) = 0$ for all k.

Definitions

- An orthonormal set A in an inner product space V is *maximal* if the only point in V which is orthogonal to every $x \in A$ is 0, *i.e.*, A cannot be extended to a larger orthonormal set.
- A set A in a normed space V is *total* (or *fundamental*) if its span is dense in V.
- If A is a total orthonormal set in an inner product space V, every $x \in V$ can be written as $x = \sum_{e\in A}(x,e)e$, and A is called an *orthonormal basis* of V.

Note. By Bessel's inequality, given $x \in V$ and an orthonormal set A, since $\sum_{e\in A}|(x,e)|^2 \leqslant \|x\|^2$, at most a countable number of terms (x,e), as e runs over A, can be non-zero: for every $n \in \mathbb{N}$, the number of terms s.t. $|(x,e)|^2 > 1/n$ can be at most $n\|x\|^2$, and $\{e \in A : (x,e) \neq 0\} = \bigcup_{n\in\mathbb{N}}\{e \in A : |(x,e)|^2 > 1/n\}$, which is at most countable. Thus, sums like $\sum_{e\in A}(x,e)e$ and $\sum_{e\in A}|(x,e)|^2$ can be reduced to sums over sequences.

Total Orthonormal Sequences (cont.)

Theorem. If A is an orthonormal set in a Hilbert space H, the following are equivalent:

(1) A is total.

(2) $\|x\|^2 = \sum_{e\in A} |(x,e)|^2$ for all $x \in H$.

(3) $(x,y) = \sum_{e\in A}(x,e)(e,y)$ for all $x,y \in H$.

(4) A is maximal.

If H is an incomplete inner product space, then (1)-(3) are still equivalent, and they imply (4), but not conversely (*see bonus slides for an example*).

Proof

(1) $\Leftrightarrow$ (2): For a given $x \in H$, sort the elements of $\{e \in A : (x,e) \neq 0\}$ into a sequence (e_n). Then, take $N \to \infty$ in $\sum_{n=1}^{N} |(x,e_n)|^2 = \|x\|^2 - \left\| x - \sum_{n=1}^{N}(x,e_n)e_n \right\|^2$.

(1) $\Rightarrow$ (3): As before, let $N \to \infty$ in $\sum_{n=1}^{N}(x,e_n)(e_n,y) = \left(\sum_{n=1}^{N}(x,e_n)e_n, y\right)$, and recall the continuity of the inner product.

(3) $\Rightarrow$ (2): (2) is a special case of (3), where $x = y$.

(2) $\Rightarrow$ (4): If A is not maximal, take a nonzero $x \perp A$. Then $\|x\|^2 > 0 = \sum_{e\in A} |(x,e)|^2$.

(4) $\Rightarrow$ (1): Given an $x \in H$, $\sum_{e\in A}(x,e)e$ is convergent (due to the completeness of H), and $x - \sum_{e\in A}(x,e)e$ is orthogonal to every $e \in A$, so by maximality of A, $x = \sum_{e\in A}(x,e)e$, which implies that A is an orthonormal basis.

Only the implication (4) $\Rightarrow$ (1) requires H to be complete. $\square$

Remarks. By Zorn's Lemma, every inner product space V has a maximal orthonormal set A (*why?*). Also, if V is complete, $\operatorname{clin} A = V$; otherwise $V = \operatorname{clin} A \oplus (\operatorname{clin} A)^\perp$, so $(\operatorname{clin} A)^\perp \neq \{0\}$ and there is an $x \in (\operatorname{clin} A)^\perp$ of unit norm, so $A \cup \{x\}$ is also orthonormal, contradicting the maximality of A.

Total Orthonormal Sequences (cont.)

Separability in Inner Product Spaces

Theorem. Let H be an inner product space. Then,

(1) If H is separable, then every orthonormal set in H is countable.

(2) If H contains a total orthonormal sequence, then H is separable.

Proof

(1) If $A \subseteq H$ is an orthonormal set, distinct points $x, y \in A$ are at a distance $\|x-y\| = \sqrt{(x-y, x-y)} = \sqrt{2}$, so if A were uncountable, a set dense in H would be uncountable too.

(2) If (e_n) is a total orthonormal set, consider the set D, consisting of all linear combinations $\lambda_1 e_1 + \cdots + \lambda_n e_n$ where $n \in \mathbb{N}$ and $\lambda_k = a_k + i b_k$ with $a_k, b_k \in \mathbb{Q}$ for $k = 1, \ldots, n$. D is a countable set dense in H (*why?*). □

Observation: A separable Hilbert space is isomorphic to $\mathbb{C}^n$ (for some n) or to ℓ_2 (*see bonus slides for proof*).

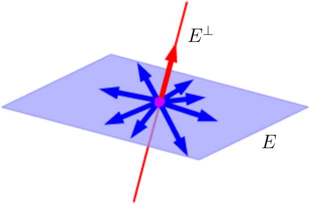
See bonus slides for examples of non-separable Hilbert spaces.

Outline

Orthogonal Complements

We can use orthogonality to decompose a Hilbert space.

Definition
Let H be a Hilbert space. The *orthogonal complement* of $E \subseteq H$ is $E^{\perp} := \{x \in H : x \perp E\}$.



Theorem
For every subset E of a Hilbert space, $E^{\perp}$ is a closed linear space. (*Exercise!*)

The projection theorem gives the following characterization of $E^{\perp}$:

Lemma
Let M be a linear subspace of an inner product space V, and let $x \in V$. Then $x \in M^{\perp}$ iff $\|x - y\| \geqslant \|x\|$ for all $y \in M$.

Definition

Let $M, N \subseteq V$, where V is a vector space. V is the *direct sum* of M and N, denoted $V = M \oplus N$, if every $x \in V$ has a *unique* decomposition $x = y + z$, where $y \in M$ and $z \in N$.

Theorem

Let M be a closed linear subspace of a Hilbert space H. Then, $H = M \oplus M^\perp$.

Proof. Let $x \in H$. Assume that $M \neq \{0\}$ (otherwise the result is trivial). Take $y \in M$ as the unique minimizer of $\inf_{m \in M} \|x - m\|$, and $z := x - y$. By the projection theorem, $z \in M^\perp$.
If $x = y' + z'$, with $y' \in M$ and $z' \in M^\perp$, then $(x - y') \perp M$, so by the projection theorem, $y' = y$, which proves the uniqueness of the decomposition. $\square$

Corollary

If M is a closed linear subspace of a Hilbert space H, then $(M^\perp)^\perp = M$.

Proof. By definition, $M \subseteq (M^\perp)^\perp$. Let $x \in (M^\perp)^\perp$, and write it as $x = y + z$ with $y \in M$ and $z \in M^\perp$. Since $x \perp M^\perp$, $0 = (x, z) = (y + z, z) = (y, z) + \|z\|^2 = \|z\|^2$, so $z = 0$ and $x \in M$. $\square$

Outline

Let $e_n(t) := (2\pi)^{-1/2} e^{int}$, $t \in [-\pi, \pi]$, $n \in \mathbb{Z}$. We want to prove that $(e_n)_{n\in\mathbb{Z}}$ is total in $L_2[-\pi, \pi]$.

We need to show that $\mathrm{clin}\{e_n : n \in \mathbb{Z}\} = L_2[-\pi, \pi]$. It is known that the closure of $C[-\pi, \pi]$ is $L_2[-\pi, \pi]$, so it is enough to show that for every $f \in C[-\pi, \pi]$ there is a sequence in $\mathrm{clin}\{e_n : n \in \mathbb{Z}\}$ converging to f. An obvious choice is $f_N = \sum_{n=-N}^{N} (f, e_n) e_n$, but it is easier to work with

$$F_m = \frac{1}{m+1}(f_0 + f_1 + \cdots + f_m), \quad m = 0, 1, \ldots \qquad (\textit{Césaro sum of the } f_N\textit{'s})$$

Since $(f, e_n) = (2\pi)^{-1/2} \displaystyle\int_{-\pi}^{\pi} f(t) e^{-int} dt$, we have

$$F_m(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau) K_m(t-\tau) d\tau, \quad \text{where} \quad K_m(x) := \frac{1}{m+1} \sum_{N=0}^{m} \sum_{n=-N}^{N} e^{-inx}. \quad (\textit{Fejér kernel})$$

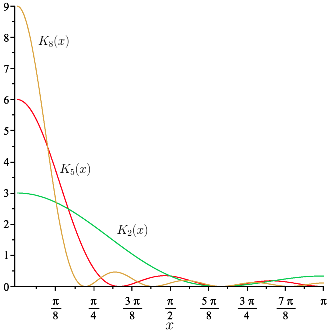
Fejér Kernel properties:

(1) $K_m(x) \geqslant 0$ for all $x \in \mathbb{R}$, $m = 0, 1, 2, \dots$

(2) $\displaystyle\int_{-\pi}^{\pi} K_m(x)dx = 2\pi$, for $m = 0, 1, 2, \dots$

(3) For all $0 < \delta < \pi$, $\displaystyle\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(x)dx \to 0$ as $m \to \infty$. (*see bonus slides for proofs*)

Therefore, $(K_m/2\pi)$ is a *Delta sequence* (it "converges" to a Dirac delta).



We will prove a strong result: $\displaystyle\lim_{m\to\infty} \sup_{t\in[-\pi,\pi]} |f(t) - F_m(t)| = 0$.

$$\Rightarrow \|f - F_m\|_2^2 = \int_{-\pi}^{\pi} |f(t) - F_m(t)|^2 dt \leqslant 2\pi \sup_{t\in[-\pi,\pi]} |f(t) - F_m(t)|^2 \to 0 \quad \text{as } m \to \infty. \text{ (}L_2\text{ convergence)}$$

Classical Fourier Series (cont.)

Take a $\delta > 0$ (to be defined more precisely later):

$$\begin{aligned}
|f(t) - F_m(t)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(t) - f(\tau)] K_m(t-\tau) d\tau \right| \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - f(\tau)| K_m(t-\tau) d\tau \\
&= \frac{1}{2\pi} \left(\int_{\substack{|t-\tau|>\delta \\ -\pi\leqslant\tau\leqslant\pi}} + \int_{t-\delta}^{t+\delta} \right) |f(t) - f(\tau)| K_m(t-\tau) d\tau.
\end{aligned}$$

For the first integral, we use the fact that f is bounded, *i.e.*, there is an $M > 0$ s.t. $\sup_{t\in[-\pi,\pi]} |f(t)| \leqslant M$, hence

$$\frac{1}{2\pi} \int_{\substack{|t-\tau|>\delta \\ -\pi\leqslant\tau\leqslant\pi}} |f(t) - f(\tau)| K_m(t-\tau) d\tau \leqslant \frac{2M}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_m(\tau) d\tau. \quad \text{(This is negligible as } m \to \infty\text{.)}$$

For the second integral, we need to recall *uniform continuity*:

Definition (reminder). Given metric spaces (X, d_X) and (Y, d_Y), $f : X \to Y$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Reminder. By Heine-Cantor's theorem, given metric spaces (X, d_X) and (Y, d_Y), if X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

Let $\varepsilon > 0$. Then, take δ as in the definition of uniform continuity, so

$$\frac{1}{2\pi} \int_{\substack{|t-\tau|<\delta \\ -\pi \leqslant \tau \leqslant \pi}} |f(t) - f(\tau)| K_m(t-\tau) d\tau \leqslant \frac{\varepsilon}{2\pi} \int_{\substack{|t-\tau|<\delta \\ -\pi \leqslant \tau \leqslant \pi}} K_m(t-\tau) d\tau \leqslant \varepsilon.$$

Therefore: $\displaystyle\sup_{t \in [-\pi, \pi]} |f(t) - F_m(t)| < \frac{M}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_m(\tau) d\tau + \varepsilon \to \varepsilon \quad \text{as} \quad m \to \infty.$

and since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \to 0$ gives $\displaystyle\lim_{m \to \infty} \sup_{t \in [-\pi, \pi]} |f(t) - F_m(t)| = 0.$ $\square$

We have actually proved

Theorem (Fejér)
Let $f : [-\pi, \pi] \to \mathbb{C}$ be continuous, $s_n(f)$ be the n-th partial sum of its Fourier series, and $\sigma_n(f)$ be the arithmetic mean of $s_0(f), \dots, s_n(f)$. Then $\sigma_n(f) \to f$ uniformly as $n \to \infty$.

Notice that $s_n(f)$ does not always converge point-wisely to continuous f. (*An example is provided in the bonus slides of Topic 8!*)

A similar result (proven analogously, with a different kernel) is

Theorem (Weierstrass theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous, where $-\infty < a < b < \infty$. For every $\varepsilon > 0$ there is a polynomial p s.t. $\sup_{t \in [a,b]} |f(t) - p(t)| < \varepsilon$.

Next Topic

Least Squares Estimation

Outline

Bonus: Example of Maximal Non-Total Orthonormal Sets

In every incomplete inner product space V there are maximal orthonormal sets which are not total, *i.e.*, whose closed linear span is not the entire space:

Proof. First note that if every proper, closed subspace M of V is s.t. $M^\perp \neq \{0\}$, then V is complete. Indeed, assume that V is incomplete, and let $\hat{V}$ be the completion of V. Pick an $x \in \hat{V} \setminus V$, and let $\hat{M} = \{x\}^\perp$ in $\hat{V}$. Then, $\hat{M} \cap V$ is closed in V (because $\hat{M}$ is closed in $\hat{V}$). If $x \perp V$, then $d(x,V) = \|x\| > 0$, and V would not be dense in $\hat{V}$; thus, $\hat{M} \cap V \neq V$, and there is a $y \in V$ s.t. $(x,y) \neq 0$, which we can normalize so that $(x,y) = 1$.

Note that $\hat{M} \cap V$ is dense in $\hat{M}$. Indeed, let $z \in \hat{M}$ and let (x_n) be a sequence in V s.t. $x_n \to z$ (which exists because $\overline{V} = \hat{V}$). Let $x'_n = x_n - (x_n,x)y$; then $x'_n \in V$, $(x'_n,x) = (x_n,x) - (x_n,x)(y,x) = 0$ so that $x'_n \in \hat{M}$, and $\|x'_n - z\| \leqslant \|x_n - z\| + |(x_n,x)|\|y\| \to 0 + |(z,x)|\|y\| = 0$, thus $x'_n \to z$. Then, $(\hat{M} \cap V)^\perp \cap V = (\overline{\hat{M} \cap V})^\perp \cap V$ $= \hat{M}^\perp \cap V = \text{lin}\{x\} \cap V = \emptyset$, so $M = \hat{M} \cap V$ is the sought proper, closed subspace of V.

Now, assume every maximal orthonormal set in an incomplete V is a basis, and let M be a closed, proper subspace of V s.t. $M^\perp = \{0\}$. Let B be a maximal orthonormal set in M, and extend it to a maximal orthonormal set $B \cup B_1$ for V. Assume $B_1 \neq \emptyset$, and let $x_1 \in B_1$; since $M^\perp = \{0\}$, there is a $y \in M$ s.t. $(y,x_1) \neq 0$. As $B \cup B_1$ is a basis, $y = \sum_k c_k y_k + \sum_k d_k x_k$ ($y_k \in B$, $x_k \in B_1$). Now, $z = \sum_k d_k x_k =$ $y - \sum_k c_k y_k \in M$, but $x_k \perp B$ for all k, hence $z \perp B$. As B is maximal in M, $z = 0$, so $(y,x_1) = d_1 = 0$, a contradiction. Hence, $B_1 = \emptyset$, B is a maximal orthonormal set for V, so B is a basis for V, *i.e.*, $M = V$, a contradiction. Thus, V contains a non-total maximal orthonormal set. $\square$

Bonus: Example of Maximal Non-Total Orthonormal Sets (cont.)

From this result, every incomplete inner product space has maximal non-total orthonormal sets. Here is a specific example:

Let $V = \ell_2$, and denote by (e_n) its standard orthonormal basis. Consider the linear subspace $Y \subseteq V$ spanned by $A = \{a, e_2, e_3, \dots\}$, where $a := \sum_{k=1}^{\infty} (1/k) e_k$. Then, $B = \{e_2, e_3, \dots\}$ is a maximal orthonormal set in Y, because if $x = \alpha_1 a + \sum_{k=2}^{N} \alpha_k e_k \in Y$ is orthogonal to B (*why is it enough to consider such an x?*), then $0 = (x, e_{N+1}) = \alpha_1/(N+1)$, and $0 = (x, e_k) = \alpha_k$ for $k = 2, \dots, N$, hence $x = 0$. However, clin B does not include a, so B is a maximal orthonormal set for Y which is not total in Y.
Note, however, that Y does have an orthonormal basis, which can be obtained by applying Gram-Schmidt to A (*see Homework 3!*).

Bonus: Characterization of Separable Hilbert Spaces

Definition. Two Hilbert spaces H, K are *isomorphic* if there is a bijective mapping $U: H \to K$ s.t., for all $x, y \in H$ and $\alpha \in \mathbb{C}$, $U(x+y) = U(x) + U(y)$, $U(\alpha x) = \alpha U(x)$ and $(U(x), U(y)) = (x, y)$. Such a mapping is a *unitary linear operator*.

Theorem. Every separable Hilbert space is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, or to ℓ_2.

Proof. Assume H is a separable Hilbert space, so it has a total orthonormal sequence. Suppose first that such sequence is finite, say, $\{e_1, \dots, e_n\}$. Then, $x = \sum_{k=1}^n (x, e_k) e_k$ for each $x \in H$. Let $U: H \to \mathbb{C}^n$ be given by $U\left(\sum_{k=1}^n \lambda_k e_k\right) = (\lambda_1, \dots, \lambda_n)$; U is bijective and linear, and if $x = \sum_{k=1}^n x_k e_k$, $y = \sum_{k=1}^n y_k e_k$, we have that $(x, y) = \sum_{k=1}^n x_k \overline{y_k} = (U(x), U(y))$, so U is unitary and H is isomorphic to $\mathbb{C}^n$.
If the total orthonormal sequence is infinite, say, $(e_k)_{k \in \mathbb{N}}$, define the mapping $U: H \to \ell_2$ by $U(x) = (\lambda_k)_{k \in \mathbb{N}}$, where $x = \sum_{k=1}^\infty \lambda_k e_k$. U is linear and unitary (as in the finite case), hence injective. By the characterization of total orthonormal sequences, $U(x) \in \ell_2$, and if $(\lambda_k)_{k \in \mathbb{N}} \in \ell_2$, $\sum_{k=1}^\infty \lambda_k e_k$ converges to an $x \in \ell_2$, so U is surjective. Thus, H is isomorphic to ℓ_2. $\square$

Bonus: Examples of Non-Separable Hilbert Spaces

1. $\ell_2(\mathbb{R})$**:** The space of all $f : \mathbb{R} \to \mathbb{R}$ s.t. $E_f = \{x \in \mathbb{R} : f(x) \neq 0\}$ is countable and $\sum_{x \in E_f} f^2(x) < \infty$ (this sum is well defined, *why?*), with inner product $(f, g) = \sum_{x \in E_f \cap E_g} f(x)\overline{g(x)}$. $\ell_2(\mathbb{R})$ is a Hilbert space (*Exercise! Hint: countable unions of countable sets are countable*). Also, the functions $f_y \in \ell_2(\mathbb{R})$, with $f_y(x) = 1$ if $x = y$ and $f_y(x) = 0$ otherwise, are an uncountable orthonormal system, so $\ell_2(\mathbb{R})$ is non-separable.

2. **Almost-periodic functions:** In an attempt to extend the classical Fourier series to non-periodic functions in $\mathbb{R}$, the following definition has been coined:

   $f : \mathbb{R} \to \mathbb{C}$ is *almost-periodic* (AP) if it is the uniform limit of functions $\sum_{k=1}^{n} a_k e^{i\lambda_k t}$, with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The set E of AP functions is a vector space, with inner product $(f, g) = \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} f(t)\overline{g(t)} dt$ (modulo an equivalence relation). The completion of E is a Hilbert space, but not all its elements can be identified as functions (*e.g.*, $\sum_{k=1}^{\infty} (1/k) e^{it/k}$). Also, $(e^{\lambda t})_{\lambda \in \mathbb{R}}$ is an uncountable orthonormal system in E, so E is non-separable.

Bonus: Proofs of Properties of Fejér Kernels

Letting $z = e^{ix}$, the Fejér kernel can be written, for every x not a multiple of 2π, as

$$K_m(x) = \frac{1}{m+1}\sum_{N=0}^{m}\sum_{n=-N}^{N} z^{-n} = \frac{1}{m+1}\sum_{N=0}^{m}\frac{z^N - z^{-N-1}}{1-z^{-1}} = \frac{1}{(m+1)(1-z^{-1})}\left[\frac{1-z^{m+1}}{1-z} - \frac{z^{-1}-z^{-m-2}}{1-z^{-1}}\right]$$

$$= \frac{1}{(m+1)(1-z^{-1})}\left[\frac{1-z^{m+1}}{1-z} + \frac{1-z^{-m-1}}{1-z}\right] = \frac{2-z^{m+1}-z^{-m-1}}{(m+1)(|1-z|^2)} = \frac{\sin^2\left(\frac{(m+1)x}{2}\right)}{(m+1)\sin^2\left(\frac{x}{2}\right)}. \qquad (*)$$

This, and the continuity of K_m, directly proves Property 1.

Since $\int_{-\pi}^{\pi} e^{inx}dx = 2\pi$ if $n = 0$ and $= 0$ otherwise, $\int_{-\pi}^{\pi} K_m(x)dx = (m+1)^{-1}\sum_{N=0}^{m}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{inx}dx = (m+1)^{-1}\sum_{N=0}^{m} 2\pi = 2\pi$, which establishes Property 2.

Finally, note that if $x \in [-\pi, -\delta) \cup (\delta, \pi]$, then $\sin^2(x/2) \geqslant \sin^2(\delta/2) > 0$. Thus, by $(*)$, for this range of values of x, $0 \leqslant K_m(x) \leqslant (m+1)^{-1}\sin^{-2}(\delta/2)$, so

$$0 \leqslant \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(x)dx \leqslant \frac{2\pi}{m+1}\sin^{-2}(\delta/2) \to 0 \quad \text{as } m \to \infty,$$

which proves Property 3. □

Bonus: Orthogonal Polynomials

Orthonormal sequences composed only of polynomials are very useful in applied mathematics, physics and engineering, due to their ease of computation.

Definition. A sequence $(p_n)_{n\in\mathbb{N}_0}$ of polynomials, where p_n has degree n, is *orthogonal* on (a,b) (which can be infinite) with respect to the weight function $w\colon (a,b)\to\infty$, if $(p_n,p_m) := \int_a^b w(x)p_n(x)p_m(x)dx = 0$ whenever $n\neq m$. If $c_n := \int_a^b w(x)p_n^2(x)dx = 1$ for all n, they are also *orthonormal*.

Orthogonal polynomials can be easily generated via the *Gram-Schmidt procedure* (*see Homework 3!*).

Examples

- *Legendre polynomials* ($w\equiv 1$ on $(-1,1)$, $c_n = 2/(2n+1)$):
  $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=(1/2)(3x^2-1)$, $p_3(x)=(1/2)(5x^3-3x)$, ...
- *Laguerre polynomials* ($w(x)=e^{-x}$ on $(0,\infty)$, $c_n=1$):
  $p_0(x)=1$, $p_1(x)=1-x$, $p_2(x)=(1/2)(x^2-4x+2)$, $p_3(x)=(1/6)(-x^3+9x^2-18x+6)$, ...
- *Hermite polynomials* ($w(x)=e^{-x^2}$ on $(-\infty,\infty)$, $c_n=\sqrt{\pi}2^n n!$):
  $p_0(x)=1$, $p_1(x)=2x$, $p_2(x)=4x^2-2$, $p_3(x)=8x^3-12x$, ...
- *Chebyshev polynomials* ($w(x)=1/\sqrt{1-x^2}$ on $(-1,1)$, $c_0=\pi$, $c_n=\pi/2$ for $n>0$):
  $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=2x^2-1$, $p_3(x)=4x^3-3x$, ...

By definition, p_n is orthogonal to every polynomial of degree lower than n, and $\mathrm{lin}\{1,x,\ldots,x^n\}=\mathrm{lin}\{p_0,\ldots,p_n\}$ (*why?*).

Bonus: Orthogonal Polynomials (cont.)

Let $(p_n)_{n\in\mathbb{N}_0}$ be a sequence of orthogonal polynomials over (a,b) with respect to w. Then, $(p_n)_{n\in\mathbb{N}_0}$ enjoys many interesting properties. Here are just a couple of them:

Property 1 (Moments). Let $\mu_i := \int_a^b x^i w(x)dx$ $(i \in \mathbb{N}_0)$. Then,

$$p_n(x) \propto \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{bmatrix}.$$

Proof. As $\mathrm{lin}\{1,x,\dots,x^n\} = \mathrm{lin}\{p_0,\dots,p_n\}$, (monic) p_n is of the form $p_n(x) = x^n - m^T(x)\alpha$, where $m(x) := [1,\dots,x^{n-1}]^T$ and $\alpha \in \mathbb{R}^n$ minimizes $\|x^n - m^T(x)\alpha\|^2$ (*why?*). Thus, α satisfies $H\alpha = \mu$, with $\alpha := [\alpha_0,\dots,\alpha_{n-1}]^T$, $\mu := [\mu_n,\dots,\mu_{2n-1}]^T$, and $H \in \mathbb{R}^{n\times n}$ s.t. $H_{i,j} = \mu_{i+j}$. This equation can be extended to $\begin{bmatrix} H & 0 \\ m^T(x) & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ p_n(x) \end{bmatrix} = \begin{bmatrix} \mu \\ x^n \end{bmatrix}$, and Cramér's rule implies the result. □

Property 2 (Zeros). The roots of p_n $(n \geqslant 1)$ are all real, simple, and lie in (a,b).

Proof. Let $q_r(x) = (x-x_1)(x-x_2)\cdots(x-x_r)$ consist of all the roots of $p_n(x) = 0$ in (a,b) (including their multiplicities). Then, q_r has degree r, and it has sign changes wherever p_n does in (a,b). Thus, $p_n(x)q_r(x)$ does not change sign in (a,b), so $\int_a^b w(x)p_n(x)q_r(x)dx \neq 0$. This can only be true if $r = n$, because p_n is orthogonal to all polynomials of lower degree (*why?*). Now, assume that some root, say, x_1, is multiple. Then, we can write $p_n(x) = (x-x_1)^2 r(x)$, where r has degree $n-2$. However, $p_n(x)r(x) = [p_n(x)/(x-x_1)]^2 \geqslant 0$, so $\int_a^b w(x)p_n(x)r(x)dx > 0$, which is again a contradiction (since p_n is orthogonal to any lower degree polynomial); hence, multiple roots cannot occur. □

Bonus: Orthogonal Polynomials (cont.)

Property 3 (Three-term recurrence). If (p_n) is orthonormal, then $p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$, where $A_n = a_{n+1}/a_n$, $C_n = a_{n+1}a_{n-1}/a_n^2$ and $B_n = (a_{n+1}/a_n)[b_{n+1}/a_{n+1} - b_n/a_n]$, with a_k and b_k being the coefficients of the k-th and $(k-1)$-th degree terms of $p_k(x)$, respectively.

Proof. With $A_n = a_{n+1}/a_n$, $q_n(x) := p_{n+1}(x) - A_n x p_n(x)$ is a polynomial of degree at most n, so $q_n = \alpha_n p_n + \cdots + \alpha_0 p_0$ for some $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$. By orthogonality, $\alpha_k = \int_a^b w(x)p_k(x)q_n(x)dx = \int_a^b w(x)p_k(x)p_{n+1}(x)dx - A_n \int_a^b w(x)p_k(x)xp_n(x)dx = 0$ for $k = 0, 1, \ldots, n-2$. Thus, the three-term relation holds with $B_n = \alpha_n$ and $C_n = -\alpha_{n-1}$. Now, write $xp_{n-1}(x) = (a_{n-1}/a_n)p_n(x) + q_{n-1}(x)$, where $q_{n-1}(x)$ has degree at most $n-1$, so $C_n = A_n \int_a^b w(x)p_n(x)xp_{n-1}(x)dx = (A_n a_{n-1}/a_n)\int_a^b w(x)p_n^2(x)dx$ $+ A_n \int_a^b w(x)p_n(x)q_{n-1}(x)dx = A_n a_{n-1}/a_n$. Finally, B_n is obtained by equating the n-th degree terms of the three-term relation. Note also that the result is valid for $n = 0$ if we define $a_{-1} := p_{-1}(x) := 0$. □

Property 4 (Christoffel-Darboux relation). If (p_n) is orthonormal, then $(a_n/a_{n+1})[p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)] = (x-y)\sum_{i=0}^n p_i(x)p_i(y)$ for all $x, y \in \mathbb{R}$.

Proof. Multiplying Property 3 by $p_n(y)$ yields $p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) - C_n p_{n-1}(x)p_n(y)$. Exchanging x and y, subtracting this identity from the previous one, and multiplying by $1/A_n$ gives

$$(x-y)p_n(x)p_n(y) = A_n^{-1}[p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)] - A_{n-1}^{-1}[p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)].$$

Summing these equations over $0, 1, \ldots, n$, and taking $a_{-1} = 0$, proves the result, as $A_n^{-1} = a_n/a_{n+1}$. □

Property 4 gives a convenient formula for the *kernel* $G_n(x,y) := \sum_{i=0}^n p_i(x)p_i(y)$ appearing, *e.g.*, in the error from approximating a function in terms of (p_n).

Other properties of (p_n) follow from the *Sturm-Liouville theory* (see Young's book).