EL3370 Mathematical Methods in Signals, Systems and Control

Homework 2

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Instructions (read carefully):

- The exercise sets are *individual*: even though discussion with your peers is encouraged, you have to provide your own personal solution to each problem.
- The solutions to some problems can possibly be found by searching in math books other than the main course book. *Try to avoid such practice*: the only way to understand the topics in the course is by working hard on the problems by yourself.
- To prove statements in the exercises, use only the notation, definitions and results proven (not those given as exercises) in the lectures.

1 p-norms

Prove that, for every $x \in \mathbb{R}^n$, $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

2 Closed linear subspace of ℓ_∞

Let c_0 denote the linear subspace of ℓ_∞ comprising all sequences (x_n) that tend to zero as $n \to \infty$. Prove that c_0 is closed in ℓ_∞ with respect to $\|\cdot\|_\infty$, and that it is the closed linear span of $\{e_n : n \in \mathbb{N}\}$, where e_n is the sequence with n-th term 1 and all other terms equal to zero.

3 Finite-dimensional subspaces

Prove that every finite-dimensional linear subspace of a (real or complex) normed space is closed.

Hint: If $\{x_1, \ldots, x_n\}$ span the linear subspace, and $y \notin \operatorname{lin}\{x_1, \ldots, x_n\}$ it is enough to show that $\inf_{a_1,\ldots,a_n \in \mathbb{R}} \|y - a_1x_1 - \cdots - a_nx_n\| > 0$ (*why?*). The norm inside the inf can be seen as a norm defined only on $\operatorname{lin}\{y, x_1, \ldots, x_n\}$, so one could use the equivalence of finite-dimensional norms to lower bound this quantity by a strictly positive number.

4 Existence of optimal approximation

Let X be a (real or complex) normed space, and let $x_1, \ldots, x_n$ be linearly independent vectors in X. Given a fixed $y \in X$, show that there are coefficients $a_1, \ldots, a_n$ minimizing $\|y - a_1x_1 - \cdots - a_nx_n\|$.

Hint: Use the fact stated in Problem 3. Note that closedness is not enough to establish this result: you may need to rely on compactness (closedness + boundedness, in the case of finite-dimensional linear spaces). To use compactness, consider a bounded subset of $\operatorname{lin}\{x_1, \ldots, x_n\}$ where the minimizer might lie.

5 Different topologies in $C[0, 1]$

Prove that in $C[0, 1]$ the norms $\|x\|_\infty = \max_{0\leqslant t\leqslant 1} |x(t)|$ and $\|x\|_2 = \sqrt{\int_0^1 |x(t)|^2 dt}$ induce different topologies, *i.e.*, find a sequence of functions (x_n) such that $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$, but $\|x_n\|_2 \to 0$ as $n \to \infty$.

6 Closedness and completeness

Show that a closed linear subspace of a complete metric space is itself complete (with respect to the same metric). Deduce then that $(c_0, \|\cdot\|_\infty)$, the normed space of sequences (x_n) in ℓ_∞ such that $x_n \to 0$ as $n \to \infty$ (see Problem 2), is a Banach space.

Note: Assume that ℓ_∞ is complete.

7 Completeness of $C(X)$

Recall the complex normed space $C(X)$, consisting of all bounded continuous functions $f : X \to \mathbb{C}$, where X is a topological space, with norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Show that $C(X)$ is a Banach space.

Hint: Follow the steps of a standard completeness proof. Note that, as part of that proof, you need to establish that if (f_n) is a sequence in $C(X)$, and $f_n \to f$ uniformly in X (*i.e.*, $\sup_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$), where $f\colon X \to \mathbb{C}$ is the limit function, then f must be continuous. To prove this, notice that

$$|f(x_1) - f(x_2)| \leqslant |f(x_1) - f_n(x_1)| + |f_n(x_1) - f_n(x_2)| + |f_n(x_2) - f(x_2)|.$$

The first and last terms can be bounded due to the uniform convergence of (f_n), and to bound the middle term use the fact that f_n is continuous.

8 Optimization in Hilbert Space

Using the projection theorem, solve the finite-dimensional problem:

$$\begin{array}{cl} \min_{x \in \mathbb{R}^n} & x^T Q x \\ \text{s.t.} & Ax = b, \end{array}$$

where $Q = Q^T \succ 0$ (non-singular), $A \in \mathbb{R}^{m \times n}$ $(m < n)$ and $b \in \mathbb{R}^m$. Assume that A has full (row) rank.

Do not use Lagrange multipliers, KKT conditions or similar techniques.

Hint: At some point of your derivation, you may need to characterize the orthogonal complement of the nullspace of A, *i.e.*, those $x \in \mathbb{R}^n$ such that $(x, v)_Q = 0$ for all $v \in \mathbb{R}^n$ such that $Av = 0$, where $(x, y)_Q = y^T Q x$. To this end, notice that, for every $w \in \mathbb{R}^m$, $0 = w^T A v = w^T A Q^{-1} Q v = (Q^{-1} A^T w, v)_Q$, so the nullspace of A consists exactly of those $v \in \mathbb{R}^n$ that are orthogonal to the columns of $Q^{-1} A^T$.