EL3370 Mathematical Methods in Signals, Systems and Control

Homework 1

Division of Decision and Control Systems
School of Electrical Engineering and Computer Science
KTH Royal Institute of Technology

Instructions (read carefully):

- The exercise sets are *individual*: even though discussion with your peers is encouraged, you have to provide your own personal solution to each problem.
- The solutions to some problems can possibly be found by searching in math books other than the main course book. *Avoid such practice*: the only way to understand the topics in the course is by working hard on the problems by yourself.
- To prove statements in the exercises, use only the notation, definitions and results proven (not those given as exercises) in the lectures.

1 Distance to a closed set

Let X be a metric space, and $M \subseteq X$ a closed set. Show that if $x \notin M$, then the distance from x to M, $d(x, M) := \inf\{d(x, y) \colon y \in M\}$, is non-zero.

Remark: Make sure that you use the assumption that M is closed, because *the result is not always true for general sets.*

2 Some properties of the closure

Consider a metric space X, and $A, B \subseteq X$. Show that $A \subseteq \overline{A}$, $\overline{\overline{A}} = \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Hint: You can either use directly the definition of closure based on neighborhoods, or its characterization based on sequences. Try to avoid appealing to accumulation points, as they may obfuscate the reasoning.

Remark: Regarding the last statement, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ in general, so make sure that your argument does not imply that $\overline{A \cap B} \supseteq \overline{A} \cap \overline{B}$.

3 Finite convergence

Prove that if X is a *finite* metric space, and (x_n) is a convergent sequence in X with limit x, then there is an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geqslant N$. Also, extend this result to an arbitrary set X with the *discrete topology* (see the slides of Topic 1, page 32, for a definition of this topology).

Remark: This simple result has some impressive consequences. For example, in the theory of Markov decision processes (as well as in model predictive control), this result implies that when the action space is finite, given an initial state and an infinite horizon problem, there is an N (called *forecast horizon*) such that a moving horizon approximation of horizon length larger than N has exactly the same optimal solutions as the original problem. Also, in social learning it leads to a phenomenon called "herding" (where an agent ignores its own observations and follows the same behavior as the other agents).

4 Super-continuous functions

A function $f \colon [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$ $(a < b)$, is said to be *Hölder continuous with coefficient* $\alpha > 0$ if there is a $C > 0$ such that $|f(x) - f(y)| \leqslant C|x - y|^\alpha$ for all $x, y \in [a, b]$.

A mathematical urban myth tells the story of a PhD student who wrote an entire thesis on Hölder continuous functions $f \colon [a, b] \to \mathbb{R}$ with $\alpha > 1$, which were called *super continuous functions.* During the defense, a member of the committee asked if the student had any examples of super continuous functions, and the student replied that unfortunately only constant functions had been found satisfying this property. A few minutes later, the committee member said that she had just proven that all super continuous functions are constant!

Prove the statement above.

Hint: For fixed $a \leqslant x < y \leqslant b$, subdivide $[x, y]$ into n intervals $[x_i, x_{i+1}]$ of equal length, apply the Hölder condition on these intervals, then bound $|f(x) - f(y)|$ by the sum of the differences $|f(x_i) - f(x_{i-1})|$, and by letting $n \to \infty$ show that $f(x) = f(y)$.

5 Linear independence and orthogonality

Let $x \neq 0$ and $y \neq 0$ belong to an inner product space.

(a) If x, y are *orthogonal*, *i.e.*, $(x, y) = 0$, show that $\{x, y\}$ is a linearly independent set.

(b) Extend the result to mutually orthogonally nonzero vectors $x_1, \ldots, x_m$.

6 A normed space

Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm defined on X by

$$\|x\| = |x_1| + |x_2|, \qquad x = (x_1, x_2),$$

from an inner product?

7 Convergence in inner product spaces

Show that for a sequence (x_n) in an inner product space, the conditions $\|x_n\| \to \|x\|$ and $(x_n, x) \to (x, x)$ imply convergence $x_n \to x$ (*i.e.*, $\|x_n - x\| \to 0$).

8 Reproducing kernels

Let $k_\alpha \in RL_2$ be defined by

$$k_\alpha(z) = \frac{1}{\overline{\alpha} z - 1},$$

where $|\alpha| \neq 1$. Show that, for every $f \in RH_2$,

$$(f, k_\alpha) = \begin{cases} f(\alpha), & \text{if } |\alpha| > 1, \\ 0, & \text{if} |\alpha| < 1. \end{cases}$$

A function k_α such that $k_\alpha \in RH_2$ and $(f, k_\alpha) = f(\alpha)$ for all $\alpha \in \mathbb{E}$ is called a *reproducing kernel* for RH_2, so RH_2 becomes a *reproducing kernel Hilbert space.*