EL3370 Mathematical Methods in Signals, Systems and Control

Topic 8: Linear Operators

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Outline

Outline

Motivation and Definitions

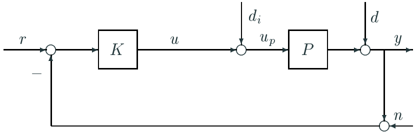
Solving Linear Equations

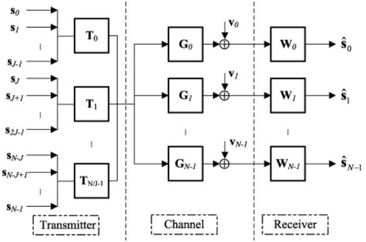
Many problems in physics and engineering involve solving linear equations $Lf = g$, where L is, *e.g.*, a differential operator. Some questions are:

(1) Is there a solution of $Lf = g$?
(2) Is it unique?
(3) How does it change if g is slightly perturbed?

Transfer functions

In systems theory, signals are represented by elements of normed spaces ($\ell_2, \ell_\infty, L_2, L_\infty, \ldots$), and systems are described by *operators* between these spaces.





Motivation and Definitions (cont.)

Definitions

If E,F are vector spaces, a *linear operator* from E to F is a mapping $T: E \to F$ s.t.

$$T(\lambda x + \mu y) = \lambda T x + \mu T y \quad \text{for all } x, y \in E \text{ and scalars } \lambda, \mu.$$

If E,F are normed, T is *bounded* if there is an $M > 0$ s.t. $\|Tx\| \leqslant M\|x\|$ for all $x \in E$. If so, the *norm* of T is the smallest such M, *i.e.*,

$$\|T\| := \sup\{\|Tx\| : x \in E, \|x\| \leqslant 1\}.$$

The *kernel*, Ker T, of $T: E \to F$ is the subspace $\{x \in E : Tx = 0\} \subseteq E$, and the *range* of T, $\mathscr{R}(T)$, is the subspace $\{Tx : x \in E\} \subseteq F$.

The operator $I_E : E \to E$, given by $I_E(x) = x$ for all $x \in E$, is the *identity operator* on E. When there is no ambiguity, it will be written simply as I.

Motivation and Definitions (cont.)

Examples

1. *Multiplication*
   Define M_f on $L_2[a,b]$ by: $(M_f x)(t) = f(t)x(t)$, where $f \in C[a,b]$. M_f is linear, and

$$\|M_f x\|^2 = \int_a^b |f(t)|^2 |x(t)|^2 dt \leqslant \sup_{\tau \in [a,b]} |f(\tau)|^2 \int_a^b |x(t)|^2 dt = \|f\|^2 \|x\|^2,$$

   so $\|M_f\| \leqslant \|f\|$. In fact, $\|M_f\| = \|f\|$ (by choosing an appropriate (x_n)).

2. *Integral operator*
   Let $a, b, c, d \in \mathbb{R}$, and $k : [c,d] \times [a,b] \to \mathbb{R}$ continuous. Then, define
   $K : L_2[a,b] \to L_2[c,d]$ as

$$(Kx)(t) = \int_a^b k(t,s)x(s)ds, \quad c \leqslant t \leqslant d.$$

   K is linear, and, by Cauchy-Schwarz, $\|Kx\|^2 \leqslant \left(\int_c^d \int_a^b |k(t,s)|^2 ds dt \right) \|x\|^2$, so K is bounded.

Examples (cont.)

3. *Differential operator*
   Let $\mathscr{D} \subseteq L_2(-\infty,\infty)$ be the space of differentiable functions $f \in L_2(-\infty,\infty)$ s.t. $f' \in L_2(-\infty,\infty)$. Then,

$$\frac{d}{dx} : \mathscr{D} \to L_2(-\infty,\infty)$$

   is a linear operator, but it is not bounded.

4. *Shift operator*
   Define S on ℓ_2 by:

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

   S is an *isometry* (*i.e.*, $\|Sx\| = \|x\|$ for all $x \in \ell_2$), so it is bounded and $\|S\| = 1$. We can also define the backward shift operator S^* on ℓ_2 by $S^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$, which is bounded and s.t. $\|S^*\| = 1$, but it is not an isometry.

Motivation and Definitions (cont.)

Theorem

Let E, F be normed spaces, and $T: E \to F$ be a linear operator. The following are equivalent:

(1) T is continuous,

(2) T is continuous at 0,

(3) T is bounded.

Proof. Similar to the case for linear functionals. □

Outline

The Banach Space $\mathscr{L}(E,F)$

Definition
Let E,F be normed spaces. $\mathscr{L}(E,F)$ is the space of bounded linear operators from E to F, and $\mathscr{L}(E) = \mathscr{L}(E,E)$.

If F is a Banach space, so is $\mathscr{L}(E,F)$ (*similar to the proof that V^* is Banach*, in Topic 7).

The *composition* of operators $A: E \to F$ and $B: F \to G$, BA, is $BA(x) = B(Ax)$ for all $x \in E$.

Theorem. If $A \in \mathscr{L}(E,F)$ and $B \in \mathscr{L}(F,G)$, then $BA \in \mathscr{L}(E,G)$, and $\|BA\| \leqslant \|B\|\|A\|$.
Proof. BA is linear, and, since A,B are continuous, so is BA. Also,

$$\|BAx\|_G = \|B(Ax)\|_G \leqslant \|B\|\|Ax\|_F \leqslant \|B\|\|A\|\|x\|_E, \qquad x \in E,$$

so $\|BA\| \leqslant \|B\|\|A\|$. □

Observation. This last result shows that $\mathscr{L}(E)$ is not only a normed space, but also a *normed algebra* (since we have defined a product). If $\mathscr{L}(E)$ is complete, we say that it is a *Banach algebra*.

Outline

Inverses of Operators

Solving an equation $Ax = y$ involves computing "$x = A^{-1}y$".

Definition. Let E, F be normed spaces. $A \in \mathscr{L}(E,F)$ is *invertible* if there is a $B \in \mathscr{L}(F,E)$ s.t. $AB = I_F$ and $BA = I_E$. In this case, B is unique (*why?*) and is called the *inverse* of A, A^{-1}.

If E, F are Banach spaces, and $A \in \mathscr{L}(E,F)$ is bijective, its inverse is necessarily bounded (*Banach-Schauder* / *Open mapping theorem*) and linear (*why?*).

Examples

1. The shift operators S and S^* on ℓ_2 satisfy $S^*S = I$, but $SS^* \neq I$ (*why?*), so S, S^* are not invertible.

2. The multiplication operator M_t on $L_2[0,1]$ given by $(M_t x)(t) = tx(t)$ $(0 \leqslant t \leqslant 1)$ is injective but not surjective:

   $M_t x = 0$ implies $tx(t) = 0$, so $x(t) = 0$ (for almost all t).

   However, there is no $x \in L_2[0,1]$ s.t. $(M_t x)(t) = 1$, since $t \mapsto 1/t \notin L_2[0,1]$.

One way to produce inverses is as follows:

Theorem. Let E be a Banach space, and $A \in \mathscr{L}(E)$ s.t. $\|A\| < 1$. Then $I - A$ is invertible (in the normed space $\mathscr{L}(E)$), and

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n = \lim_{N\to\infty} \left(I + A + A^2 + \cdots + A^N\right).$$

Proof. Let $x \in E$. Then $((I + A + A^2 + \cdots + A^n)x)$ is Cauchy: If $m > n$,

$$\left\| \sum_{k=0}^{m} A^k x - \sum_{k=0}^{n} A^k x \right\| = \left\| \sum_{k=n+1}^{m} A^k x \right\| \leqslant \sum_{k=n+1}^{m} \|A\|^k \|x\| \leqslant \frac{\|A\|^{n+1}}{1-\|A\|} \|x\| \to 0 \quad \text{as } n, m \to \infty \quad (m > n), \qquad (*)$$

so $\sum_{k=0}^{n} A^k x \to Tx$. T is linear, and letting $m \to \infty$ in $(*)$ gives $\left\| Tx - \sum_{k=0}^{n} A^k x \right\| \leqslant \frac{\|A\|^{n+1}}{1-\|A\|} \|x\|$, hence $Tx - \sum_{k=0}^{n} A^k x$ is bounded, and so is T.

Proof (cont.)

Also, $\left\| T - \sum_{k=0}^{n} A^k \right\| \leqslant \dfrac{\|A\|^{n+1}}{1-\|A\|}$, so $\sum_{k=0}^{\infty} A^k = T$.

Finally, since $\|A^n x\| \leqslant \|A\|^n \|x\| \to 0$ as $n \to \infty$ (so $\lim A^n x = 0$),

$$(I-A)Tx = (I-A)\lim \sum_{k=0}^{n} A^k x = \lim \sum_{k=0}^{n} (A^k - A^{k+1})x = x - \lim(A^{n+1}x) = x,$$

and similarly $T(I-A) = I$. Therefore $T = (I-A)^{-1}$. □

Corollary. If E is a Banach space, the set of invertible operators on E is open in $\mathscr{L}(E)$.

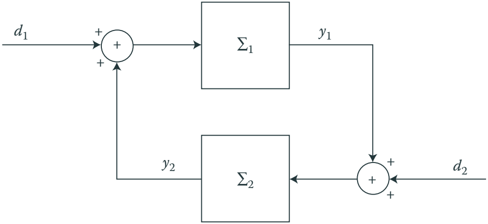
Proof. Let $A \in \mathscr{L}(E)$ be invertible. Then for every $B \in \mathscr{L}(E)$ s.t. $\|B\| \leqslant 1/\|A^{-1}\|$, we have that $I + A^{-1}B$ is invertible, since $\|A^{-1}B\| \leqslant \|A^{-1}\|\|B\| < 1$, and $[(I+A^{-1}B)^{-1}A^{-1}](A+B) = (I+A^{-1}B)^{-1}(I+A^{-1}B) = I$, while $(A+B)[(I+A^{-1}B)^{-1}A^{-1}] = A(I+A^{-1}B)[(I+A^{-1}B)^{-1}A^{-1}] = AA^{-1} = I$, so $A+B$ is invertible and it has inverse $(A+B)^{-1} = (I+A^{-1}B)^{-1}A^{-1}$. This means that every invertible element of $\mathscr{L}(E)$ has a nbd of invertible elements, hence the set of invertible operators on E is open in $\mathscr{L}(E)$. □

Application to small gain theorem in control, and to structured SVD

The previous theorem allows us to derive a simple sufficient criterion for stability of feedback systems:

Theorem (Small Gain)

Consider two stable (with respect to the ℓ_2 norm), causal and linear systems Σ_1, Σ_2 in a feedback interconnection as shown below. The closed loop system, with d_1, d_2 as inputs and y_1, y_2 as outputs, is ℓ_2-stable if $\|\Sigma_1\|\|\Sigma_2\| < 1$.



Application to small gain theorem in control, and to structured SVD (cont.)

Proof. The feedback interconnection yields, $y_2 = \Sigma_2(d_2 + y_1) = \Sigma_2 d_2 + \Sigma_2\Sigma_1 d_1 + \Sigma_2\Sigma_1 y_2$. This means that the closed loop system is stable iff $I - \Sigma_2\Sigma_1$ is invertible, since in that case

$$y_2 = [I - \Sigma_2\Sigma_1]^{-1}(\Sigma_2 d_2 + \Sigma_2\Sigma_1 d_1).$$

The previous theorem tells us that a sufficient condition for $I - \Sigma_2\Sigma_1$ to be invertible is that $\|\Sigma_2\Sigma_1\| < 1$, and this condition is fulfilled if $\|\Sigma_1\|\|\Sigma_2\| < 1$, since $\|\Sigma_2\Sigma_1\| \leqslant \|\Sigma_1\|\|\Sigma_2\|$. □

In multivariable control, Σ_1 may correspond to a feedback loop, while Σ_2 represents a source of uncertainty in the plant being controlled. If only the norm of Σ_2 were known, the small gain theorem states that Σ_1 should satisfy $\|\Sigma_1\|\|\Sigma_2\| < 1$ to ensure stability.

If Σ_2 had a known structure, *e.g.*, $\Sigma_2 = \text{diag}(\delta_1, \dots, \delta_n)$, one can define the *structured singular value* $\mu(\Sigma_1) = \sup\left\{\|\Sigma_2\|^{-1} : \Sigma_2 = \text{diag}(\delta_1, \dots, \delta_n), \|\Sigma_1\Sigma_2\| \geqslant 1\right\}$, so the condition $\mu(\Sigma_1) < 1$ implies that $\|\Sigma_1\Sigma_2\| < 1$ for all $\Sigma_2 = \text{diag}(\delta_1, \dots, \delta_n)$ with $\|\Sigma_2\| < 1$, and thus, by the small gain theorem, (Σ_1, Σ_2) is stable for those Σ_2.

Outline

Adjoint Operators

The transpose of a matrix $A \in \mathbb{R}^{n \times n}$ satisfies $(Ax, y) = y^T Ax = (A^T y)^T x = (x, A^T y)$ for $x, y \in \mathbb{R}^n$.
We can generalize the transpose to general normed spaces:

Theorem. Let $A \in \mathscr{L}(E,F)$, where E,F are normed spaces. Then there is a unique $A^* \in \mathscr{L}(F^*,E^*)$ s.t. $\langle Ax, y^* \rangle_F = \langle x, A^* y^* \rangle_E$ for all $x \in E$, $y^* \in F^*$, and $\|A\| = \|A^*\|$.
Proof. Fix $y^* \in F^*$. $x \mapsto \langle Ax, y^* \rangle_F$ is a linear functional on E. Also, $|\langle Ax, y^* \rangle| \leqslant \|y^*\| \|Ax\| \leqslant \|y^*\| \|A\| \|x\|$, so $x \mapsto \langle Ax, y^* \rangle_F$ is a bounded linear functional, say, $x^* \in E^*$. Define $A^* y^* = x^*$. A^* is unique and linear (*why?*). Furthermore, $|\langle x, A^* y^* \rangle_E| = |\langle Ax, y^* \rangle_F| \leqslant \|y^*\| \|Ax\| \leqslant \|y^*\| \|A\| \|x\|$, so $\|A^* y^*\| \leqslant \|A\| \|y^*\|$, *i.e.*, $\|A^*\| \leqslant \|A\|$, and if $x_0 \in E$ is non-zero, by Corollary 2 of Hahn-Banach, there is a $y_0^* \in F^*$, $\|y_0^*\| = 1$, s.t. $\langle Ax_0, y_0^* \rangle_F = \|Ax_0\|$, so $\|Ax_0\| = |\langle x_0, A^* y_0^* \rangle_E| \leqslant \|A^* y_0^*\| \|x_0\| \leqslant \|A^*\| \|x_0\|$, thus $\|A\| \leqslant \|A^*\|$. Thus, $\|A\| = \|A^*\|$. □

A^* is the *adjoint* of A. It can be shown that, when E,F are reflexive, $A^{**} = A$.

Note. If E,F are inner product spaces, one can also define the *inner product adjoint* of $A \in \mathscr{L}(E,F)$ via $(Ax, y) = (x, A^* y)$ for all $x \in E$, $y \in F$; this differs from the normed adjoint in that $(\alpha A)^* = \overline{\alpha} A^*$ for the inner product adjoint, while $(\alpha A)^* = \alpha A^*$ for the normed adjoint.

Properties of the Adjoint

(1) $I^* = I$.
(2) If $A_1, A_2 \in \mathscr{L}(E,F)$, then $(A_1 + A_2)^* = A_1^* + A_2^*$.
(3) If $A \in \mathscr{L}(E,F)$ and $\alpha \in \mathbb{C}$, then $(\alpha A)^* = \alpha A^*$. For inner product adjoints, $(\alpha A)^* = \overline{\alpha} A^*$.
(4) If $A \in \mathscr{L}(E,F)$, $B \in \mathscr{L}(F,G)$, then $(A_2 A_1)^* = A_1^* A_2^*$.
(5) If $A \in \mathscr{L}(E,F)$ and A has a bounded inverse, then $(A^{-1})^* = (A^*)^{-1}$.

Proof

Properties (1)-(4) are straightforward. Regarding (5), assume $A \in \mathscr{L}(E,F)$ has a bounded inverse A^{-1}. To show that A^* has an inverse, we will establish that A^* is injective and surjective. If $y_1^*, y_2^* \in F^*$, $y_1^* \neq y_2^*$, then $\langle x, A^* y_1^* \rangle - \langle x, A^* y_2^* \rangle = \langle Ax, (y_1^* - y_2^*) \rangle \neq 0$ for some $x \in E$, so $A^* y_1^* \neq A^* y_2^*$ and A^* is injective. Now, given some $x^* \in E^*$, and $x \in E$, $Ax = y$, we have $\langle x, x^* \rangle = \langle A^{-1} y, x^* \rangle = \langle y, (A^{-1})^* x^* \rangle = \langle Ax, (A^{-1})^* x^* \rangle = \langle x, A^* (A^{-1})^* x^* \rangle$, so $x^* \in \mathscr{R}(A^*)$, and also $(A^*)^{-1} = (A^{-1})^*$. □

Adjoint Operators (cont.)

Examples

1. Consider the multiplication operator on $L_2[a,b]$, $(M_f x)(t) = f(t)x(t)$:

$$(x, M_f^* y) = (M_f x, y) \quad \Leftrightarrow \quad \int_a^b x(t)\overline{[M_f^* y](t)}dt = \int_a^b f(t)x(t)\overline{y(t)}dt \quad \Leftrightarrow \quad [M_f^* y](t) = \overline{f(t)}y(t).$$

2. Consider the integral operator $K: L_2[a,b] \to L_2[c,d]$ with kernel k. Then

$$\begin{aligned}
(x, K^* y) = (Kx, y) \quad \Leftrightarrow \quad \int_a^b x(t)\overline{[K^* y](t)}dt &= \int_c^d Kx(t)\overline{y(t)}dt \\
&= \int_c^d \int_a^b k(t,s)x(s)\overline{y(t)}ds dt \\
&= \int_a^b x(s) \int_c^d k(t,s)\overline{y(t)}dt ds \\
\Leftrightarrow \quad (K^* y)(t) = \int_c^d \overline{k(s,t)}y(s)ds.&
\end{aligned}$$

3. The adjoint of the shift operator S on ℓ_2 is the backward shift operator S^*.

Outline

Self-Adjoint and Non-Negative Operators

Definition
Let H be a Hilbert space. $A \in \mathscr{L}(H)$ is *self-adjoint* (or *Hermitian*) if $A = A^*$.
An operator $A \in \mathscr{L}(H)$ is *non-negative* ($A \geqslant 0$) if $(Ax, x) \geqslant 0$ for all $x \in H$, and it is *positive* if, in addition, $(Ax, x) = 0$ implies that $x = 0$. $A \leqslant B$ means that $(Ax, x) \leqslant (Bx, x)$ for all $x \in H$.

Examples

1. The multiplication operator in $L_2[a, b]$ where f is real valued is self-adjoint, and non-negative if $f(x) \geqslant 0$ for all $x \in [a, b]$.
2. The integral operator in $L_2[a, b]$ with kernel k is self-adjoint iff $k(t, s) = \overline{k(s, t)}$, $t, s \in [a, b]$.

Theorem. If $A \in \mathscr{L}(H)$ is self-adjoint, then $\|A\| = \sup_{\|x\|=1} |(Ax, x)|$.
Proof (for real H). For every $x \in H$, $\|x\| = 1$, $|(Ax, x)| \leqslant \|Ax\| \|x\| \leqslant \|A\|$, hence $m := \sup_{\|x\|=1} |(Ax, x)| \leqslant \|A\|$. On the other hand, $(A(x \pm y), x \pm y) = (Ax, x) \pm 2(Ax, y) + (y, y)$, so

$$|(Ax, y)| = \frac{1}{4} |(A(x+y), x+y) - (A(x-y), x-y)| \leqslant \frac{m}{4} \left(\|x+y\|^2 + \|x-y\|^2 \right) \leqslant \frac{m}{2} \left(\|x\|^2 + \|y\|^2 \right).$$

Taking $y = (\|x\|/\|Ax\|)Ax$ gives $\|x\| \|Ax\| \leqslant m \|x\|^2$, or $\|Ax\| \leqslant m$ whenever $\|x\| = 1$, so $\|A\| \leqslant m$. □

Self-Adjoint and Non-Negative Operators (cont.)

Theorem. If $A \in \mathscr{L}(H)$, where H is a *complex* Hilbert space, and $(Ax, x) = 0$ for all $x \in H$, then $A = 0$.

Proof. Since $(A(x+y), x+y) = 0$, we have that $(Ay, x) + (Ax, y) = 0$ for all $x, y \in H$. Replacing y by iy yields $i(Ay, x) - i(Ax, y) = 0$, *i.e.*, $(Ay, x) - (Ax, y) = 0$. Adding these expressions gives $(Ay, x) = 0$, which holds for every $x, y \in H$; therefore, $Ay = 0$ for all $y \in H$, *i.e.*, $A = 0$. □

Corollary. If $A \in \mathscr{L}(H)$ is non-negative, where H is a complex Hilbert space, then it is also self-adjoint.

Proof. If $A \in \mathscr{L}(H)$ is non-negative, (Ax, x) is real, so $(x, A^* x) = (Ax, x) = (x, Ax)$, *i.e.*, $(x, [A - A^*]x) = 0$ for every $x \in H$, so by the theorem above, $A = A^*$. □

Outline

Spectrum

Goal: Extend the concept of eigenvalues to linear operators on a Banach space E.

Motivating example: Separation of variables in PDEs
To solve the differential equation $\dot{x}(t) = Ax(t)$, with $x(t) \in \mathbb{R}^n$, one can decompose the matrix A as $A = TDT^{-1}$, where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ has the eigenvalues of A (assumed distinct) and $T = [v_1 \ \cdots \ v_n]$ the corresponding eigenvectors as columns, which satisfy $Av_k = \lambda_k v_k$ for $k = 1, \dots, n$. Then, re-defining $x(t) = Ty(t)$, one obtains $\dot{y}(t) = Dy(t)$, so $y_k(t) = c_k \exp(\lambda_k t)$ and the general solution is

$$x(t) = c_1 v_1 \exp(\lambda_1 t) + \cdots + c_n v_n \exp(\lambda_n t).$$

Consider now a partial differential equation (PDE) such as

$$\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2} \qquad \text{heat equation in } y(x,t); \quad x, t \in \mathbb{R}$$

subject to an initial condition $y(x,0)$ s.t. $\lim_{x \to \pm\infty} y(x,0) = 0$.

Motivating example: Separation of variables in PDEs (cont.)

This equation can be solved in a similar manner if one consider $\underline{y}(t) = y(\cdot, t)$ as an "infinite-dimensional vector" or function for each fixed t. Then, the PDE can be written as $\dot{\underline{y}} = A\underline{y}$, where A is a linear operator satisfying

$$(A\underline{y}(t))(x) = k\frac{\partial^2 y(x,t)}{\partial x^2}.$$

One can then diagonalize A by solving the equation $Av_\lambda = \lambda v_\lambda$ for $v_\lambda : x \mapsto v_\lambda(x)$, or $kv_\lambda'' = \lambda v_\lambda$, which gives $v_\lambda(x) = a_\lambda \exp(\sqrt{\lambda/k}x) + b_\lambda \exp(-\sqrt{\lambda/k}x)$. Under the given initial condition, $\lambda < 0$, so the general solution of the PDE is, informally,

$$y(x,t) = \int_0^\infty \left\{ \tilde{a}(\lambda)\exp\left(i\sqrt{-\frac{\lambda}{k}}x\right) + \tilde{b}(\lambda)\exp\left(-i\sqrt{-\frac{\lambda}{k}}x\right)\right\}\exp(-\lambda t)d\lambda,$$

where the functions $\tilde{a}, \tilde{b}$ are determined from the initial condition $y(\cdot, 0)$.

This is the standard method of *separation of variables for solving PDEs*! To formalize it, one needs to extend the notion of eigenvalues and eigenvectors to infinite dimensional spaces.

Spectrum (cont.)

Some operators do not have eigenvalues! (λ's for which $(\lambda I - A)x = 0$ for some $x \neq 0$).
Recall the shift operator S on ℓ_2: $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ If $Sx = \lambda x$, then $x = 0$!

Definition
The *spectrum* of $A \in \mathscr{L}(E)$ is $\sigma(A) := \{\lambda \in \mathbb{C}\colon \lambda I - A$ does not have an inverse in $\mathscr{L}(E)\}$.

$\sigma(A) \neq \emptyset$, and may have not only *eigenvalues*.

Example
Consider the multiplication operator $M_f \in \mathscr{L}(L_2[a,b])$ for an $f \in C[a,b]$. Then $\sigma(M_f) = \mathscr{R}(f)$:

If $\lambda \notin f([a,b])$, then $\lambda I - M_f$ has a bounded inverse $M_{(\lambda - f)^{-1}}$, so $\lambda \notin \sigma(M_f)$. Conversely, if $\lambda = f(t)$ for some $t_0 \in [a,b]$, and $\lambda I - M_f$ had an inverse $T \in L_2[a,b]$, then consider a sequence (x_n) in $L_2[a,b]$, $x_n(t) \geqslant 0$ s.t. $x_n(t) \to 0$ for $t \neq t_0$ and $\int_a^b |x_n(t)|^2 dt = 1$: $(\lambda I - M_f)x_n \to 0$ but $T(\lambda I - M_f)x_n = x_n$, even though $\|x_n\| = 1$! This means that $\lambda = \sigma(M_f)$.

Hence, $\sigma(M_f) = \mathscr{R}(f)$. However, for many f's, M_f does not have eigenvalues (*e.g.*, $f(t) = t$).

Theorem. $\sigma(A)$ is compact, and it is contained in $\overline{B(0,\|A\|)}$.

Proof. Define $F:\mathbb{C}\to\mathscr{L}(E)$ as $F(\lambda)=\lambda I-A$. Since $\|F(\lambda)-F(\mu)\|=|\lambda-\mu|$, F is continuous. Therefore, since $\sigma(A)=F^{-1}(G^c)$, where G is the set of invertible operators in $\mathscr{L}(E)$, which is open, we have that $F^{-1}(G^c)$ is closed.
Let $|\lambda|>\|A\|$. Then, $\|\lambda^{-1}A\|<1$, so $I-\lambda^{-1}A$ is invertible, and hence $\lambda I-A$ is invertible. Therefore, $\lambda\notin\sigma(A)$. In other words, $\sigma(A)\subseteq\overline{B(0,\|A\|)}$.

Since $\sigma(A)$ is closed and bounded in $\mathbb{C}$, it is compact (by Heine-Borel). $\square$

It can also be shown that $\sigma(A)\neq\emptyset$ using complex analysis: if $\sigma(A)=\emptyset$, pick an $f\in\mathscr{L}(E)^*$ s.t. $f(A^{-1})\neq 0$. It can be shown that $g(\lambda)=f([\lambda I-A]^{-1})$ is analytic in $\lambda\in\mathbb{C}$. Since $g(\lambda)\to 0$ as $|\lambda|\to\infty$, g is bounded and analytic, so by Liouville's theorem (from complex analysis), $g=0$, which contradicts the fact that $g(0)=f(A^{-1})\neq 0$, thus $\sigma(A)\neq\emptyset$.

Self-adjoint and non-negative operators have similar spectral properties to Hermitian and positive semi-definite matrices, which can be deduced using the following lemma:

Lemma. If for a self-adjoint operator $A \in \mathscr{L}(H)$, where H is a Hilbert space, there is a $\delta > 0$ s.t. $\|Ax\| \geqslant \delta \|x\|$ for all $x \in H$, then A is invertible.

Proof. The inequality implies that T is injective (*why?*). Now, $x \in \text{Ker } A$ iff $0 = (Ax, y) = (x, Ay)$ for all $y \in H$, *i.e.*, iff $x \in \mathscr{R}(A)^\perp$, so $\mathscr{R}(A)^\perp = \{0\}$, that is, $\mathscr{R}(A)$ is dense in H. On the other hand, $\mathscr{R}(A)$ is closed, since if (y_n), $y_n = Ax_n$, is a sequence in $\mathscr{R}(A)$ that converges to, say, $y \in H$, then (y_n) is Cauchy, and so is (x_n) (by the stated inequality), so $x_n \to x \in H$, say, and by continuity $y = Ax \in \mathscr{R}(A)$. Therefore, A is bijective, and its inverse is bounded due to the inequality, so A is invertible. □

Theorem. If $A \in \mathscr{L}(H)$ is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$. Furthermore, if $A \geqslant 0$, $\sigma(A) \subseteq [0, \infty)$.
Proof. Since $A = A^*$, if $\lambda = a + bi \in \sigma(A)$, then $\|(A - \lambda I)x\|^2 = \|Ax - ax\|^2 + b^2 \|x\|^2$ for every $x \in H$, so $\|(A - \lambda I)x\| \geqslant |b| \|x\|$. If $b \neq 0$, then $A - \lambda I$ by the lemma above, so $\lambda \notin \sigma(A)$ If $A \geqslant 0$, then for every $\lambda < 0$ one has that $|\lambda| \|x\|^2 = (-\lambda x, x) \leqslant ([A - \lambda I]x, x) \leqslant \|(A - \lambda I)x\| \|x\|$ for every $x \in H$, so $|\lambda| \|x\| \leqslant \|(A - \lambda I)x\|$, and by the lemma above $A - \lambda I$ is invertible, hence $\lambda \notin \sigma(A)$. □

The previous result can be strengthened to

Theorem. If $A \in \mathscr{L}(H)$ is self-adjoint, $m := \inf_{\|x\|=1}(Ax,x)$, $M := \sup_{\|x\|=1}(Ax,x)$, then $\sigma(A) \subseteq [m,M]$, and $m, M \in \sigma(A)$.

Proof. Let $\lambda > M$. Since $(Ax,x) \leqslant M(x,x)$ for all $x \in H$, we have that $\|(\lambda I - A)x\|\|x\| \geqslant (\lambda x - Ax, x) \geqslant (\lambda - M)\|x\|^2$, where $\lambda - M > 0$, or $\|(\lambda I - A)x\| \geqslant (\lambda - M)\|x\|$, so $\lambda I - A$ is invertible, *i.e.*, $\lambda \notin \sigma(A)$. Similarly, if $\lambda < m$ then $\lambda \notin \sigma(A)$, so $\sigma(A) \subseteq [m,M]$.

To prove that $M \in \sigma(A)$, consider the bilinear form $a(x,y) := (Mx - Ax, y)$, which is symmetric (because A is self-adjoint) and s.t. $a(x,x) = (Mx,x) - (Ax,x) \geqslant 0$ for all $x \in H$. Cauchy-Schwarz applied to a yields $|a(x,y)| \leqslant \sqrt{a(x,x)}\sqrt{a(y,y)}$, or $|(Mx - Ax, y)| \leqslant \sqrt{(Mx - Ax, x)}\sqrt{(My - Ay, y)}$. Taking sup over $\|y\| = 1$, we obtain

$$\|Mx - Ax\| \leqslant C\sqrt{(Mx - Ax, x)} \text{ for all } x \in H, \qquad (*)$$

where $C = \sup_{\|y\|=1}\sqrt{(My - Ay, y)}$. By definition of M, there is a sequence (x_n) s.t. $\|x_n\| = 1$ and $(Ax_n, x_n) \to M$. From $(*)$, $\|Mx_n - Ax_n\| \to 0$, so $M \in \sigma(A)$, since otherwise $MI - A$ would be invertible, so $x_n = (MI - A)^{-1}(Mx_n - Ax_n) \to 0$, a contradiction. Similarly, $m \in \sigma(A)$. □

Corollary. If $A \in \mathscr{L}(H)$ is self-adjoint and $\sigma(A) \subseteq [0,\infty)$, then A is non-negative.

Outline

Infinite Matrices

Linear operators in infinite dimensions can be represented by infinite matrices, resembling linear algebra.

Definition. Let E, F be separable Hilbert spaces, and $A \in \mathscr{L}(E,F)$. The *matrix* of A with respect to orthonormal bases (e_n) and (f_n) of E, F, respectively, is the array $[a_{jk}]_{j,k=1}^{\infty}$ of complex numbers given by $a_{jk} = (Ae_k, f_j)$.

It is difficult to determine from a matrix representation if an operator is bounded.

Infinite Matrices (cont.)

Example (Linear system)
Let $k \in C[-\pi,\pi]$ be 2π-periodic, and consider the integral operator K on $L_2[-\pi,\pi]$ given by

$$(Kx)(t) = \int_{-\pi}^{\pi} k(t-s)x(s)ds.$$

If $(e_n)_{n\in\mathbb{Z}}$ denotes the Fourier basis of $L_2[-\pi,\pi]$, then

$$(Ke_n)(t) = \int_{-\pi}^{\pi} k(t-s)e_n(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} k(s-t)e^{ins}ds = \frac{e^{int}}{\sqrt{2\pi}}\int_{-\pi}^{\pi} k(\tau)e^{-in\tau}d\tau = c_n e_n(t),$$

where c_n is the n-th Fourier coefficient of k. Therefore, the matrix of K with respect to (e_n) is $[a_{jk}]$ with $a_{jk} = (Ae_k, e_j) = c_k\delta_{j-k}$:

$$[A] = \begin{bmatrix} \ddots & & & \\ & c_{-1} & & 0 & \\ & & c_0 & & \\ & 0 & & c_1 & \\ & & & & \ddots \end{bmatrix}. \qquad \text{(diagonal matrix)}$$

Next Topic

Optimization of Functionals

Outline

Bonus: Applications of the Adjoint

Let $A \in \mathscr{L}(E,F)$, where E,F are Hilbert spaces.

Theorem. Let $y \in F$. Then the vector $x \in E$ minimizes $\|y - Ax\|$ iff $A^*Ax = A^*y$.
Proof. By the projection theorem, $x \in E$ minimizes $\|y - Ax\|$ iff $(y - Ax, A\tilde{x}) = 0$ for all $\tilde{x} \in E$. However, $(y - Ax, A\tilde{x}) = (A^*[y - Ax], \tilde{x})$, so the latter holds iff $A^*[y - Ax] = 0$. □

Theorem (Fredholm Alternative). $[\mathscr{R}(A)]^\perp = \text{Ker } A^*$.
Proof. $x \in \text{Ker } A^*$ iff $A^*x = 0$, *i.e.*, iff $(x, Ay) = (A^*x, y) = 0$ for all y, that is, iff $x \in [\mathscr{R}(A)]^\perp$. □

Corollary. Assume that $\mathscr{R}(A^*)$ is closed and $y \in \mathscr{R}(A)$. The vector $x \in E$ of minimum norm s.t. $Ax = y$ is given by $x = A^*z$, where $z \in E$ is any solution of $AA^*z = y$.
Proof. Every $x \in E$ satisfying $Ax = y$ is of the form $x = x_0 + m$, where $Ax_0 = y$ and $m \in \text{Ker } A$. By Fredholm's Alternative, $\text{Ker } A = [\mathscr{R}(A^*)]^\perp$, and by the minimum norm theorem, the sought $x \in E$ satisfies $x \perp [\mathscr{R}(A^*)]^\perp$, or $x \in [\mathscr{R}(A^*)]^{\perp\perp} = \mathscr{R}(A^*)$ (since $\mathscr{R}(A^*)$ is closed), so $x = A^*z$ for some $z \in E$, and plugging this expression into $Ax = y$ gives $AA^*z = y$. □

Example (control)

Consider a linear system of the form $\dot{x}(t) = Ax(t) + Bu(t)$. We want to drive $x(0) = 0$ to $x(T) = x_0$ by designing a control input $u(t)$ of minimum energy $\displaystyle\int_0^T u^2(t)dt$.

Let $u \in L_2[0,T]$. We know that $x(T) = \int_0^T e^{A(T-t)}Bu(t)dt$, so let us define an operator $\Phi\colon L_2[0,T] \to \mathbb{R}^n$ as

$$\Phi u = \int_0^T e^{A(T-t)}Bu(t)dt.$$

The problem is to find a $u \in L_2[0,T]$ of minimum norm s.t. $\Phi u = x_0$. Since $\mathscr{D}(\Phi^*) = \mathbb{R}^n$, the range of Φ^* is finite dimensional, and hence it is closed, so by the last corollary we have that the optimal solution is $u^{\text{opt}} = \Phi^* z$, where $\Phi\Phi^* z = x_0$

$\ldots$ so we need expressions for Φ^* and $\Phi\Phi^*$.

Bonus: Applications of the Adjoint (cont.)

Example (control) (cont.)
For every $u \in L_2[0,T]$ and $y \in \mathbb{R}^n$,

$$(\Phi u, y) = y^T \int_0^T e^{A(T-t)} B u(t) dt = \int_0^T y^T e^{A(T-t)} B u(t) dt = (u, \Phi^* y),$$

so $(\Phi^* y)(t) = B^T e^{A^T(T-t)} y$, and

$$\Phi\Phi^* y = \int_0^T e^{A(T-t)} BB^T e^{A^T(T-t)} y dt = \underbrace{\int_0^T e^{A(T-t)} BB^T e^{A^T(T-t)} dt}_{\in \mathbb{R}^{n\times n} \text{ (Controllability Gramian)}} y.$$

The optimal control is given by

$$u^{\text{opt}}(t) = (\Phi^* [\Phi\Phi^*]^{-1} x_0)(t) = B^T e^{A^T(T-t)} \left[\int_0^T e^{A(T-\tau)} BB^T e^{A^T(T-\tau)} d\tau \right]^{-1} x_0,$$

assuming that the inverse exists. Notice that $\mathscr{R}(\Phi\Phi^*)$ corresponds to the states reachable from the origin in T seconds/minutes/..., and that $\mathscr{R}(\Phi\Phi^*) = \mathscr{R}(\Phi)$ (*why?*).

Bonus: Uniform Boundedness Principle

Together with the Hahn-Banach theorem, the Uniform Boundedness principle, the Closed-Graph theorem and the Open Mapping theorem are considered to be the cornerstones of Banach space theory.

Theorem (Uniform Boundedness Principle / Banach-Steinhaus)
Let $\mathscr{F}$ be a family of bounded linear operators from a Banach space X to a normed space Y. If $\sup_{A \in \mathscr{F}} \|Ax\| < \infty$ for every $x \in X$, then $\sup_{A \in \mathscr{F}} \|A\| < \infty$.

Proof. Assume that $\sup_{A \in \mathscr{F}} \|A\| = \infty$, and choose a sequence (A_n) in $\mathscr{F}$ s.t. $\|A_n\| \geqslant 4^n$. Set $x_0 = 0 \in X$ and, for $n \in \mathbb{N}$, choose $x_n \in X$ as follows: note that for every $\|\xi\| \leqslant 3^{-n}$,

$$\max\{\|A_n(x_{n-1}+\xi)\|, \|A_n(x_{n-1}-\xi)\|\} \geqslant \frac{1}{2}\|A_n(x_{n-1}+\xi)\| + \frac{1}{2}\|A_n(x_{n-1}-\xi)\| \geqslant \|A_n \xi\|,$$

so taking sup over $\|\xi\| \leqslant 3^{-n}$ shows that there is a $\|\xi_n\| \leqslant 3^{-n}$ s.t., say, $\|A_n(x_{n-1}+\xi_n)\| \geqslant (2/3)3^{-n}\|A_n\|$; choose $x_n = x_{n-1} + \xi_n$. On the other hand, (x_n) is a Cauchy sequence (*why?*), which converges to, say, $x \in X$, and in addition, $\|x - x_n\| \leqslant (1/2)3^{-n}$, hence

$$\|A_n x\| = \|A_n(x - x_n) + A_n x_n\| \geqslant |\|A_n x_n\| - \|A_n(x - x_n)\|| \geqslant \left|\frac{2}{3}3^{-n}\|A_n\| - \frac{1}{2}3^{-n}\|A_n\|\right| \geqslant \frac{1}{6}(4/3)^n,$$

which tends to ∞ as $n \to \infty$. □

Application to divergence of Fourier series

From Topic 5, the Fourier series of an $f \in C[-\pi,\pi]$, truncated to N terms, is

$$f_N(x) = \sum_{n=-N}^{N} (f, e_n) e_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+y) D_N(y) dy, \qquad D_N(y) := \frac{\sin([N+1/2]y)}{\sin(y/2)}.$$

Define $T_N : C[-\pi,\pi] \to \mathbb{R}$ by $T_N f = f_N(0) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(y) D_N(y) dy$, whose norm is $\|T_N\| = (2\pi)^{-1} \int_{-\pi}^{\pi} |D_N(y)| dy$. However,

$$\int_{-\pi}^{\pi} |D_N(y)| dy = \int_{-\pi}^{\pi} \left| \frac{\sin([N+1/2]y)}{\sin(y/2)} \right| dy \geqslant 4 \int_0^{\pi} \left| \frac{\sin([N+1/2]y)}{y} \right| dy = 4 \int_0^{(N+1/2)\pi} |\sin(y)| \frac{dy}{y}$$
$$> 4 \sum_{k=1}^{N} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin(y)| dy = \frac{4}{\pi} \sum_{k=1}^{N} \frac{1}{k} \to \infty \quad \text{as } N \to \infty,$$

so by the uniform boundedness principle: *there is an $f \in C[-\pi,\pi]$ s.t. $f_N(0)$ diverges*.

Bonus: Closed Graph Theorem

Definitions

- The *graph* of a function $T : \mathscr{D}(T) \subseteq X \to Y$ is $\mathscr{G}(T) = \{(x, T(x)) \in X \times Y : x \in \mathscr{D}(T)\}$. If X, Y are vector spaces and T is linear, then $\mathscr{G}(T)$ is a linear subspace of $X \times Y$.
- If X, Y are normed spaces, a norm can be introduced in $X \times Y$, *e.g.*, $\|(x, y)\| = \|x\| + \|y\|$. An operator $T : \mathscr{D}(T) \subseteq X \to Y$ is *closed* if $\mathscr{G}(T)$ is closed in $X \times Y$; equivalently, T is closed iff whenever (x_n) is a sequence in $\mathscr{D}(T)$ s.t. $x_n \to x \in \mathscr{D}(T)$ and $y_n := T(x_n) \to y \in Y$, then $y = T(x)$.
- An *adjoint* of a linear (but not necessarily bounded) operator $T : \mathscr{D}(T) \subseteq X \to Y$ is an operator $T^* : \mathscr{D}(T^*) \subseteq Y^* \to X^*$ s.t. $\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$ for all $x \in \mathscr{D}(T)$, $y^* \in \mathscr{D}(T^*)$. Adjoints in general are non-unique, unless $\mathscr{D}(T)$ is dense in X, and $\mathscr{D}(T^*)$ consists of those $y^* \in Y^*$ for which $x \mapsto \langle Tx, y^* \rangle$ is bounded on $\mathscr{D}(T)$.

If $T : \mathscr{D}(T) \to Y$ is linear and closed, where X, Y are Banach spaces, $\mathscr{D}(T)$ is itself a Banach space under the *graph norm* $\|x\|_g := \|x\| + \|T(x)\|$, since $x \mapsto (x, T(x))$ is an isometry from $\mathscr{D}(T)$ to $\mathscr{G}(T)$, which is complete (*why?*). Also, T is bounded under this norm.

As $\langle (x, -Tx), (T^* y^*, y^*) \rangle = \langle x, T^* y^* \rangle - \langle Tx, y^* \rangle = 0$, $\mathscr{G}'(T^*) = \mathscr{G}(-T)^\perp$ if $\mathscr{D}(T) \subseteq X$ is dense, where $\mathscr{G}'(T^*) := \{(T^* y^*, y^*) : y^* \in \mathscr{D}(T^*)\}$ is the *reversed graph* of T^*, so T^* is always closed.

Bonus: Closed Graph Theorem (cont.)

Lemma. Let $T: X \to Y$ be linear and closed, where X, Y are Banach spaces. Then, $\mathscr{D}(T^*) = Y^*$.

Proof. First we will show that $\mathscr{D}(T^*)$ is weak*-dense in Y^*. If not, there is a $y \in Y \setminus \{0\}$ s.t. $\langle y, y^* \rangle = 0$ for all $y^* \in \mathscr{D}(T^*)$. But then $(0, y) \in {}^{\perp}\mathscr{G}'(-T^*) = \mathscr{G}(T)$ (since $\mathscr{G}(T)$ is closed), *i.e.*, $T(0) = y \neq 0$, which is impossible because T is linear.
Next we will show that $\mathscr{D}(T^*)$ is weak*-closed, which implies that $\mathscr{D}(T^*) = Y^*$. By Krein-Smulian, it suffices to show that $V = \mathscr{D}(T^*) \cap \{y^* \in Y^* : \|y^*\| \leqslant 1\}$ is weak*-closed. Now, $\sup_{y^* \in V} |\langle x, T^* y^* \rangle| = \sup_{y^* \in V} |\langle Tx, y^* \rangle| \leqslant \|Tx\|$, hence $\sup_{y^* \in V} \|T^* y^*\| =: K < \infty$ by uniform boundedness. Thus, $|\langle Tx, y^* \rangle| = |\langle x, T^* y^* \rangle| \leqslant K\|x\|$ for all $x \in X$, $y^* \in V$; since $y^* \mapsto \langle Tx, y^* \rangle$ is weak*-continuous, $|\langle Tx, y^* \rangle| \leqslant K\|x\|$ for all y^* in the weak*-closure of V, $\overline{V}$, *i.e.*, $x \mapsto \langle Tx, y^* \rangle$ is bounded on $\overline{V}$, so V is weak*-closed. □

Theorem (Closed graph theorem)

Let $T: X \to Y$ be linear and closed, where X, Y are Banach spaces. Then, T is bounded.

Proof. Assume T is unbounded. Then, there is a (x_n) in X, $\|x_n\| = 1$, s.t. $\|Tx_n\| \to \infty$, but $\sup_n |\langle Tx_n, y^* \rangle| = \sup_n |\langle x_n, T^* y^* \rangle| \leqslant \|T^* y^*\|$. Thus, (Tx_n) is a point-wise bounded but norm-unbounded family in X^{**}, which contradicts uniform boundedness. Thus, T is bounded. □

Corollary (Hellinger-Toeplitz theorem)

Let $T: H \to H$ be a linear self-adjoint operator in a Hilbert space H. Then, T is bounded.

Proof. Let (x_n) is in H, s.t. $x_n \to x \in H$ and $Tx_n \to y \in H$. For every $z \in H$, $(Tx, z) = (x, Tz) = \lim\, (x_n, Tz)$ $= \lim\, (Tx_n, z) = (y, z)$, so $Tx = y$ and T is closed. Then, by the closed graph theorem, T is bounded. □

Bonus: Open Mapping and Banach Inverse Theorems

Theorem (Banach inverse theorem)
Let $T \in \mathscr{L}(X,Y)$, where X,Y are Banach spaces. If T is bijective, then T^{-1} is continuous.

Proof. Since $T\colon X \to Y$ is bounded, its graph $\mathscr{G}(T)$ is closed in $X \times Y$: indeed, if (x_n) is a sequence in X converging to, say, $x \in X$, and (y_n), where $y_n = Tx_n$, converges to, say, $y \in Y$, then by continuity $y = Tx$, so $\mathscr{G}(T)$ is closed. Then, $\mathscr{G}(T^{-1}) = \mathscr{G}'(T)$ is closed in $Y \times X$, and by the closed graph theorem, T^{-1} is continuous. □

Corollary (Open mapping / Banach-Schauder)
Let $T \in \mathscr{L}(X,Y)$ be surjective, where X,Y are Banach spaces. Then, T is an *open mapping*, *i.e.*, $T(U)$ is open in Y whenever U is open in X.

Proof. Define an equivalence relation on X, where $x \sim y$ iff $x - y \in \text{Ker } T$. Since T is bounded, $\text{Ker } T \subseteq X$ is closed, so the set of equivalence classes, $X/\text{Ker } T$, is a Banach space with norm $\|[x]\| := \inf_{k \in \text{Ker } T} \|x + k\|$ (*exercise!*). T induces a bijective bounded linear operator $\bar{T}\colon X/\text{Ker } T \to Y$ by $\bar{T}([x]) = T(x)$, so by the Banach inverse theorem, $\bar{T}^{-1}$ is continuous, *i.e.*, $\bar{T}$ maps open sets onto open sets. Also, $T = \bar{T} \circ \pi$, where $\pi\colon X \to X/\text{Ker } T$, given by $\pi(x) = [x]$, is linear, surjective and open (because if $\|[x - y]\| < \varepsilon$, then $\varepsilon > \inf_{m \in \text{Ker } T} \|x - y - m\|$, so there is an $m^* \in \text{Ker } T$ such that $\|x - y - m^*\| < \varepsilon$, thus $B([x], \varepsilon) \subseteq \pi(B(x, \varepsilon))$), and the composition of open maps is open, hence T is open. □

Bonus: Spectral Theorem

Spectral theorems correspond to a class of results that allow one to "diagonalize" a linear operator (thus resembling the eigenvalue decomposition result from linear algebra). Here we will establish one version for self-adjoint operators, based on the following facts:

(1) *Bounded monotone sequences of self-adjoint operators converge to a self-adjoint operator.*
Assume $0 \leqslant A_1 \leqslant A_2 \leqslant \cdots \leqslant I$, and let $B = A_{n+k} - A_n$ for some $n, k \in \mathbb{N}$. Note that $0 \leqslant B \leqslant I$, so Cauchy-Schwarz applies to the bilinear form (Bx, y); in particular, $(Bx, Bx)^2 \leqslant (Bx, x)(B^2x, Bx) \leqslant (Bx, x)(Bx, Bx)$, so $\|Bx\|^2 = (Bx, Bx) \leqslant (Bx, x)$. Thus, $\|A_{n+k}x - A_n x\|^2 \leqslant (A_{n+k}x, x) - (A_n x, x)$ for every $x \in H$. Now, since $((A_n x, x))_{n \in \mathbb{N}}$ is a bounded monotone sequence in $\mathbb{R}$, it converges, so $(A_n x)$ is Cauchy in H, and $\lim_{n \to \infty} A_n x = Ax$ exists. A is linear, and by uniform boundedness, it is bounded. Furthermore, letting $n \to \infty$ in $(A_n x, y) = (x, A_n y)$ shows that A is self-adjoint. □

Let $\mathbb{R}[t]$ ($\mathbb{C}[t]$) be the set of polynomials in t with real (complex) coefficients. If $p \in \mathbb{C}[t]$, where $p(t) = p_n t^n + p_{n-1} t^{n-1} + \cdots + p_1 t + p_0$, one can define, for every $A \in \mathscr{L}(H)$,

$$\tilde{p}(A) = p_n A^n + p_{n-1} A^{n-1} + \cdots + p_1 A + p_0 I.$$

Bonus: Spectral Theorem (cont.)

(2) *Every operator $A \geqslant 0$ has a unique non-negative square root $A^{1/2}$: $(A^{1/2})^2 = A$.*
Firstly, we can assume w.l.o.g., by scaling A, that $0 \leqslant A \leqslant I$. Consider the sequence of operators $(T_n)_{n\in\mathbb{N}}$ given by $T_1 = 0$ and $T_{n+1} = T_n + (1/2)[A - T_n^2]$ for $n \in \mathbb{N}$. Note that $0 = T_1 \leqslant I$, $T_2 - T_1 = (1/2)A \geqslant 0$, and that if $0 \leqslant T_n \leqslant I$ and $T_n \leqslant T_{n+1}$, then $I - T_n \geqslant 0$, so $0 \leqslant (1/2)(I - T_n)^2 + (1/2)(I - A) = I - T_n - (1/2)(A - T_n^2) = I - T_{n+1}$, *i.e.*, $T_{n+1} \leqslant I$, and $T_{n+2} - T_{n+1} = T_{n+1} + (1/2)[A - T_{n+1}^2] - T_n - (1/2)[A - T_n^2] = (1/2)(T_{n+1} - T_n)(I - T_{n+1} + I - T_n) \geqslant 0$, so $T_{n+1} \leqslant T_{n+2}$. Hence, from (1), $T_n \to T$, where $T = T + (1/2)[A - T^2]$, or $T^2 = A$. Let $A^{1/2} := T$.
Consider another operator $B \geqslant 0$ s.t. $B^2 = A$. Then, $BA = B^3 = AB$, so $BA^n = A^n B$ for every $n \in \mathbb{N}$, thus $BT_n = T_n B$, and taking $n \to \infty$, $BA^{1/2} = A^{1/2}B$. Let $M = (A^{1/2})^{1/2}$ and $N = B^{1/2}$. Then, given $x \in H$, let $y = (A^{1/2} - B)x$. We have that $\|My\|^2 + \|Ny\|^2 = (M^2 y, y) + (N^2 y, y) = ([A^{1/2} + B]y, y) = ([A - B^2]x, y) = 0$, so $My = Ny = 0$ and $M^2 y = N^2 y = 0$, *i.e.*, $A^{1/2} y = By = 0$, so $\|(A^{1/2} - B)x\|^2 = ([A^{1/2} - B]^2 x, x) = ([A^{1/2} - B]y, x) = 0$, that is, $A^{1/2} = B$. □

(3) *Let A, B be commuting non-negative, linear, bounded operators. Then, $AB \geqslant 0$.*
From the proof of (2), since $AB = BA$, also $AB^{1/2} = B^{1/2}A$ holds. Thus, for all $x \in H$, $(ABx, x) = (AB^{1/2}B^{1/2}x, x) = (B^{1/2}AB^{1/2}x, x) = (AB^{1/2}x, B^{1/2}x) \geqslant 0$. □

Bonus: Spectral Theorem (cont.)

The map $\phi\colon \mathbb{C}[t] \to \mathscr{L}(H)$ given by $\phi(p) = \tilde{p}(A)$ is linear, *multiplicative* (*i.e.*, $\phi(pq) = \phi(p)\phi(q)$) and *unital* (*i.e.*, $\phi(1) = I$). ϕ is also *order-preserving*:

(4) *If $p \in \mathbb{R}[t]$ satisfies $p(t) \geqslant 0$ for all $t \in [m, M]$, and the self-adjoint operator A satisfies $mI \leqslant A \leqslant MI$, then $\tilde{p}(A) \geqslant 0$.*
p can be factorized as $p(t) = c\prod_j (t - \alpha_j)\prod_k (\beta_k - t)\prod_l \left[(t - \gamma_l)^2 + \delta_l^2\right]$, where $c > 0$, $\alpha_j \leqslant m \leqslant M \leqslant \beta_k$ and $\gamma_l, \delta_l \in \mathbb{R}$. By (3), we have that $\tilde{p}(A) \geqslant 0$. □

Corollary. The map ϕ can be extended to $C[m, M]$. Moreover, if $f \in C[m, M]$, $\|\tilde{f}(A)\| \leqslant \|f\|$.

Proof. Since $\mathbb{C}[t]$ is dense in $C[m, M]$, ϕ can be extended uniquely by continuity. The inequality follows because, for every $p \in \mathbb{C}[t]$, $\|p\| \pm p$ is a non-negative polynomial in $[m, M]$, so $\|p\| I \geqslant \pm\tilde{p}(A)$, *i.e.*, $\|p\| \geqslant \|\tilde{p}(A)\|$; this inequality extends by continuity to $C[m, M]$. □

The extension of ϕ to $C[m, M]$ defines a *functional calculus* for operators, *i.e.*, given a self-adjoint $A \in \mathscr{L}(H)$, and $f \in C[m, M]$, $\tilde{f}(A)$ is another self-adjoint operator in H.

Given a self-adjoint operator $A \in \mathscr{L}(H)$, where H is a separable Hilbert space, a *cyclic vector* of A is an element $\xi \in H$ s.t. $\text{lin}\{A^k\xi : k \in \mathbb{N}_0\} = \text{lin}\{\tilde{p}(A)\xi : p \in \mathbb{C}[t]\}$ is dense in H.

Next we present a version of the Spectral Theorem for self-adjoint operators in a separable Hilbert space:

Spectral Theorem

If the self-adjoint operator $A \in \mathscr{L}(H)$, where H is a separable Hilbert space, has a cyclic vector ξ, then there is a unitary operator $U : H \to L_2(l)$ identifying H with $L_2(l)$ for some $l \in C[m,M]^*$, s.t. $UAU^* = M_t$, where $M_t : L_2(l) \to L_2(l)$ is the multiplication operator $(M_t x)(t) = t\,x(t)$ for $t \in [m,M]$, and $m, M \in \mathbb{R}$ are s.t. $m\|x\|^2 \leqslant (Ax,x) \leqslant M\|x\|^2$ for all $x \in H$.

$L_2(l)$ is the completion of $C[m,M]$, with inner product $(f,g) = l(f\overline{g})$, where $l \in C[m,M]^*$ is *positive* (*i.e.*, $l(f) \geqslant 0$ if $f(t) \geqslant 0$ for all $t \in [m,M]$). To ensure that $(f,f) > 0$ if $f \neq 0$, one actually considers $C[m,M]/N$ instead of $C[m,M]$, where $N = \{f \in C[m,M] : l(\tilde{f}^2) = 0\}$.

An operator $A \in \mathscr{L}(E,F)$ is *unitary* if $AA^* = A^*A = I$; thus, $(Ax,Ay)_F = (x,y)_E$ for all $x, y \in E$.

Bonus: Spectral Theorem (cont.)

Proof. Define the linear functional $l \in C[m,M]^*$ by $l(f) := (\tilde{f}(A)\xi, \xi)$ for all $f \in C[m,M]$. Note that $l \geqslant 0$, since $f(A) \geqslant 0$ if $f(x) \geqslant 0$ on $[m,M]$, and that $(f,g) := l(f\overline{g}) = (\tilde{f}(A)\xi, \tilde{g}(A)\xi)$ defines an inner product in $C[m,M]/N$, where $N = \{f \in C[m,M]\colon l(\tilde{f}^2) = 0\}$. Denote by $L_2(l)$ the completion of $C[m,M]/N$.
Define the operator $U\colon H \to L_2(l)$ by $U\tilde{p}(A)\xi = p$ for all $p \in \mathbb{C}[t]$, which specifies it on a dense set of H (since ξ is cyclic). This operator is well defined, since $\tilde{p}_1(A)\xi = \tilde{p}_2(A)\xi$ iff $0 = \|\tilde{p}_1(A)\xi - \tilde{p}_2(A)\xi\|^2 = l([p_1 - p_2]^2)$, *i.e.*, $p_1 - p_2 \in N$. Also, U has the following properties:

(1) U is *isometric*: $(U\tilde{p}_1(A)\xi, U\tilde{p}_2(A)\xi)_H = (p_1, p_2)$ for every $p_1, p_2 \in \mathbb{C}[t]$.

(2) $\mathscr{R}(U)$ is dense in $L_2(l)$, since is contains all polynomials in $[m,M]$ modulo N. This property, together with (1), show that the extension of U to H by continuity is a unitary operator.

(3) $(UA\tilde{p}(A)\xi)(t) = tp(t) = t(U\tilde{p}(A)\xi)(t)$, so, by the density of the polynomials and the cyclic nature of ξ, $UAv = M_t Uv$ for all $v \in H$, *i.e.*, $UAU^* = M_t$. Note in particular that $U\xi = 1$. □

Note. Assuming that A has a cyclic vector is not very restrictive, since otherwise one can pick a ξ_1 from a complete orthonormal sequence (e_n) in H, and define $H_1 = \text{clin}\{A^n\xi\colon n \in \mathbb{N}\}$; if $H_1 \neq H$, apply iteratively this procedure to $(H_1 \oplus \cdots \oplus H_{k-1})^{\perp}$, so H can be written as a countable direct sum, $H = H_1 \oplus H_2 \oplus \cdots$. The spectral theorem can then be applied to each of these subspaces individually.

Bonus: Application to SOS Optimization

Motivation: Minimization of (*non-convex*) polynomials subject to polynomial constraints:

$$\begin{aligned} \min_{x=(x_1,\ldots,x_n)} \quad & p_0(x) \\ \text{s.t.} \quad & p_k(x) \geqslant 0, \quad k=1,\ldots,m \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min_{t\in\mathbb{R}} \quad & t \\ \text{s.t.} \quad & t - p_0(x) \geqslant 0 \text{ for all } x \text{ s.t. } p_k(x) \geqslant 0,\ k=1,\ldots,m. \end{aligned}$$

We need to characterize which polynomials $p \in \mathbb{R}[x]$ are positive, *i.e.*, $p(x) \geqslant 0$, either in $\mathbb{R}^n$ or in a set defined by other polynomials, *e.g.*, $\{x \in \mathbb{R}^n : p_k(x) \geqslant 0 \text{ for all } k = 1,\ldots,m\}$.

Definitions

- $p \in \mathbb{R}[x]$ ($x \in \mathbb{R}^n$) is a *sum-of-squares* (SOS) polynomial if $p(x) = (q(x))^2$ for some $q \in \mathbb{R}[x]$.
- The set of SOS polynomials in $\mathbb{R}[x]$ is denoted $\Sigma^2\mathbb{R}[x]$.
- The set of polynomials $p \in \mathbb{R}[x]$ which are non-negative in $\mathbb{R}^n$ is denoted $\mathscr{P}_+(\mathbb{R}^n)$.
- The *quadratic module generated by a finite set of polynomials* $F = \{f_1,\ldots,f_N\} \subseteq \mathbb{R}[x]$ is

$$\mathrm{QM}(F) = \sum_{f\in F\cup\{1\}} f\Sigma^2\mathbb{R}[x] = \left\{q_0^2(x) + f_1(x)q_1^2(x) + \cdots + f_N(x)q_N^2(x) \colon q_k \in \mathbb{R}[x]\right\}.$$

- A quadratic module is *Archimedean* if there is a $C > 0$ s.t. $C - x_1^2 - \cdots - x_n^2 \in \mathrm{QM}(F)$.

Bonus: Application to SOS Optimization (cont.)

In general $\Sigma^2\mathbb{R}[x] \subseteq \mathscr{P}_+(\mathbb{R}^n)$, and both sets are typically strictly different (Hilbert, 1888).

While $\mathscr{P}_+(\mathbb{R}^n)$ may be difficult to characterize, the coefficients of SOS polynomials have a simple, convex characterization (Parrilo, 2000): Since $p \in \Sigma^2\mathbb{R}[x]$ iff $p(x) = q^2(x)$, and a polynomial $q \in \mathbb{R}[x]$ can be written as a linear combination of *monomials* (*e.g.*, $q(x) = x_1^2 + 3x_1x_2 + 4x_2^2 = [1\ 3\ 4][x_1^2\ \ x_1x_2\ \ x_2^2]^T =: \alpha^T m(x)$), one has that

$$p(x) = m(x)^T \underbrace{\alpha\alpha^T}_{A} m(x).$$

The coefficients of p appear in $A \succeq 0$. Conversely, if $p(x) = m(x)^T A m(x)$ for some matrix $A \succeq 0$, decomposing A as $v_1v_1^T + \cdots + v_mv_m^T$ yields $p(x) = [v_1^T m(x)]^2 + \cdots + [v_m^T m(x)]^2$, so $p \in \Sigma^2\mathbb{R}[x]$.

Note. The decomposition $p(x) = m(x)^T A m(x)$ is not unique: $x_1^2 + 2x_1x_2 + x_2^2$ can be written as $[x_1\ \ x_2]\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}[x_1\ \ x_2]^T$ or $[x_1\ \ x_2]\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}[x_1\ \ x_2]^T$; however, the set of all A that yield p is a linear subspace (*e.g.*, $\{A \in \mathbb{R}^{2\times 2} : a_{11} = a_{22} = 1, a_{12} + a_{21} = 2\}$), so the characterization of an SOS polynomial in terms of A is convex.

Bonus: Application to SOS Optimization (cont.)

An impressive result, due to M. Putinar (1993), shows that, under mild conditions, the set of polynomials which are strictly positive on a set $\mathscr{D}_F := \{x \in \mathbb{R}^n : f(x) \geqslant 0 \text{ for all } f \in F\}$ defined by a finite set $F \subseteq \mathbb{R}[x]$ can be characterized in terms of SOS polynomials:

Theorem (Putinar's *Positivstellensatz*)
Consider a finite set $F \subseteq \mathbb{R}[x]$, $x \in \mathbb{R}^n$, s.t. QM(F) is Archimedean. Then, every polynomial strictly positive on $\mathscr{D}_F$ is in QM(F).

In other words, every p which is strictly positive on $\mathscr{D}_F$ can be written as

$$p(x) = p_0(x) + f_1(x)p_1(x) + \cdots + f_N(x)p_N(x), \qquad F = \{f_1, \ldots, f_N\},$$

where $p_0, \ldots, p_N$ are SOS polynomials, so if one fixes the degrees of these polynomials, it is possible to characterize p in a convex manner!

The assumption of QM(F) being Archimedean implies that $\mathscr{D}_F$ should be compact, and is easy to fulfill by adding to F the polynomial $C - x_1^2 - \cdots - x_n^2$, with $C \geqslant 1$ sufficiently large.

Putinar's Positivstellensatz is a purely algebraic result from real semi-algebraic geometry, but we will provide a functional analytical proof, based on Hahn-Banach and some spectral properties. However, first we need to generalize the notion of spectrum to a set of operators, and establish the *spectral mapping theorem* :

Definition. Let $A_1,\dots,A_n \in \mathscr{A} \subseteq \mathscr{L}(H)$, where $\mathscr{A}$ is a *commutative algebra* of operators on a Hilbert space H, *i.e.*, a subset of $\mathscr{L}(H)$ s.t. if $A,B \in \mathscr{A}$ and $\alpha \in \mathbb{C}$, then $AB = BA$ and $A+B, \alpha A, AB \in \mathscr{A}$. The *joint spectrum* of $A = (A_1,\dots,A_n)$ in $\mathscr{A}$, denoted $\sigma(A)$, is the set of $\lambda \in \mathbb{C}^n$ for which there exist no $B_1,\dots,B_n \in \mathscr{A}$ s.t. $B_1(A_1 - \lambda_1 I) + \cdots + B_n(A_n - \lambda_1 I) = I$. Note that $\sigma(A) \subseteq \sigma(A_1) \times \cdots \times \sigma(A_n)$.

If $f\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial of the form $f(x) = \sum_{i_1,\dots,i_n \in \mathbb{N}_0} \alpha_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$, and $A_1,\dots,A_n \in \mathscr{L}(H)$ are commuting operators, let $\tilde{f}\colon \mathscr{L}(H)^n \to \mathscr{L}(H)$ be given by $\tilde{f}(A) = \sum_{i_1,\dots,i_n \in \mathbb{N}_0} \alpha_{i_1 \cdots i_n} A_1^{i_1} \cdots A_n^{i_n}$, where $A = (A_1,\dots,A_n) \in \mathscr{L}(H)^n$. This definition extends to systems of polynomials $f\colon \mathbb{C}^n \to \mathbb{C}^m$.

Theorem (Spectral Mapping)
Let $A = \{A_1,\dots,A_n\}$ be a subset of a commutative algebra of operators $\mathscr{A}$ on a Hilbert space H, and $f\colon \mathbb{C}^n \to \mathbb{C}^m$ a system of polynomials. Then, $f(\sigma(A)) = \sigma(\tilde{f}(A))$.

Bonus: Application to SOS Optimization (cont.)

Lemma. If $A \in \mathscr{L}(H)$, and $\lambda \in \partial\sigma(A)$, then there is a sequence (T_n) in $\mathscr{L}(H)$ s.t. T_n is invertible and $\|T_n\| = 1$ for all $n \in \mathbb{N}$, and $(A - \lambda I)T_n \to 0$.

Proof. Since $\lambda \in \partial\sigma(A)$, pick a sequence (λ_n) in $\sigma(A)^c$ s.t. $\lambda_n \to \lambda$, and let $R_n := (A - \lambda_n I)^{-1}$. Then, $R_n(A - \lambda I) - I = R_n(A - \lambda_n I + (\lambda_n - \lambda)I) - I = (\lambda_n - \lambda)R_n$. Then, $(\|R_n\|)$ is unbounded; otherwise there is an $M > 0$ s.t. $\|R_n\| \leqslant M$ for all n, and $\|R_n(A - \lambda I) - I\| = |\lambda_n - \lambda|\|R_n\| \to 0$, so $\|R_{n^*}(A - \lambda I) - I\| < 1$ for some n^*, thus $R_{n^*}(A - \lambda I)$ is invertible, and so is $A - \lambda I = (A - \lambda_n I)R_{n^*}(A - \lambda I)$, a contradiction. Thus, assume that $\|R_n\| \to \infty$, and let $T_n := R_n/\|R_n\|$, so $\|T_n\| = 1$. Then, $\|(A - \lambda I)T_n\| = \|(A - \lambda I)R_n\|/\|R_n\| = \|I/\|R_n\| + (\lambda_n - \lambda)T_n\| \leqslant 1/\|R_n\| + |\lambda_n - \lambda|\|T_n\| \to 0$. □

Proof of Spectral Mapping Theorem (Harte, 1972). If $f_k : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, then by the remainder theorem, for every $\lambda \in \mathbb{C}^n$, $\tilde{f}_k(A) - f_k(\lambda)I = \sum_j B_j(A_j - \lambda_j I)$ for some $B_1, \dots, B_n \subseteq \mathscr{A}$, so if $f(\lambda) \notin f(\sigma(A))$, then $\lambda \notin \sigma(A)$, *i.e.*, $f(\sigma(A)) \subseteq \sigma(\tilde{f}(A))$.

To prove the converse, we will show that if $C = (C_1, \dots, C_m) \in \mathscr{A}^m$, and $\mu \in \sigma(C) \subseteq \mathbb{C}^m$, then there exists a $\lambda \in \mathbb{C}^n$ s.t. $(\lambda, \mu) \in \sigma(A, C)$. This is done by induction on n, so we will only consider $n = 1$:

Let $\mathscr{N} := \overline{\left\{\sum_j B_j(C_j - \mu_j I) : B_1, \dots, B_m \in \mathscr{A}\right\}}$. Note that $A\mathscr{N} \subseteq \mathscr{N}$ for every $A \in \mathscr{A}$ and that $I \neq \mathscr{N}$ (since $\mu \in \sigma(C)$), so $\mathscr{A}/\mathscr{N} \neq \{[0]\}$. Define $L_{A_1} : \mathscr{A}/\mathscr{N} \to \mathscr{A}/\mathscr{N}$ as $L_{A_1}([B]) = [A_1 B]$. $\sigma(L_{A_1}) \neq \emptyset$ is compact, so pick a $\lambda_1 \in \partial\sigma(L_{A_1})$. Then, by the lemma above, there is a sequence (T_n) of invertible operators in $\mathscr{A}/\mathscr{N}$ s.t. $\|[T_n]\|_{\mathscr{A}/\mathscr{N}} = 1$ for all n and $\|[(A_1 - \lambda_1 I)T_n]\|_{\mathscr{A}/\mathscr{N}} = \inf_{N \in \mathscr{N}} \|(A_1 - \lambda_1 I)T_n + N\| \to 0$.

Proof (cont.)

Based on this result, we claim that $(\lambda_1, \mu) \in \sigma(A_1, C)$, since otherwise there would be $A'_1, C'_1, \ldots, C'_n \in \mathscr{A}$ s.t. $A'_1(A_1 - \lambda_1 I) + C'_1(C_1 - \lambda_1 I) + \cdots + C'_n(C_n - \lambda_n I) = I$, hence for an arbitrary $D \in \mathscr{A}$ we have that $D = A'_1(A_1 - \lambda_1 I)D + C'_1(C_1 - \lambda_1 I)D + \cdots + C'_n(C_n - \lambda_n I)D \in A'_1(A_1 - \lambda_1 I)D + \mathscr{N}$, but then $\|[D]\|_{\mathscr{A}/\mathscr{N}} = \inf_{N \in \mathscr{N}} \|A'_1(A_1 - \lambda_1 I)D + N\| \leqslant \inf_{N \in \mathscr{N}} \|A'_1(A_1 - \lambda_1 I)D + A'_1 N\| = \inf_{N \in \mathscr{N}} \|A'_1[(A_1 - \lambda_1 I)D + N]\| \leqslant \|A'_1\| \|[(A_1 - \lambda_1 I)D]\|_{\mathscr{A}/\mathscr{N}}$, which contradicts the properties of (T_n). Thus, $(\lambda_1, \mu) \in \sigma(A_1, C)$.

Therefore, in general, for every $\mu \in \sigma(\tilde{f}(A))$ there is a $\lambda \in \mathbb{C}^n$ s.t. $(\lambda, \mu) \in \sigma(A, \tilde{f}(A))$. Since $\sigma(A, \tilde{f}(A)) \subseteq \sigma(A) \times \sigma(\tilde{f}(A))$, $\lambda \in \sigma(A)$. We just need to show that $\mu \in f(\lambda)$. Consider the system of polynomials $g: \mathbb{C}^{n+m} \to \mathbb{C}^m$ given by $g(\lambda, \mu) = \mu - f(\lambda)$. Then, by our first result, $\mu - f(\lambda) = g(\lambda, \mu) \in g(\sigma(A, \tilde{f}(A))) \subseteq \sigma(\tilde{g}(A, \tilde{f}(A))) = \sigma(0) = \{0\}$, *i.e.*, $\mu = f(\lambda)$, so $\sigma(\tilde{f}(A)) \subseteq f(\sigma(A))$.
In conclusion, $f(\sigma(A)) \subseteq \sigma(\tilde{f}(A))$ and $\sigma(\tilde{f}(A)) \subseteq f(\sigma(A))$, thus $\sigma(\tilde{f}(A)) = f(\sigma(A))$. □

Bonus: Application to SOS Optimization (cont.)

Definition. Let K be a convex set in a vector space V. $x \in K$ is an *algebraic interior point of K relative to V* if for every $v \in V$ there is an $\varepsilon > 0$ s.t. $x + tv \in K$ for all $t \in [0, \varepsilon]$. The set of all algebraic interior points of K is called the *algebraic interior* of K, aint K.

To establish Putinar's Positivstellensatz, note that Eidelheit's separating hyperplane theorem can be modified to this "algebraic" version: If K_1 and K_2 are convex sets in a real vector space V s.t. aint $K_1 \neq \emptyset$ and $K_2 \cap \text{aint } K_1 = \emptyset$. Let $x_0 \in \text{aint } K_1$. Then there is a linear functional $l: V \to \mathbb{R}$ s.t. $l(x) \leqslant 0$ for all $x \in K_2$, $l(x) \geqslant 0$ for all $x \in K_1$, and $l(x_0) > 0$. (*Exercise!*)

Lemma. 1 is an algebraic interior point of an Archimedean QM(F).

Proof. Since $C - x_1^2 - \cdots - x_n^2 \in \text{QM}(F)$ for some $C \geqslant 1$, and QM(F) is a convex set,

- $C - x_i^2 = C - x_1^2 - \cdots - x_n^2 + \sum_{j \neq i} x_j^2 \in \text{QM}(F)$ for all $i = 1, \ldots, n$.
- $C \pm x_i = \frac{1}{2}[(C-1) + (C - x_i^2) + (x_i \pm 1)^2] \in \text{QM}(F)$ for all $i = 1, \ldots, n$.
- If $K \pm q \in \text{QM}(F)$ ($q \in \mathbb{R}[x]$, $K > 0$), then $K^2 - q^2 = \frac{1}{2K}[(K+q)^2(K-q) + (K-q)^2(K+q)] \in \text{QM}(F)$.
- If $K_1 \pm q_1, K_2 \pm q_2 \in \text{QM}(F)$, then $K_1 + K_2 - (q_1 \pm q_2) \in \text{QM}(F)$, and $\frac{(C_1+C_2)^2}{4} \pm q_1 q_2 = \frac{(C_1+C_2)^2}{4} \pm \frac{1}{4}(q_1+q_2)^2 \mp \frac{1}{4}(q_1 - q_2)^2 \in \text{QM}(F)$.
- From the previous properties, for every $p \in \mathbb{R}[x]$ there is a $K > 0$ s.t. $N \pm p \in \text{QM}(F)$ for all $N \geqslant K$, *i.e.*, $1 \pm \varepsilon p \in \text{QM}(F)$ for all $\varepsilon \in [0, 1/K]$. Thus, 1 is an algebraic interior point of QM(F). □

Bonus: Application to SOS Optimization (cont.)

Proof of Putinar's Positivstellensatz (Helton and Putinar, 2008)
Firstly notice that $\mathrm{QM}(F)$ is a convex set. Assume, to the contrary, that p is a strictly positive polynomial in $\mathscr{D}_F$, but $p \notin \mathrm{QM}(F)$. By the modified separating hyperplane theorem, there is a linear functional l on $\mathbb{R}[x]$ s.t. $l(1) > 0$, $l(q) \geqslant 0$ for all $q \in \mathrm{QM}(F)$, and $l(p) \leqslant 0$; extend l algebraically to $\mathbb{C}[x]$.
Construct a Hilbert space $L_2(l)$ as the completion of $\mathbb{C}[x]/N$, where $N = \{q \in \mathbb{C}[x] \colon l(q) = 0\}$, and $(q, r) = l(q\bar{r})$. Consider the tuple of multiplication operators $M = (M_{x_1}, \dots, M_{x_n})$ on $L_2(l)$ where $M_{x_k} q(x) = x_k q(x)$, which are self-adjoint and commute with each other. Furthermore, these operators are bounded, since $([C - x_1^2 - \cdots - x_n^2]q, q) = l([C - x_1^2 - \cdots - x_n^2]q^2) \geqslant 0$ by the Archimedean property (*i.e.*, $[C - x_1^2 - \cdots - x_n^2]q^2 \in \mathrm{QM}(F)$) and this implies that $(M_{x_k} q, q) \leqslant C(q, q)$ for every $q \in \mathbb{C}[x]$.
For every $f \in F$, since $(\tilde{f}(M)p, p) = (fp, p) \geqslant 0$ for every $p \in \mathbb{C}[x]$, thus $\tilde{f}(M)$ is non-negative, *i.e.*, $\sigma(\tilde{f}(M)) \subseteq [0, \infty)$, so the spectral mapping theorem implies that $f(\sigma(M)) = \sigma(\tilde{f}(M)) \subseteq [0, \infty)$ for all $f \in F$, that is, $\sigma(M) \subseteq \mathscr{D}_F$.
Therefore, for every $q \in \mathbb{C}[x]$ s.t. $q(x) \geqslant 0$ on $\mathscr{D}_F$, it holds by the spectral mapping theorem that $\sigma(\tilde{q}(M)) = q(\sigma(M)) \subseteq [0, \infty)$, so, by the Corollary in Slide 29, $\tilde{q}(M)$ is non-negative, thus $l(q) = (q, 1) = (\tilde{q}(M)1, 1) \geqslant 0$, *i.e.*, l is a positive functional on $\mathbb{R}[x]$.
Since $\mathscr{D}_F$ is compact, there is an $\varepsilon > 0$ s.t. $p(x) \geqslant \varepsilon$ for all $x \in \mathscr{D}_F$, so $l(p) \geqslant \varepsilon l(1) > 0$, a contradiction.
Therefore, all strictly positive polynomials in $\mathscr{D}_F$ belong to $\mathrm{QM}(F)$. $\square$

Example (from slides by C. Scherer and S. Weiland)

Consider the problem of testing whether the following polynomials are Hurwitz (*i.e.*, have all their roots inside the unit disk):

$$\{s^3 + (3 - \delta_1^2 + \delta_2)s^2 + (3 + \delta_1)s + (0.9 + \delta_1\delta_2) \colon \delta_1 \in [-1,1],\ \delta_2 \in [-1,1]\}.$$

By the Routh-Hurwitz criterion, this amounts to checking

$$\left.\begin{array}{l} 3 - \delta_1^2 + \delta_2 \geqslant 0, \text{ and} \\ (3 + \delta_1 + \delta_2)(3 + \delta_1) - (0.9 + \delta_1\delta_2) \geqslant 0 \end{array}\right\} \quad \text{for all } \delta_1, \delta_2 \text{ s.t. } \delta_1^2 \leqslant 1 \text{ and } \delta_2^2 \leqslant 1.$$

By Putinar's Positivstellensatz, the positivity of the first condition is equivalent to

$$3 - \delta_1^2 + \delta_2 = p_0(\delta_1, \delta_2) + p_1(\delta_1, \delta_2)(1 - \delta_1^2) + p_2(\delta_1, \delta_2)(1 - \delta_2^2) \qquad (*)$$

for some SOS polynomials $p_0, p_1, p_2 \in \Sigma^2\mathbb{R}[\delta_1, \delta_2]$. By setting upper bounds on the degrees of these polynomials, $(*)$ corresponds to an LMI feasibility problem that can be solved using standard convex optimization tools (CVX/Yalmip via Sedumi, SDPT3, Mosek, . . .).