EL3370 Mathematical Methods in Signals, Systems and Control

Topic 5: Orthogonal Expansions

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Outline

Outline

Orthonormal Sets

The notion of basis is very important, since it allows to define "coordinates" in a space, thus allowing explicit computations in Hilbert spaces.

Definition

In an inner product space V, a family $(e_\alpha)_{\alpha\in I}$ in $V \setminus \{0\}$ is an *orthogonal set* if $e_\alpha \perp e_\beta$ for $\alpha \neq \beta$. If also $\|e_\alpha\| = 1$ for all $\alpha \in I$, $(e_\alpha)_{\alpha\in I}$ is an *orthonormal set*. In case I is finite, $\mathbb{N}$ or $\mathbb{Z}$, (e_α) is an *orthogonal*/*orthonormal sequence*.

Examples of orthonormal sets

1. In $\mathbb{C}^n$, take the standard basis vectors.
2. In ℓ_2, take $(e_n)_{n\in\mathbb{N}}$ with $e_n = (0,\dots,0,1,0,\dots)$. (The 1 is in the n-th position.)
3. In $L_2[-\pi,\pi]$, take $(e_n)_{n\in\mathbb{Z}}$, with $e_n(t) = (2\pi)^{1/2}e^{int}$ for $n \in \mathbb{Z}$. (*Fourier basis*)

Orthonormal Sets (cont.)

Definition

If (e_n) is an orthonormal sequence in a Hilbert space H, then, for every $x \in H$, (x, e_n) is the *n-th Fourier coefficient of x w.r.t.* (e_n), and $\sum_{n=1}^{\infty} (x, e_n) e_n$ is the *Fourier series w.r.t.* (e_n).

Lemma

Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space V; $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x \in V$. Then, $\left\| x - \sum_{k=1}^{n} \lambda_k e_k \right\|^2 = \|x\|^2 + \sum_{k=1}^{n} |\lambda_k - c_k|^2 - \sum_{k=1}^{n} |c_k|^2$, where $c_k := (x, e_k)$.
(*Exercise!*)

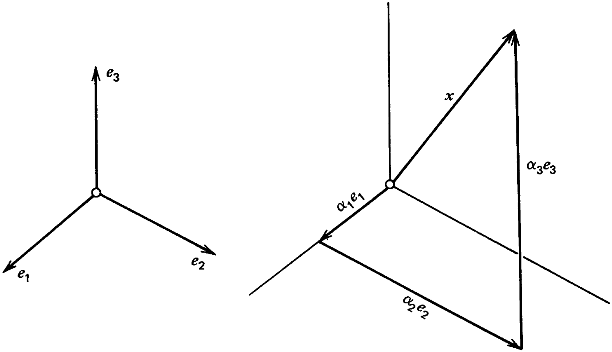
Since $\{e_1, \ldots, e_n\}$ span $\text{lin}\{e_1, \ldots, e_n\}$, we have

Theorem

Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space V. The closest point y of $\text{lin}\{e_1, \ldots, e_n\}$ to a point $x \in V$ is $y = \sum_{k=1}^{n} (x, e_k) e_k$, and $\|x - y\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, e_k)|^2$.

Orthonormal Sets (cont.)

Corollary
If $x \in \text{lin}\{e_1, \ldots, e_n\}$, then $x = \sum_{k=1}^{n} (x, e_k) e_k$, and $\|x\|^2 = \sum_{k=1}^{n} |(x, e_k)|^2$.



$\alpha_k = (x, e_k)$

Outline

Bessel Inequality

Theorem (Bessel Inequality)
If (e_n) is an orthonormal sequence in an inner product space V, and $x \in V$, then

$$\sum_{n=1}^{\infty} |(x, e_n)|^2 \leqslant \|x\|^2.$$

Proof. For $N \in \mathbb{N}$, $\sum_{k=1}^{N} |(x, e_k)|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^{N} (x, e_k) e_k \right\|^2 \leqslant \|x\|^2$. Take $N \to \infty$. □

We want to study the meaning of $\sum_{k=1}^{\infty} (x, e_k) e_k$.

Definition (Infinite sum in a normed space)
Let (x_n) be a sequence in a normed space V. We say that $\sum_{n=1}^{\infty} x_n$ *converges* and has *sum* x (*i.e.*, $\sum_{n=1}^{\infty} x_n = x$) if $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$, *i.e.*, $\left\| x - \sum_{n=1}^{N} x_n \right\| \to 0$ as $N \to \infty$.

Bessel Inequality (cont.)

Theorem

Let (e_n) is an orthonormal sequence in a Hilbert space H, and let (λ_n) be a sequence in $\mathbb{C}$. Then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in H iff $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

Proof

($\Rightarrow$) Let $x = \sum_{n=1}^{\infty} \lambda_n e_n$ and $x_N = \sum_{n=1}^{N} \lambda_n e_n$. Then, $(x_N, e_n) = \lambda_n$ for $n \leqslant N$, and taking $N \to \infty$ gives $(x, e_n) = \lambda_n$. Then, by Bessel inequality: $\sum_{n=1}^{\infty} |\lambda_n|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 \leqslant \|x\|^2 < \infty$.

($\Leftarrow$) Assume that $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$, and let $x_N = \sum_{n=1}^{N} \lambda_n e_n$. Then,

$$\|x_{N+P} - x_N\|^2 = \left\| \sum_{n=N+1}^{N+P} \lambda_n e_n \right\|^2 = \sum_{n=N+1}^{N+P} \|\lambda_n e_n\|^2 = \sum_{n=N+1}^{N+P} |\lambda_n|^2 \to 0 \quad \text{as } N \to \infty.$$

Therefore, (x_n) is Cauchy, and it converges in H. □

Observation

If $H = L_2[a,b]$, then the above convergence is *in norm* (or L_2 convergence).
A *different* type is *point-wise convergence*: $\sum_{n=1}^{\infty} x_n(t) = x(t)$ for all $t \in [a,b]$.

Outline

Goal: When does $\sum_{n=1}^{\infty}(x, e_n)e_n = x$? $\Rightarrow$ We need conditions on (e_n).

Approximation error: $y = x - \sum_{n=1}^{\infty}(x, e_n)e_n$.
Notice that $(y, e_k) = 0$ for all k. We want to ensure $y = 0$.

Definitions

- An orthonormal set A in an inner product space V is *maximal* if the only point in V which is orthogonal to every $x \in A$ is 0, *i.e.*, A cannot be extended to a larger orthonormal set.
- A set A in a normal space V is *total* (or *fundamental*) if its span is dense in V.
- If A is a total orthonormal set in an inner product space V, every $x \in V$ can be written as $x = \sum_{e \in A}(x, e)e$, and A is called an *orthonormal basis* of V.

Note. By Bessel's inequality, given $x \in V$ and an orthonormal set A, since $\sum_{e \in A} |(x, e)|^2 \leqslant \|x\|$, at most a countable number of terms (x, e), as e runs over A, can be non-zero: for every $n \in \mathbb{N}$, the number of terms s.t. $|(x, e)|^2 > 1/n$ can be at most $n\|x\|^2$, and $\{e \in A : (x, e) \neq 0\} = \bigcup_{n \in \mathbb{N}}\{e \in A : |(x, e)|^2 > 1/n\}$, which is at most countable. Thus, sums like $\sum_{e \in A}(x, e)e$ and $\sum_{e \in A} |(x, e)|^2$ can be reduced to sums over sequences.

Total Orthonormal Sequences (cont.)

Theorem. If A is an orthonormal set in a Hilbert space H, the following are equivalent:

(1) A is total.

(2) $\|x\|^2 = \sum_{e \in A} |(x,e)|^2$ for all $x \in H$.

(3) A is maximal.

If H is an incomplete inner product space, then (1) and (2) are still equivalent, and they imply (3), but not conversely (*see bonus slides for an example*).

Proof

(1) ⇔ (2): For a given $x \in H$, sort the elements of $\{e \in A : (x,e) \neq 0\}$ into a sequence (e_n). Then, take $N \to \infty$ in $\sum_{n=1}^{N} |(x,e_n)|^2 = \|x\|^2 - \left\| x - \sum_{n=1}^{N} (x,e_n)e_n \right\|^2$.

(2) ⇒ (3): If A is not maximal, take a nonzero $x \perp A$. Then $\|x\|^2 > 0 = \sum_{e \in A} |(x,e)|^2$.

(3) ⇒ (1): Given an $x \in H$, $\sum_{e \in A} (x,e)e$ is convergent (due to the completeness of H), and $x - \sum_{e \in A} (x,e)e$ is orthogonal to every $e \in A$, so by maximality of A, $x = \sum_{e \in A} (x,e)e$, which implies that A is an orthonormal basis.

Only the implication (3) ⇒ (1) requires H to be complete. □

Total Orthonormal Sequences (cont.)

Theorem. Let H be an inner product space. Then,

(1) If H is separable, then every orthonormal set in H is countable.

(2) If H contains a total orthonormal sequence, then H is separable.

Proof

(1) If $A \subseteq H$ is an orthonormal set, distinct points $x, y \in A$ are at a distance $\sqrt{(x-y, x-y)} = \sqrt{2}$, so if A were uncountable, a set dense in H would be uncountable too.

(2) If (e_n) is a total orthonormal set, consider the set D, consisting of all linear combinations $\lambda_1 e_1 + \cdots + \lambda_n e_n$ where $n \in \mathbb{N}$ and $\lambda_k = a_k + i b_k$ with $a_k, b_k \in \mathbb{Q}$ for $k = 1, \dots, n$. D is a countable set dense in H (*why?*). □

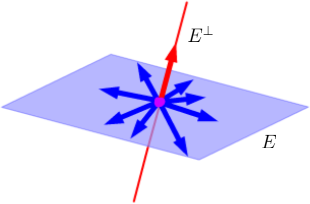
Observation: A separable Hilbert space is isomorphic to $\mathbb{C}^n$ (for some n) or to ℓ_2 (*see bonus slides for proof*).

See bonus slides for examples of non-separable Hilbert spaces.

Outline

We can use orthogonality to decompose a Hilbert space.

Definition
Let H be a Hilbert space. The *orthogonal complement* of $E \subseteq H$ is $E^\perp := \{x \in H : x \perp E\}$.



Theorem
For every subset E of a Hilbert space, $E^\perp$ is a closed linear space. (*Exercise!*)

The projection theorem gives the following characterization of $E^\perp$:

Lemma
Let M be a linear subspace of an inner product space V, and let $x \in V$. Then $x \in M^\perp$ iff $\|x - y\| \geqslant \|x\|$ for all $y \in M$.

Orthogonal Complements (cont.)

Definition

Let $M, N \subseteq V$, where V is a vector space. V is the *direct sum* of M and N, denoted $V = M \oplus N$, if every $x \in V$ has a *unique* decomposition $x = y + z$, where $y \in M$ and $z \in N$.

Theorem

Let M be a closed linear subspace of a Hilbert space H. Then, $H = M \oplus M^{\perp}$.

Proof. Let $x \in H$. Assume that $M \neq \emptyset$ (otherwise the result is trivial). Take $y \in M$ as the unique minimizer of $\inf_{m \in M} \|x - m\|$, and $z := x - y$. By the projection theorem, $z \in M^{\perp}$.
If $x = y' + z'$, with $y' \in M$ and $z' \in M^{\perp}$, then $(x - y') \perp M$, so by the projection theorem, $y' = y$, which proves the uniqueness of the decomposition. □

Corollary

If M is a closed linear subspace of a Hilbert space H, then $(M^{\perp})^{\perp} = M$.

Proof. By definition, $M \subseteq (M^{\perp})^{\perp}$. Let $x \in (M^{\perp})^{\perp}$, and write it as $x = y + z$ with $y \in M$ and $z \in M^{\perp}$. Since $x \perp M^{\perp}$, $0 = (x, z) = (y + z, z) = (y, z) + \|z\|^2 = \|z\|^2$, so $z = 0$ and $x \in M$. □

Outline

Let $e_n(t) := (2\pi)^{-1/2} e^{int}$, $t \in [-\pi, \pi]$, $n \in \mathbb{Z}$. We want to prove that $(e_n)_{n \in \mathbb{Z}}$ is total in $L_2[-\pi, \pi]$.

We need to show that $\text{clin}\{e_n : n \in \mathbb{Z}\} = L_2[-\pi, \pi]$. It is known that the closure of $C[-\pi, \pi]$ is $L_2[-\pi, \pi]$, so it is enough to show that for every $f \in C[-\pi, \pi]$ there is a sequence in $\text{clin}\{e_n : n \in \mathbb{Z}\}$ converging to f. An obvious choice is $f_N = \sum_{n=-N}^{N} (f, e_n) e_n$, but it is easier to work with

$$F_m = \frac{1}{m+1}(f_0 + f_1 + \cdots + f_m), \quad m = 0, 1, \ldots \qquad \text{(\textit{Césaro sum of the } } f_N\text{\textit{'s})}$$

Since $(f, e_n) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(t) e^{-int} dt$, we have

$$F_m(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau) K_m(t - \tau) d\tau, \quad \text{where} \quad K_m(x) := \frac{1}{m+1} \sum_{N=0}^{m} \sum_{n=-N}^{N} e^{-inx}. \quad \text{(\textit{Fejér kernel})}$$

Classical Fourier Series (cont.)

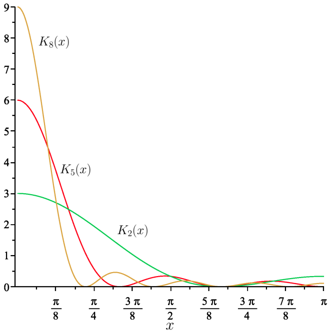
Fejér Kernel properties:

(1) $K_m(x) \geqslant 0$ for all $x \in \mathbb{R}$, $m = 0, 1, 2, \ldots$

(2) $\displaystyle\int_{-\pi}^{\pi} K_m(x)dx = 2\pi$, for $m = 0, 1, 2, \ldots$

(3) For all $0 < \delta < \pi$, $\displaystyle\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(x)dx \to 0$ as $m \to \infty$. *(see bonus slides for proofs)*

Therefore, $(K_m/2\pi)$ is a *Delta sequence* (it "converges" to a Dirac delta).



We will prove a strong result: $\displaystyle\lim_{m\to\infty} \sup_{t\in[-\pi,\pi]} |f(t) - F_m(t)| = 0.$

$$\Rightarrow \|f - F_m\|_2^2 = \int_{-\pi}^{\pi} |f(t) - F_m(t)|^2 dt \leqslant 2\pi \sup_{t\in[-\pi,\pi]} |f(t) - F_m(t)|^2 \to 0 \quad \text{as } m \to \infty. \text{ (L_2 convergence)}$$

Classical Fourier Series (cont.)

Take a $\delta > 0$ (to be defined more precisely later):

$$\begin{aligned}
|f(t) - F_m(t)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(t) - f(\tau)] K_m(t-\tau) d\tau \right| \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - f(\tau)| K_m(t-\tau) d\tau \\
&= \frac{1}{2\pi} \left(\int_{\substack{|t-\tau|>\delta \\ -\pi\leqslant\tau\leqslant\pi}} + \int_{t-\delta}^{t+\delta} \right) |f(t) - f(\tau)| K_m(t-\tau) d\tau.
\end{aligned}$$

For the first integral, we use the fact that f is bounded, *i.e.*, there is an $M > 0$ s.t. $\sup_{t\in[-\pi,\pi]} |f(t)| \leqslant M$, hence

$$\frac{1}{2\pi} \int_{\substack{|t-\tau|>\delta \\ -\pi\leqslant\tau\leqslant\pi}} |f(t) - f(\tau)| K_m(t-\tau) d\tau \leqslant \frac{2M}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_m(\tau) d\tau. \quad \text{(This is negligible as } m \to \infty.\text{)}$$

Classical Fourier Series (cont.)

For the second integral, we need to recall *uniform continuity*:

Definition (reminder). Given metric spaces (X, d_X) and (Y, d_Y), $f : X \to Y$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Reminder. By Heine-Cantor's theorem, given metric spaces (X, d_X) and (Y, d_Y), if X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

Let $\varepsilon > 0$. Then, take δ as in the definition of uniform continuity, so

$$\frac{1}{2\pi}\int_{\substack{|t-\tau|<\delta \\ -\pi\leqslant\tau\leqslant\pi}} |f(t)-f(\tau)|K_m(t-\tau)d\tau \leqslant \frac{\varepsilon}{2\pi}\int_{\substack{|t-\tau|<\delta \\ -\pi\leqslant\tau\leqslant\pi}} K_m(t-\tau)d\tau \leqslant \varepsilon.$$

Therefore: $\sup_{t\in[-\pi,\pi]} |f(t)-F_m(t)| < \frac{M}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right)K_m(\tau)d\tau + \varepsilon \to \varepsilon \quad \text{as} \quad m \to \infty.$

and since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \to 0$ gives $\lim_{m\to\infty} \sup_{t\in[-\pi,\pi]} |f(t)-F_m(t)| = 0.$ $\square$

Classical Fourier Series (cont.)

We have actually proved

Theorem (Fejér)
Let $f : [-\pi, \pi] \to \mathbb{C}$ be continuous, $s_n(f)$ be the n-th partial sum of its Fourier series, and $\sigma_n(f)$ be the arithmetic mean of $s_0(f), \dots, s_n(f)$. Then $\sigma_n(f) \to f$ uniformly as $n \to \infty$.

Notice that $s_n(f)$ does not always converge point-wisely to continuous f. (*An example is provided in the bonus slides of Topic 9!*)

A similar result (proven analogously, with a different kernel) is

Theorem (Weierstrass theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous, where $-\infty < a < b < \infty$. For every $\varepsilon > 0$ there is a polynomial p s.t. $\sup_{t \in [a,b]} |f(t) - p(t)| < \varepsilon$.

Least Squares Estimation

Outline

Bonus: Example of Maximal Non-Total Orthonormal Sets

By Zorn's Lemma, every inner product space V has a maximal orthonormal set A (*why?*). If V is complete, $\text{clin } A = V$; otherwise $V = \text{clin } A \oplus (\text{clin } A)^{\perp}$, so $(\text{clin } A)^{\perp} \neq \{0\}$ and there is an $x \in (\text{clin } A)^{\perp}$ of unit norm, so $A \cup \{x\}$ is also orthonormal, contradicting the maximality of A.

However, in every incomplete inner product space V there are maximal orthonormal sets which are not total, *i.e.*, whose closed linear span is not the entire space:

Proof. First note that if every proper, closed subspace M of V is s.t. $M^{\perp} \neq \{0\}$, then V is complete. Indeed, assume that V is incomplete, and let $\hat{V}$ be the completion of V. Pick an $x \in \hat{V} \setminus V$, and let $\hat{M} = \{x\}^{\perp}$ in $\hat{V}$. Then, $\hat{M} \cap V$ is closed in V (because $\hat{M}$ is closed in $\hat{V}$). If $x \perp V$, then $d(x, V) = \|x\| > 0$, and V would not be dense in $\hat{V}$; thus, $\hat{M} \cap V \neq V$, and there is a $y \in V$ s.t. $(x, y) \neq 0$, which we can normalize so that $(x, y) = 1$.
Note that $\hat{M} \cap V$ is dense in $\hat{M}$. Indeed, let $z \in \hat{M}$ and let (x_n) be a sequence in V s.t. $x_n \to z$ (which exists because $\overline{V} = \hat{V}$). Let $x'_n = x_n - (x_n, x)y$; then $x'_n \in V$, $(x'_n, x) = (x_n, x) - (x_n, x)(y, x) = 0$ so that $x'_n \in \hat{M}$, and $\|x'_n - z\| \leqslant \|x_n - z\| + |(x_n, x)| \|y\| \to 0 + |(z, x)| \|y\| = 0$, thus $x'_n \to z$. Then, $(\hat{M} \cap V)^{\perp} \cap V = (\overline{\hat{M} \cap V})^{\perp} \cap V$ $= \hat{M}^{\perp} \cap V = \text{lin}\{x\} \cap V = \emptyset$, so $M = \hat{M} \cap V$ is the sought proper, closed subspace of V.
Now, assume every maximal orthonormal set in an incomplete V is a basis, and let M be a closed, proper subspace of V s.t. $M^{\perp} = \{0\}$. Let B be a maximal orthonormal set in M, and extend it to a maximal orthonormal set $B \cup B_1$ for V. Assume $B_1 \neq \emptyset$, and let $x_1 \in B_1$; since $M^{\perp} = \{0\}$, there is a $y \in M$ s.t. $(y, x_1) \neq 0$. As $B \cup B_1$ is a basis, $y = \sum_k c_k y_k + \sum_k d_k x_k$ ($y_k \in B$, $x_k \in B_1$). Now, $z = \sum_k d_k x_k = y - \sum_k c_k y_k \in M$, but $x_k \perp B$ for all k, hence $z \perp B$. As B is maximal in M, $z = 0$, so $(y, x_1) = d_1 = 0$, a contradiction. Hence, $B_1 = \emptyset$, B is a maximal orthonormal set for V, so B is a basis for V, *i.e.*, $M = V$, a contradiction. Thus, V contains a non-total maximal orthonormal set. □

Bonus: Example of Maximal Non-Total Orthonormal Sets (cont.)

From this result, every incomplete inner product space has maximal non-total orthonormal sets. Here is a specific example:

Let $V = \ell_2$, and denote by (e_n) its standard orthonormal basis. Consider the linear subspace $Y \subseteq V$ spanned by $A = \{a, e_2, e_3, \ldots\}$, where $a := \sum_{k=1}^{\infty} (1/k) e_k$. Then, $B = \{e_2, e_3, \ldots\}$ is a maximal orthonormal set in Y, because if $x = \alpha_1 a + \sum_{k=2}^{N} \alpha_k e_k \in Y$ is orthogonal to B (*why is it enough to consider such an x?*), then $0 = (x, e_{N+1}) = \alpha_1/(N+1)$, and $0 = (x, e_k) = \alpha_k$ for $k = 2, \ldots, N$, hence $x = 0$. However, clin B does not include a, so B is a maximal orthonormal set for Y which is not total in Y.
Note, however, that Y does have an orthonormal basis, which can be obtained by applying Gram-Schmidt to A (*see exercise set 3!*).

Definition. Two Hilbert spaces H, K are *isomorphic* if there is a bijective mapping $U: H \to K$ s.t., for all $x, y \in H$ and $\alpha \in \mathbb{C}$, $U(x+y) = U(x) + U(y)$, $U(\alpha x) = \alpha U(x)$ and $(U(x), U(y)) = (x, y)$. Such a mapping is a *unitary linear operator*.

Theorem. Every separable Hilbert space is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, or to ℓ_2.

Proof. Assume H is a separable Hilbert space, so it has a total orthonormal sequence. Suppose first that such sequence is finite, say, $\{e_1, \dots, e_n\}$. Then, $x = \sum_{k=1}^{n} (x, e_k) e_k$ for each $x \in H$. Let $U: H \to \mathbb{C}^n$ be given by $U\left(\sum_{k=1}^{n} \lambda_k e_k\right) = (\lambda_1, \dots, \lambda_n)$; U is bijective and linear, and if $x = \sum_{k=1}^{n} x_k e_k$, $y = \sum_{k=1}^{n} y_k e_k$, we have that $(x, y) = \sum_{k=1}^{n} x_k \overline{y_k} = (U(x), U(y))$, so U is unitary and H is isomorphic to $\mathbb{C}^n$.
If the total orthonormal sequence is infinite, say, $(e_k)_{k \in \mathbb{N}}$, define the mapping $U: H \to \ell_2$ by $U(x) = (\lambda_k)_{k \in \mathbb{N}}$, where $x = \sum_{k=1}^{\infty} \lambda_k e_k$. U is linear and unitary (as in the finite case), hence injective. By the characterization of total orthonormal sequences, $U(x) \in \ell_2$, and if $(\lambda_k)_{k \in \mathbb{N}} \in \ell_2$, $\sum_{k=1}^{\infty} \lambda_k e_k$ converges to an $x \in \ell_2$, so U is surjective. Thus, H is isomorphic to ℓ_2. □

Bonus: Examples of Non-Separable Hilbert Spaces

1. $\ell_2(\mathbb{R})$**:** The space of all $f: \mathbb{R} \to \mathbb{R}$ s.t. $E_f = \{x \in \mathbb{R}: f(x) \neq 0\}$ is countable and $\sum_{x \in E_f} f^2(x) < \infty$ (this sum is well defined, *why?*), with inner product $(f,g) = \sum_{x \in E_f \cap E_g} f(x)\overline{g(x)}$. $\ell_2(\mathbb{R})$ is a Hilbert space (*Exercise! Hint: countable unions of countable sets are countable*). Also, the functions $f_y \in \ell_2(\mathbb{R})$, with $f_y(x) = 1$ if $x = y$ and $f_y(x) = 0$ otherwise, are an uncountable orthonormal system, so $\ell_2(\mathbb{R})$ is non-separable.
2. **Almost-periodic functions:** In an attempt to extend the classical Fourier series to non-periodic functions in $\mathbb{R}$, the following definition has been coined:

   $f: \mathbb{R} \to \mathbb{C}$ is *almost-periodic* (AP) if it is the uniform limit of functions $\sum_{k=1}^n a_k e^{i\lambda_k t}$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The set E of AP functions is a vector space, with inner product $(f,g) = \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} f(t)\overline{g(t)} dt$ (modulo an equivalence relation). The completion of E is a Hilbert space, but not all its elements can be identified as functions (*e.g.*, $\sum_{k=1}^{\infty} (1/k) e^{it/k}$). Also, $(e^{\lambda t})_{\lambda \in \mathbb{R}}$ is an uncountable orthonormal system in E, so E is non-separable.

Bonus: Proofs of Properties of Fejér Kernels

Letting $z = e^{ix}$, the Fejér kernel can be written, for every x not a multiple of 2π, as

$$K_m(x) = \frac{1}{m+1}\sum_{N=0}^{m}\sum_{n=-N}^{N} z^{-n} = \frac{1}{m+1}\sum_{N=0}^{m}\frac{z^N - z^{-N-1}}{1-z^{-1}} = \frac{1}{(m+1)(1-z^{-1})}\left[\frac{1-z^{m+1}}{1-z} - \frac{z^{-1}-z^{-m-2}}{1-z^{-1}}\right]$$

$$= \frac{1}{(m+1)(1-z^{-1})}\left[\frac{1-z^{m+1}}{1-z} + \frac{1-z^{-m-1}}{1-z}\right] = \frac{2-z^{m+1}-z^{-m-1}}{(m+1)(|1-z|^2)} = \frac{\sin^2\left(\frac{(m+1)x}{2}\right)}{(m+1)\sin^2\left(\frac{x}{2}\right)}. \qquad (*)$$

This, and the continuity of K_m, directly proves Property 1.

Since $\int_{-\pi}^{\pi} e^{inx}dx = 2\pi$ if $n = 0$ and $= 0$ otherwise, $\int_{-\pi}^{\pi} K_m(x)dx = (m+1)^{-1}\sum_{N=0}^{m}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{inx}dx = (m+1)^{-1}\sum_{N=0}^{m} 2\pi = 2\pi$, which establishes Property 2.

Finally, note that if $x \in [-\pi,-\delta)\cup(\delta,\pi]$, then $\sin^2(x/2) \geqslant \sin^2(\delta/2) > 0$. Thus, by $(*)$, for this range of values of x, $0 \leqslant K_m(x) \leqslant (m+1)^{-1}\sin^{-2}(\delta/2)$, so

$$0 \leqslant \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(x)dx \leqslant \frac{2\pi}{m+1}\sin^{-2}(\delta/2) \to 0 \quad \text{as } m \to \infty,$$

which proves Property 3. □