EL3370 Mathematical Methods in Signals, Systems and Control

Topic 4: Hilbert and Banach Spaces

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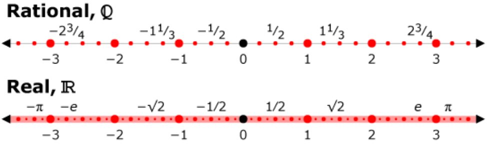
Outline

Outline

Completeness

A property that many important metric spaces share is *completeness*: a sequence that seems to converge, actually does. This, however, does not always hold (recall the ℓ_0 example from last topic). This property is what distinguishes $\mathbb{R}$ from $\mathbb{Q}$!



Definition

(x_n) is a *Cauchy sequence* in a metric space (X,d) if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. for all $n,m \geqslant N$, $d(x_n, d_m) < \varepsilon$.

(X,d) is a *complete metric space* if every Cauchy sequence in X is convergent.

Observation

Every convergent sequence in a metric space is a Cauchy sequence (*exercise!*).

Example 1

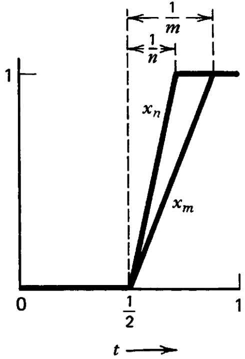
$\mathbb{R}$ and $\mathbb{C}$ are complete, while $\mathbb{Q}$ is not: take $x_n = 1 + 1/1! + 1/2! + \cdots + 1/n!$; (x_n) is a Cauchy sequence in $\mathbb{Q}$, since it is convergent in $\mathbb{R}$ (and hence Cauchy in $\mathbb{R}$), but its limit in $\mathbb{R}$ is e, which is not rational (*see bonus slides for proof*).

Example 2

Consider $C[0,1]$, with inner product $(x,y) = \int_0^1 x(t)\overline{y(t)}dt$. This space is *not* complete: consider the sequence (x_n) in the figure. This sequence is Cauchy, since $d(x_n, x_m) < \varepsilon$ when $n, m > 1/\varepsilon$. However, (x_n) is not convergent: for every $x \in C[0,1]$,

$$d^2(x_n, x) = \int_0^{1/2} |x(t)|^2 dt + \int_{1/2}^{1/2+1/n} |x_n(t) - x(t)|^2 dt + \int_{1/2+1/n}^{1} |1 - x(t)|^2 dt,$$

thus $d(x_n, x) \to 0$ means that $x(t) = 0$ for $t < 1/2$ and $x(t) = 1$ for $t > 1/2$, but then x cannot be continuous!



Completeness (cont.)

Recall: in a metric space, a set is *closed* if it contains the limits of its convergent sequences (this means that such a set is "closed with respect to taking limits"). Unfortunately, this property is relative, *i.e.*, it depends on the underlying topological space in which it is embedded.

Example. $\mathbb{Q}$ is not closed in $\mathbb{R}$, but it is closed in $\mathbb{Q}$.

Reason: $\mathbb{Q}$ has more convergent sequences as part of $\mathbb{R}$ than as part of $\mathbb{Q}$.

Completeness is a property that resembles closedness, but is intrinsic/hereditary, *i.e.*, it only depends on the set itself, not on the space in which it is embedded (unfortunately it can only be defined on metric spaces, not on general topologies). It is based on an intrinsic property of convergent sequences, that of being Cauchy (which only depends on the distances between elements of the sequences, not on other points of the space).

In fact, a complete subset of a metric space is always closed (*why?*), no matter the underlying space, and conversely, a closed subset of a complete space is always complete (*this is part of Exercise Set 2!*).

Completeness (cont.)

Theorem

$\mathbb{C}^n$ and ℓ_2 are complete metric spaces.

Proof (for ℓ_2). Most completeness proofs follow these steps:

(1) Take a Cauchy sequence (x^n).
(2) Postulate a candidate limit x.
(3) Show that x belongs to the metric space.
(4) Show that $x^n \to x$.

Steps:

(1) Pick a Cauchy sequence (x^n) in ℓ_2, where $x^n = (x_1^n, x_2^n, \dots)$.
(2) Consider a fixed index k. Since (x^n) is Cauchy, for a given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t. $|x_k^n - x_k^m| \leqslant \sqrt{\sum_{s=1}^{\infty} |x_s^n - x_s^m|^2} = d(x^n, x^m) < \varepsilon$ when $n, m \geqslant N$. Hence, $(x_k^n)_n$ is Cauchy in $\mathbb{C}$, so it converges to, say, x_k. Therefore, consider the candidate limit $x = (x_1, x_2, \dots)$.

Proof (cont.)

(3) We will show that $x^n - x \in \ell_2$ for some n. As $x^n \in \ell_2$, this implies that $x = x^n - (x^n - x) \in \ell_2$. Since (x^n) is Cauchy, given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t. $d(x^n, x^m) < \varepsilon$ for $n, m \geqslant N$. Fix $M \in \mathbb{N}$. Then, for $n, m \geqslant N$,

$$\sum_{k=1}^{M} |x_k^n - x_k^m|^2 \leqslant \sum_{k=1}^{\infty} |x_k^n - x_k^m|^2 = d^2(x^n, x^m) < \varepsilon^2.$$

Taking the limit $m \to \infty$, we obtain $\sum_{k=1}^{M} |x_k^n - x_k|^2 \leqslant \varepsilon^2$, and taking $M \to \infty$ we finally obtain $\|x^n - x\|^2 = \sum_{k=1}^{\infty} |x_k^n - x_k|^2 \leqslant \varepsilon^2$, hence $x^n - x \in \ell_2$, and $x \in \ell_2$.

(4) We have shown in Step 3 that there is an $N \in \mathbb{N}$ s.t. $\|x^n - x\| \leqslant \varepsilon$ for $n \geqslant N$, so $x^n \to x$.

We have proven that every Cauchy sequence in ℓ_2 is convergent. This means that ℓ_2 is complete. $\square$

Definition

Hilbert Space: Inner product space which is complete (as a metric space).

Banach Space: Normed space which is complete (as a metric space).

Every Hilbert space is a Banach space, but not conversely. *E.g.*, ℓ_∞ is a Banach space, but not a Hilbert space, since the ℓ_∞-norm does not satisfy the parallelogram law (*exercise!*).

Examples

1. The ℓ_p spaces (for $1 \leqslant p \leqslant \infty$) are Banach spaces. Of them, only ℓ_2 is a Hilbert space.
2. Every finite-dimensional normed space is Banach (*prove it! this follows from Exercise Set 2*). In particular, $\mathbb{R}^n$ is a Banach space for every p-norm ($1 \leqslant p \leqslant \infty$).

If V is a Banach space, and M is a closed subspace of V, then the quotient normed space V/M is also a Banach space (*exercise!*).

Outline

We have seen that $C[0,1]$ with inner product $(x,y) = \displaystyle\int_0^1 x(t)\overline{y(t)}dt$ is not complete. The problem lies in requiring that the elements of $C[0,1]$ should be continuous functions. Removing this requirement leads to

Definition

Let $-\infty \leqslant a < b \leqslant \infty$. $L_2[a,b]$ is the vector space of Lebesgue measurable functions $f: [a,b] \to \mathbb{C}$ s.t. $\int_a^b |f(t)|^2 dt$, with inner product $(x,y) = \int_a^b x(t)\overline{y(t)}dt$.

Subtlety

$(x,x) = \int_a^b |x(t)|^2 dt = 0$ for some $x \neq 0$! (*e.g.*, $\int_a^b |x(t)|^2 dt = 0$ if $x(t) \neq 0$ only for finite # of t's).

To solve this, define an equivalence relation on $L_2[a,b]$:

$$x \sim y \quad \text{iff} \quad \int_a^b |x(t) - y(t)|^2 dt = 0.$$

This relation is compatible with the operations in $L_2[a,b]$ (*i.e.*, for all $\lambda \in \mathbb{C}$, if $x_1 \sim x_2$ and $y_1 \sim y_2$, then $x_1 + y_1 \sim x_2 + y_2$, $\lambda x_1 \sim \lambda x_2$ and $(x_1, y_1) = (x_2, y_2)$).

Digression (Density and separability)
Let X be a topological space, and $D \subseteq X$. D is *dense* in X if $\overline{D} = X$, *i.e.*, if every point in X can be approximated arbitrarily well by points in D.
E.g., $\mathbb{Q}$ is dense in $\mathbb{R}$, and the set of polynomials in $[a,b]$ is dense in $C[a,b]$ (this will be proven in Topic 5).

A topological space is *separable* if it contains a countable subset which is dense in it. Many proofs in functional analysis can be simplified if the underlying space is separable.

Alternative characterization of separability for normed spaces:
A normed space V is separable iff there is a countable l.i. subset $B \subseteq V$ s.t. clin $B = V$.

Proof. If $B = \{e_n : n \in \mathbb{N}\}$ is a countable l.i. subset of V s.t. clin $B = V$, one can form the set D, consisting of all linear combinations $\lambda_1 e_1 + \cdots + \lambda_n e_n$ where $n \in \mathbb{N}$ and $\lambda_k = a_k + i b_k$ with $a_k, b_k \in \mathbb{Q}$ for $k = 1, \ldots, n$, which is a countable set dense in V. Conversely, if $D = \{x_n : n \in \mathbb{N}\}$ is a countable dense subset of V, consider the set B formed by adding, inductively, x_n to B (by calling it, *e.g.*, e_m) iff $\{e_1, \ldots, e_m\}$ is l.i.; the resulting set B is l.i. and clin $B = V$. □

Examples of separable spaces

1. $\mathbb{C}$ is separable: a countable dense set in $\mathbb{C}$ is $\{x+iy\colon x,y\in\mathbb{Q}\}$. Thus, $\mathbb{R}$ is separable too.
2. ℓ_p $(1\leqslant p<\infty)$ is separable:
   The set M of sequences of the form $x=(x_1,\dots,x_n,0,0,\dots)$ for some $n\in\mathbb{N}$, and $x_k\in\mathbb{Q}$, is countable and dense in ℓ_p (*exercise!*).
3. ℓ_∞ is *not* separable:
   Let $A\subseteq\ell_\infty$ be the set of sequences $x=(x_1,x_2,\dots)$ where $x_k\in\{0,1\}$. Each $x\in A$ can be mapped to a number $\hat{x}=x_1/2+x_2/2^2+x_3/2^3+\cdots\in[0,1]$. The mapping is bijective (except for the countable set of numbers with finite binary expansion, since they can be written in more than one way: *e.g.*, $0.1=0.01111\dots$ in binary form), and $[0,1]$ is uncountable, so A is uncountable. If $x,y\in A$ and $x\neq y$, then $\|x-y\|_\infty=1$, hence every dense set in ℓ_∞ must be uncountable (*why?*).

The Space $L_2[a,b]$ (cont.)

It can be proven that $L_2[a,b]$ is a Hilbert space, and that $C[a,b]$ is a dense subspace of $L_2[a,b]$.

Similarly, we can define the Banach spaces $L_p[a,b]$ ($1 \leqslant p \leqslant \infty$) of Lebesgue measurable functions $x\colon [a,b] \to \mathbb{C}$ s.t. $\int_a^b |x(t)|^p dt < \infty$ (for $p < \infty$) or $\operatorname{ess\,sup}_{t\in[a,b]} |x(t)| < \infty$ (for $p = \infty$), with norm

$$\|x\|_p := \begin{cases} \left(\int_a^b |x(t)|^p dt \right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{t\in[a,b]} |x(t)|, & p = \infty. \end{cases}$$

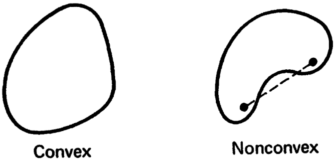
Note. The *essential supremum* of a Lebesgue measurable function $f\colon [a,b] \to \mathbb{C}$ is $\operatorname{ess\,sup}_{t\in[a,b]} f(t) := \inf\{\sup_{t\in[a,b]} g(t) \colon g \sim f\}$, where $g \sim f$ iff $\int_a^b |g(t) - f(t)| dt = 0$, so it is independent of the representative f of class $[f]$ being chosen.

Outline

The Closest Point Property

Do you remember the "projection theorem" from geometry? (the shortest distance from a point to a plane is achieved by the perpendicular) This result is fundamental in $\mathbb{R}^n$, and it also holds in Hilbert spaces!

Definition
A subset A of a vector space is *convex* if, for all $x, y \in A$ and $0 \leqslant \lambda \leqslant 1$, $\lambda x + (1-\lambda)y \in A$.

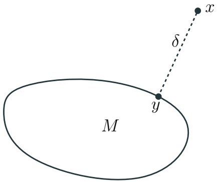


Definition
In an inner product space V, $x, y \in V$ are *orthogonal*, denoted $x \perp y$, if $(x, y) = 0$. x is orthogonal to a subset $S \subseteq V$, $x \perp S$, if $x \perp s$ for all $s \in S$.

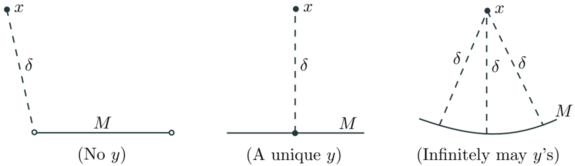
Lemma (Pythagorean theorem)
If x, y are orthogonal in an inner product space, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. (*Exercise!*)

The Closest Point Property (cont.)

Theorem (Closest point property)
Let M be a non-empty closed convex set in a Hilbert space V. For every $x \in V$ there is a unique point $y \in M$ which is closer to x than any other point of M (*i.e.*, $\|x-y\| = \inf_{m \in M} \|x-m\|$).



Why closed and convex?



In a Banach space there may be infinitely many closest points, or none at all.

Proof

(1) *Existence*:
Let $\delta = \inf_{m \in M} \|x - m\|$, which is finite (because $M \neq \emptyset$). Then, for each $n \in \mathbb{N}$ there is a $y_n \in M$ s.t. $\|x - y_n\|^2 \leqslant \delta^2 + 1/n$. We will show that (y_n) is Cauchy: For n, m,

$$\begin{aligned}\|(x - y_n) - (x - y_m)\|^2 + \|(x - y_n) + (x - y_m)\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 \quad \text{(parallelogram law)} \\ &< 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right),\end{aligned}$$

$$\begin{aligned}\text{or} \quad \|y_n - y_m\|^2 &< 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\left\|x - \left(\frac{y_n + y_m}{2}\right)\right\|^2 \\ &< 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\delta^2 \qquad \left(\text{because } M \text{ is convex, so } \frac{y_n + y_m}{2} \in M\right) \\ &< 2\left(\frac{1}{n} + \frac{1}{m}\right),\end{aligned}$$

so (y_n) is Cauchy, and it converges, say, to $y \in M$ (since M is closed). Taking the limit $n \to \infty$ in $\|x - y_n\|^2 \leqslant \delta^2 + 1/n$, we obtain $\|x - y\| \leqslant \delta$, so $\|x - y\| = \inf_{m \in M} \|x - m\|$.

Proof (cont.)

(2) *Uniqueness*:
Let $y_1, y_2 \in M$ be s.t. $\|x - y_1\| = \|x - y_2\| = \delta$. Then $(y_1 + y_2)/2 \in M$, so $\|x - (y_1 + y_2)/2\| \geqslant \delta$. By the parallelogram law,

$$\|y_1 - y_2\|^2 = 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - 4\left\|x - \frac{y_1 + y_2}{2}\right\|^2 \leqslant 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Therefore, $y_1 = y_2$. □

Corollary (Projection Theorem)
If, in the theorem, M is a closed convex set, then $y \in M$ is the minimizer of $\|x-m\|$ over all $m \in M$ iff $(x-y, m-y) \leqslant 0$ for all $m \in M$.

Proof. Assume that y is the minimizer, but that there is an $m \in M$ s.t. $(x-y, m-y) = \varepsilon > 0$. Let $y_\alpha := (1-\alpha)y + \alpha m$ for $\alpha \in [0,1]$; since M is convex, $y_\alpha \in M$. Then,

$$\|x-y_\alpha\|^2 = (1-\alpha)^2\|x-y\|^2 + 2\alpha(1-\alpha)(x-y, x-m) + \alpha^2\|x-m\|^2,$$

which is differentiable in α, with derivative $-2(x-y, m-y) = -2\varepsilon < 0$ at $\alpha = 0$, so for α sufficiently small, $\|x-y_\alpha\|^2 < \|x-y\|^2$, which contradicts the optimality of y.

Conversely, assume that $(x-y, m-y) \leqslant 0$ for all $m \in M$. Then, for every $m \in M$, $m \neq y$,

$$\|x-m\|^2 = \|x-y+y-m\|^2 = \|x-y\|^2 + \|y-m\|^2 - 2(x-y, m-y) > \|x-y\|^2.$$

□

Note. If M is a closed linear subspace, the condition above becomes $(x-y) \perp M$ (*why?*).

Outline

Banach Fixed Point Theorem

Let (X,d) be a metric space. A mapping $A: X \to X$ is a *contraction with rate* $0 \leqslant \alpha < 1$ if $d(Ax, Ay) \leqslant \alpha d(x,y)$ for all $x, y \in X$.

Theorem (Banach Fixed Point Theorem)
Every contraction mapping A on a complete metric space X has a unique fixed point, *i.e.*, $Ax = x$ has a unique solution x in X.
Also, if Y is another metric space, and $\lambda \mapsto A_\lambda$ is a *weak***-continuous* mapping in Y, where A_λ is a contraction with rate α, *i.e.*, for every $x \in X$ and $\lambda_0 \in Y$, $\lim_{\lambda \to \lambda_0} d(A_\lambda(x), A_{\lambda_0}(x)) = 0$, then the fixed point of A_λ is continuous in λ.

Proof

(1) *Existence*: Take $x_0 \in X$, and define by induction $x_n = A(x_{n-1}) = A^n(x_0)$ for every $n \in \mathbb{N}$. If $m \geqslant n$,

$$d(x_n, x_m) = d(A^n(x_0), A^m(x_0)) \leqslant \alpha^n d(x_0, x_{m-n}) \leqslant \alpha^n \sum_{k=0}^{m-n-1} d(x_i, x_{i+1}) \leqslant \alpha^n d(x_0, x_1) \sum_{k=0}^{m-n-1} \alpha^k \leqslant \frac{\alpha^n d(x_0, x_1)}{1-\alpha},$$

hence (x_n) is Cauchy, so it converges to, say, $x \in X$. Now, a contraction is continuous (*why?*), so $A(x) = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} x_{n+1} = x$, thus x is a fixed point of A.

Banach Fixed Point Theorem (cont.)

Proof (cont.)

(2) *Uniqueness*: Assume that $A(x) = x$ and $A(y) = y$ for some $x, y \in X$. Then $d(x,y) = d(A(x), A(y)) \leqslant \alpha d(x,y)$, where $0 \leqslant \alpha < 1$, so $d(x,y) = 0$, *i.e.*, $x = y$.

(3) *Continuity*: Note that, if x_λ is the fixed point of A_λ,

$$\begin{aligned} d(x_\lambda, x_{\lambda_0}) &= d(A_\lambda(x_\lambda), A_{\lambda_0}(x_{\lambda_0})) \\ &\leqslant d(A_\lambda(x_\lambda), A_\lambda(x_{\lambda_0})) + d(A_\lambda(x_{\lambda_0}), A_{\lambda_0}(x_{\lambda_0})) \\ &\leqslant \alpha d(x_\lambda, x_{\lambda_0}) + d(A_\lambda(x_{\lambda_0}), A_{\lambda_0}(x_{\lambda_0})), \end{aligned}$$

so $d(x_\lambda, x_{\lambda_0}) \leqslant (1-\alpha)^{-1} d(A_\lambda(x_{\lambda_0}), A_{\lambda_0}(x_{\lambda_0})) \to 0$ as $\lambda \to \lambda_0$. ☐

Banach Fixed Point Theorem (cont.)

Application: Value iteration in dynamic programming

Consider a Markov chain (x_n) on a finite set X s.t. $\mathrm{P}\{x_{n+1} = j | x_n = k, a_n = a\} = p_{jk}(a)$, where (a_n) is a sequence of actions on a finite set A. The problem is to find $a_{k'}$, as a function of $x_{n'}$, s.t. $\mathrm{E}\left\{\sum_{n=1}^{\infty} \alpha^n R(x_n)\right\}$ is maximized, where $R: X \to \mathbb{R}_0^+$ is a *reward* function.

To solve this, define the *value function* $V(x) = \max_{(a_1, a_2, \ldots)} \left[R(x) + \sum_{n=2}^{\infty} \alpha^n \mathrm{E}\{R(x_n)\}\right]$, which satisfies the *dynamic programming (DP) equation*

$$V(x) = R(x) + \alpha \max_{a \in A} \sum_{k \in X} V(k) p_{kx}(a).$$

Once V is found, the optimal action is $a^{\mathrm{opt}}(x) := \arg\max_{a \in A} \sum_{k \in X} V(k) p_{kx}(a)$.

To solve the DP equation, one can use the *value iteration algorithm*:

(1) Let $V_0(x) := 0$.

(2) For $n = 1, 2, \ldots$, let $V_n(x) := R(x) + \alpha \max_{a \in A} \sum_{k \in X} V_{n-1}(k) p_{kx}(a)$.

Application: Value iteration in dynamic programming (cont.)

The value iteration algorithm yields a sequence of value functions (V_n) that converges to the unique solution of the DP equation, since it is defined by a contraction mapping!

Indeed, define the *Bellman operator* $A\colon C(X) \to C(X)$ as $A(V)(x) := R(x) + \alpha \max_{a \in A} \sum_{k \in X} V(k) p_{kx}(a)$. Then, for $V_1, V_2 \in C(X)$,

$$\begin{aligned}
|A(V_1)(x) - A(V_2)(x)| &= \alpha \left| \max_{a \in A} \sum_{k \in X} V_1(k) p_{kx}(a) - \max_{a \in A} \sum_{k \in X} V_2(k) p_{kx}(a) \right| \\
&\leqslant \alpha \max_{a \in A} \left| \sum_{k \in X} V_1(k) p_{kx}(a) - \sum_{k \in X} V_2(k) p_{kx}(a) \right| \\
&\leqslant \alpha \max_{a \in A} \sum_{k \in X} |V_1(k) - V_2(k)| p_{kx}(a) \\
&\leqslant \alpha \max_{k \in X} |V_1(k) - V_2(k)|,
\end{aligned}$$

so A is a contraction.

Next Topic

Orthogonal Expansions

Outline

Completeness

The Space $L_2[a,b]$

The Closest Point Property

Banach Fixed Point Theorem

Bonus Slides

Bonus: Irrationality of e

Euler's number is defined as $e := \sum_{n=0}^{\infty} 1/n!$. We will show that e is irrational.

Assume, to the contrary, that e is rational, and in particular that $e = a/b$ where $a, b \in \mathbb{N}$. Let

$$x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right).$$

Notice that:

(1) $x = b!\sum_{n=b+1}^{\infty} 1/n! > 0$.

(2) $x = b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) = a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}$ is an integer.

(3) For $n \geqslant b+1$, $\frac{b!}{n!} = \frac{1}{(b+1)\cdots n} \leqslant \frac{1}{(b+1)^{n-b}}$, so

$$x = \sum_{n=b+1}^{\infty} \frac{b!}{n!} < \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}} = \sum_{n=1}^{\infty} \frac{1}{(b+1)^{n}} = \frac{1/(b+1)}{1 - 1/(b+1)} = \frac{1}{b} < 1,$$

which is a contradiction, since there is no integer between 0 and 1, so e is irrational. □

Bonus: Completion of Metric Spaces

Let (X,d) be a metric space. If X is not complete, it can be "embedded" into a complete metric space $(\tilde{X},\tilde{d})$, *i.e.*, there exists a complete metric space $(\tilde{X},\tilde{d})$ and a mapping $T\colon X\to\tilde{X}$ which is an *isometry*, that is, s.t. $\tilde{d}(T(x),T(y))=d(x,y)$ for all $x,y\in X$, and s.t. $\mathscr{R}(T)$ is dense in $\tilde{X}$. Such an $(\tilde{X},\tilde{d})$ is a *completion* of (X,d).

Theorem. Every metric space (X,d) has a unique completion, up to isometry, *i.e.*, if $\tilde{X}$ and X' are two completions of X, they are isometric.

Proof

(1) *Existence*: Let (x_n) and (x'_n) be two Cauchy sequences in X. (x_n) and (x'_n) are said to be equivalent, $(x_n)\sim(x'_n)$, if $d(x_n,x'_n)\to 0$ as $n\to\infty$. Let $\tilde{X}$ consist of all equivalence classes of Cauchy sequences in X, and define the metric $\tilde{d}$ as $\tilde{d}((x_n),(y_n))=\lim_{n\to\infty}d(x_n,y_n)$. This metric is well defined since $|d(x_n,y_n)-d(x_m,y_m)|\leqslant d(x_n,x_m)+d(y_n,y_m)\to 0$ as $n,m\to\infty$, and if $(x_n)\sim(x'_n)$ and $(y_n)\sim(y'_n)$, then $|d(x_n,y_n)-d(x_n,y'_n)|\leqslant d(x_n,x'_n)+d(y_n,y'_n)\to 0$ as $n\to\infty$. $\tilde{d}$ is a metric because (i) if $\tilde{d}((x_n),(y_n))=0$ then $d(x_n,y_n)\to 0$, so $(x_n)\sim(y_n)$, (ii) $d(x_n,y_n)=d(y_n,x_n)$, so taking $n\to\infty$ gives $\tilde{d}((x_n),(y_n))=\tilde{d}((y_n),(x_n))$, and (iii) $d(x_n,y_n)\leqslant d(x_n,z_n)+d(z_n,y_n)$, so $n\to\infty$ gives $\tilde{d}((x_n),(y_n))\leqslant\tilde{d}((x_n),(z_n))+\tilde{d}((z_n),(y_n))$.

Proof (cont.)

An isometry $T: X \to \tilde{X}$ is given by $T(x) = (x_n)$ with $x_n = x$ for all n. Clearly, $\tilde{d}(T(x), T(y)) = d(x,y)$. Now, take an (x_n) in $\tilde{X}$, and let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ s.t. $d(x_n, x_N) < \varepsilon$ for all $n \geqslant N$; then, $(y_n) \in \mathscr{R}(T)$ given by $y_n = x_N$ satisfies $\tilde{d}((x_n),(y_n)) = \lim_{n\to\infty} d(x_n, x_N) \leqslant \varepsilon$, so $\mathscr{R}(T)$ is dense in $\tilde{X}$.

To prove the completeness of $\tilde{X}$, consider a Cauchy sequence $(\tilde{x}^n)$ in $\tilde{X}$, where $\tilde{x}^n = (\tilde{x}^n_k)_k$. Since $\mathscr{R}(T)$ is dense in $\tilde{X}$, for every $\tilde{x}^n$ there is a $\tilde{y}^n = (\tilde{y}^n_k)_k$ in $\mathscr{R}(T)$, with $\tilde{y}^n_k = y_n \in X$, s.t. $\tilde{d}(\tilde{x}^n, \tilde{y}^n) < 1/n$, and $\tilde{d}(\tilde{y}^n, \tilde{y}^m) \leqslant \tilde{d}(\tilde{y}^n, \tilde{x}^n) + \tilde{d}(\tilde{x}^n, \tilde{x}^m) + \tilde{d}(\tilde{x}^m, \tilde{y}^m) < 1/n + \tilde{d}(\tilde{x}^n, \tilde{x}^m) + 1/m$, so $(\tilde{y}^n)$ is Cauchy. Consider $\tilde{x} = (y_n)$; since $d(y_n, y_m) = \tilde{d}(\tilde{y}^n, \tilde{y}^m)$, $\tilde{x}$ is Cauchy, so $\tilde{x} \in \tilde{X}$, and $\tilde{d}(\tilde{x}^n, \tilde{x}) \leqslant \tilde{d}(\tilde{x}^n, \tilde{y}^n) + \tilde{d}(\tilde{y}^n, \tilde{x}) < 1/n + \lim_{k\to\infty} d(y_n, y_k) \to 0$ as $n \to \infty$, so $\tilde{x}^n \to \tilde{x}$ and $\tilde{X}$ is complete.

(2) *Uniqueness*: If (X', d') is another completion of (X, d), with T' being an isometry of X into X', then for every $x', y' \in X'$ there are sequences $(x'^n), (y'^n)$ in $\mathscr{R}(T')$, *i.e.*, for all n, $x'^n = T'(x_n)$ and $y'^n = T'(y_n)$ for some $x_n, y_n \in X$, s.t. $x'^n \to x'$ and $y'^n \to y'$. Hence, $|d'(x', y') - d'(x'^n, y'^n)| \leqslant d'(x', x'^n) + d'(y', y'^n) \to 0$ as $n \to \infty$. Since $\mathscr{R}(T')$ and $\mathscr{R}(T)$ are isometric to X, for each $x', y' \in X'$ there are Cauchy sequences $(\tilde{x}^n), (\tilde{y}^n)$ in $\mathscr{R}(T)$, where $\tilde{x}^n = T(x_n)$ and $\tilde{y}^n = T(y_n)$ for all n, whose limits $\tilde{x}, \tilde{y}$ are uniquely defined and satisfy $\tilde{d}(\tilde{x}, \tilde{y}) = d'(x', y')$, so $\tilde{X}$ and X' are isometric. □

Bonus: Completion of Metric Spaces (cont.)

Application to $\mathbb{R}$ as a completion of $\mathbb{Q}$
The previous theorem yields a constructive means to complete a metric space. In particular, the set of rational numbers $\mathbb{Q}$ can be completed by constructing the set of all Cauchy sequences in $\mathbb{Q}$. This is a representation (*model*) of the set of real numbers, $\mathbb{R}$.

All operations in $\mathbb{R}$ can be defined point-wisely: given $x = (x_n), y = (y_n) \in \mathbb{R}^{\mathbb{N}}$,

$$d(x,y) := [(|x_n - y_n|)],$$

$$x \pm y := [(x_n \pm y_n)], \quad x \cdot y := [(x_n \cdot y_n)], \quad \textit{etc.}$$