EL3370 Mathematical Methods in Signals, Systems and Control

Homework 4

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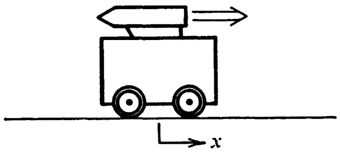
Instructions (read carefully):

- The exercise sets are *individual*: even though discussion with your peers is encouraged, you have to provide your own personal solution to each problem.
- The solutions to some problems can possibly be found by searching in math books other than the main course book. *Avoid such practice*: the only way to understand the topics in the course is by working hard on the problems by yourself.
- To prove statements in the exercises, use only the notation, definitions and results proven (not those given as exercises) in the lectures.

1 A rocket problem

Suppose we wish to bring a rocket car of unit mass, and subject only to the force of the rocket thrust, to rest at $x = 0$ in minimum time by proper choice of the rocket thrust program. The position of the rocket is described by the equation $\ddot{x} = u$, where u is the available thrust, which is limited to $|u(t)| \leqslant 1$ for each t. Assume that initially $x(0) = 0$ and $\dot{x}(0) = 1$.



(a) Produce an argument that converts this problem to a minimum norm problem on a fixed interval $[0, T]$.

(b) Solve the problem.

2 Existence of solution to a system of linear equations

Let X be a real vector space and let $f_1, \dots, f_n$ be linear functionals on X. Show that, for fixed α_k's the system of equations

$$f_k(x) = \alpha_k, \qquad k = 1, \dots, n,$$

has a solution $x \in X$ iff for every $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ the relation $\sum_{k=1}^n \lambda_k f_k = 0$ implies $\sum_{k=1}^n \lambda_k \alpha_k = 0$.
Hint: In $\mathbb{R}^n$, consider the linear subspace consisting of all points of the form $(f_1(x), \dots, f_n(x))$ as x varies over X, and apply the Hahn-Banach theorem.
Note: The expression $\sum_{k=1}^n \lambda_k f_k = 0$ is an equality between functions, *i.e.*, it is equivalent to $\sum_{k=1}^n \lambda_k f_k(x) = 0$ for all $x \in X$.

3 Bounded operators

Let $(e_n)_{n \in \mathbb{N}}$ be a total orthonormal sequence in a complex Hilbert space H, and let $\lambda_n \in \mathbb{C}$ for $n \in \mathbb{N}$.

(a) Show that there is a bounded linear operator D on H such that $De_n = \lambda_n e_n$ for all $n \in \mathbb{N}$ iff (λ_n) is a bounded sequence. What is $\|D\|$, when defined?
Note: Remember that a bounded sequence does not necessarily achieve its maximum (so do not confuse sup with max).

(b) Show that D, assumed bounded, is invertible iff $\inf_n |\lambda_n| > 0$. What is $\|D^{-1}\|$, when applicable?
Note: D being invertible means that D^{-1} exists as a function and is a *bounded* linear operator.

(c) Find the adjoint of D, when (λ_n) is a bounded sequence.

(d) Find the spectrum of D, assuming that D is bounded.
Note: Remember that the spectrum is the complement of the set of those $\lambda \in \mathbb{C}$ such that $(\lambda I - D)^{-1}$ exists and is a *bounded* linear operator.

4 Projection operators

Let $M \neq \{0\}$ be a closed subspace of a Hilbert space H. The operator P defined by $Px = m$, where $x = m + n$ is the unique representation of $x \in H$ with $m \in M$, $n \in M^\perp$, is called the *projection operator onto* M.

(a) Show that a projection is linear with $\|P\| = 1$.

(b) Show that a bounded linear operator on H is a projection iff

1. $P^2 = P$ (idempotent), and
2. $P^* = P$ (self-adjoint).

(c) Two projections P_1 and P_2 on H are said to be *orthogonal* if $P_1 P_2 = 0$. Show that two projections are orthogonal iff their ranges are orthogonal.

(d) Show that the sum of two projections is a projection iff they are orthogonal.

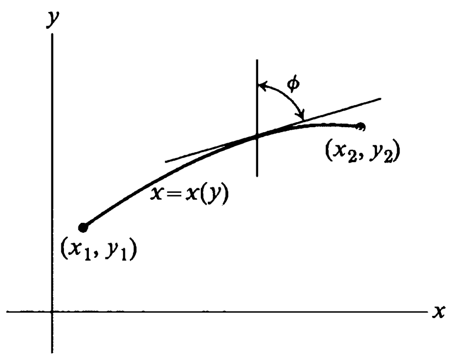
5 Fermat's principle

Fermat's principle of optics states: If the velocity of light is given by the continuous function $u = u(y)$, the actual light path connecting the points (x_1, y_1) and (x_2, y_2) in a plane is the one which *extremizes* (*i.e.*, yields a local minimum or maximum of) the time integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{u} = \int_{y_1}^{y_2} \frac{\sqrt{1 + [dx(y)/dy]^2}}{u(y)} dy$$

(Actually, refinements are needed to make this formulation of Fermat's principle hold for all cases.)

(a) Derive Snell's law from Fermat's principle; that is, prove that $(\sin\phi)/u$ is constant, where ϕ is the angle shown in the following figure:



(b) Suppose that light travels in the xy-plane in such a way that its speed is proportional to y. Then, prove that the light rays emitted from any point are circle arcs with their centers on the x-axis.
Hint: Solve the Euler-Lagrange differential equation (you may use a computer algebra package to compute the resulting integrals).