EL3370 Mathematical Methods in Signals, Systems and Control

Homework 3

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Instructions (read carefully):

- The exercise sets are *individual*: even though discussion with your peers is encouraged, you have to provide your own personal solution to each problem.
- The solutions to some problems can possibly be found by searching in math books other than the main course book. *Try to avoid such practice*: the only way to understand the topics in the course is by working hard on the problems by yourself.
- To prove statements in the exercises, use only the notation, definitions and results proven (not those given as exercises) in the lectures.

1 Gram-Schmidt method

Let $(x_1, x_2, \dots)$ be a sequence of linearly independent vectors in an (infinite dimensional) inner product space. Define vectors e_n inductively as follows:

$$\begin{aligned} e_1 &= \frac{x_1}{\|x_1\|}, \\ f_n &= x_n - \sum_{k=1}^{n-1} (x_n, e_k) e_k, \quad n \geqslant 2, \\ e_n &= \frac{f_n}{\|f_n\|}, \quad n \geqslant 2. \end{aligned}$$

Show that (e_n) is an orthonormal sequence having the same closed linear span as $(x_1, x_2, \dots)$.

2 Approximation in RH_2

Given an $f \in RL_2$, with Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, we want to find the best approximation, f^+, to f in RH_2, *i.e.*, $\|f - f^+\| = \min_{g \in RH_2} \|f - g\|$. To this end, follow these steps:

(a) Show that $(e_n)_{n=0,1,2,\dots}$, where $e_n(z) = z^n$, is a total orthonormal sequence in RH_2 (this means that $\operatorname{clin}\{e_n\} = RH_2$).
Hint: Notice that $f \in RH_2$ can be expanded as a series of non-negative powers of z, whose coefficients a_n decay to 0 as $n \to \infty$ (expand f in partial fractions, and write each fraction as a geometric series in z).

(b) Prove that if H is an inner product space, and M is a closed linear subspace of H with a total orthonormal sequence $(e_n)_{n=0,1,\dots}$, then, for every $x \in H$, if

$$y = \sum_{n=0}^{\infty} (x, e_n) e_n$$

exists in H, then y is the best approximation to x in M.

Remark: RH_2 is **not** a Hilbert space, so not all the statements of the closest point property apply (in particular, the existence of a minimizer is not guaranteed). However, some conclusions of this result are still valid, such as the uniqueness of the minimizer, and its characterization in the projection theorem.

(c) Apply steps (a) and (b) to show that $f^+(z) = \sum_{n=0}^{\infty} a_n z^n$.

3 Channel equalization

In communication systems, we often expect a transmission system to be a pure time delay so that there is no distortion introduced by the channel. However, physical limitations often prevent us from designing an exact pure delay system. In this case, we strive for a system to be as close to a pure delay as possible, by designing a suitable channel equalizer. This leads to the following optimization problem:

$$J(d) = \inf_{Q \in RH_2} \|T - FQ\|, \tag{1}$$

where $F, T \in RL_2$ and $d \in \mathbb{N}$. Here, F is the channel transfer function, Q is a channel equalizer, and $T(z) = z^d$ is the desired transfer function of the equalized channel[1].

[1] To follow the mathematical convention for RH_2, z here corresponds to a time delay rather than to a forward time shift.

(a) Assume that $F \in RH_2$ and that $F(e^{i\omega}) \neq 0$ for all $\omega \in [-\pi, \pi]$. In case $F^{-1} \notin RH_2$ (*i.e.*, if F is of non-minimum-phase), then F can be factorized as $F = F_I F_O$, where $|F_I(e^{i\omega})|^2$ is constant in ω (*i.e.*, F_I is an *all-pass filter*) and $F_O, F_O^{-1} \in RH_2$. Assuming this factorization, show that the problem can be written in the form:

$$J(d) = \inf_{Q \in RH_2} \alpha \|F_O Q - G\|. \tag{2}$$

What are α and G?

(b) Prove that the orthogonal complement of RH_2 in RL_2, denoted $RH_2^\perp$, is the space of all functions $f \in RL_2$ that are analytic in $\mathbb{D}^c = \{z \in \mathbb{C} \colon |z| \geqslant 1\}$ and such that $\lim_{z\to\infty} f(z) = 0$.

(c) If G in (a) is split as $G = G_+ + G_-$ according to $RL_2 = RH_2 \oplus RH_2^\perp$, solve problem (2). *Hint:* Use the Pythagorean theorem.

(d) Solve problem (1) for $F(z) = (z^{-1} - 2)/z^{-1}$ as a function of $d \in \mathbb{N}$, and show that $\lim_{d\to\infty} J(d) = 0$.

4 Minimum norm problem on a polytope

The projection theorem seen in class states that, given a closed non-empty convex subset M of a Hilbert space H, and a point $x \in H$, the point $y \in M$ closest to x satisfies $(x - y, z - y) \leqslant 0$ for all $z \in M$. Using this result (and the closest point property), we will address the problem of finding the vector x of minimum norm in a real Hilbert space H satisfying

$$(x, y_k) \geqslant c_k, \quad \text{for all } k = 1, 2, \ldots, n,$$

where the y_k's are linearly independent.

(a) Show that this problem has a unique solution.

(b) Show that the solution of this problem is of the form $x = \sum_{k=1}^n a_k y_k$, for some $a_1, \ldots, a_n \in \mathbb{R}$. Also, prove that a necessary and sufficient condition for $x = \sum_{k=1}^n a_k y_k$ to be the solution is that the vector $a \in \mathbb{R}^n$ with components a_k satisfies[2]

$$\begin{aligned} Ga &\geqslant c, \\ a &\geqslant 0, \end{aligned}$$

and that $a_k = 0$ if $(x, y_k) > c_k$. Here, $c := [c_1, \ldots, c_n]^T$, and G is the *Gram matrix* of $(y_1, \ldots, y_n)$, *i.e.*,

$$G := \begin{bmatrix} (y_1, y_1) & (y_1, y_2) & \cdots & (y_1, y_n) \\ (y_2, y_1) & (y_2, y_2) & \cdots & (y_2, y_n) \\ \vdots & \vdots & & \vdots \\ (y_n, y_1) & (y_n, y_2) & \cdots & (y_n, y_n) \end{bmatrix}.$$

Hint: First, decompose H as $H = N \oplus N^\perp$, where $N = \text{lin}\{y_1, \ldots, y_n\}$, and prove that the optimal x belongs to N. Also, check the example on page 71 of Luenberger's book for inspiration.

[2]Here, the order relation $\geqslant$ is considered to hold *component-wisely*, *i.e.*, $a \geqslant b$ iff $a_k \geqslant b_k$ for all k.