DISCRETE ANALOGUES OF THE LAGUERRE INEQUALITIES AND A CONJECTURE OF I. KRASIKOV

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Abstract. A conjecture of I. Krasikov is proved. Several discrete analogues of classical polynomial inequalities are derived, along with results which allow extensions to a class of transcendental entire functions in the Laguerre-Pólya class.

1. Introduction

The classical Laguerre inequality for polynomials states that a polynomial of degree $n$ with only real zeros, $p(x) \in \mathbb{R}[x]$, satisfies $(n-1)p'(x)^2 - np''(x)p(x) \geq 0$ for all $x \in \mathbb{R}$ (see [3, 13]). Thus, the classical Laguerre inequality is a necessary condition for a polynomial to have only real zeros. Our investigation is inspired by an interesting paper of I. Krasikov [8]. He proves several discrete polynomial inequalities, including useful versions of generalized Laguerre inequalities [17], and shows how to apply them by obtaining bounds on the zeros of some Krawtchouk polynomials. In [8], I. Krasikov conjectures a new discrete Laguerre inequality for polynomials. After establishing this conjecture, we generalize the inequality to transcendental entire functions (of order $\rho < 2$, and minimal type of order $\rho = 2$) in the Laguerre-Pólya class (see Definition 1.1).

Definition 1.1. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is said to belong to the Laguerre-Pólya class, written $\varphi \in \mathcal{L}$-$\mathcal{P}$, if it can be expressed in the form

$$\varphi(x) = cx^m e^{-ax^2 + bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k} \quad (0 \leq \omega \leq \infty),$$

where $b, c, x_k \in \mathbb{R}$, $m$ is a non-negative integer, $a \geq 0$, $x_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$.

The significance of the Laguerre-Pólya class stems from the fact that functions in this class, and only these, are uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros [12, Chapter VIII].

Definition 1.2. We denote by $\mathcal{L}$-$\mathcal{P}_n$ the set of polynomials of degree $n$ in the Laguerre-Pólya class; that is, $\mathcal{L}$-$\mathcal{P}_n$ is the set of polynomials of degree $n$ having only real zeros.

The minimal spacing between neighboring zeros of a polynomial in $\mathcal{L}$-$\mathcal{P}_n$ is a scale that provides a natural criterion for the validity of discrete polynomial inequalities.

Definition 1.3. Suppose $p(x) \in \mathcal{L}$-$\mathcal{P}_n$ has zeros $\{\alpha_k\}_{k=1}^{n}$, repeated according to their multiplicities, and ordered such that $\alpha_k \leq \alpha_{k+1}$, $1 \leq k \leq n-1$. We define the mesh size, associated with the zeros of $p$, by

$$\mu(p) := \min_{1 \leq k \leq n-1} |\alpha_{k+1} - \alpha_k|.$$
With the above definition of mesh size, we can now state a conjecture of I. Krasikov, which is proved in Section 2.

**Conjecture 1.4.** (I. Krasikov [8]) If \( p(x) \in \mathcal{LP}_n \) and \( \mu(p) \geq 1 \), then
\[
(n - 1)[p(x + 1) - p(x - 1)]^2 - 4np(x)[p(x + 1) - 2p(x) + p(x - 1)] \geq 0
\]
holds for all \( x \in \mathbb{R} \).

The classical Laguerre inequality is found readily by differentiating the logarithmic derivative of a polynomial \( p(x) \) with only real zeros \( \{\alpha_i\}_{i=1}^n \), to give
\[
\left( \frac{p''(x)p(x) - (p'(x))^2}{(p(x))^2} \right) = \left( \frac{p'(x)}{p(x)} \right)' = \left( \sum_{k=1}^n \frac{1}{x - \alpha_k} \right)' = -\sum_{k=1}^n \frac{1}{(x - \alpha_k)^2}.
\]
Since the right-hand side is non-positive,
\[
(p'(x))^2 - p''(x)p(x) \geq 0.
\]
This inequality is also valid for an arbitrary function in \( \mathcal{LP} \) [3]. A sharpened form of the Laguerre inequality for polynomials can be obtained with the Cauchy-Schwarz inequality,
\[
\left( \sum_{k=1}^n \frac{1}{x - \alpha_k} \right)^2 \leq n \sum_{k=1}^n \frac{1}{(x - \alpha_k)^2}.
\]
In terms of \( p \), (3) becomes
\[
\left( \frac{p''(x)}{(p(x))^2} \right)^2 \leq n \sum_{k=1}^n \frac{1}{(x - \alpha_k)^2},
\]
and with (2) yields the sharpened version of the Laguerre inequality for polynomials on which Conjecture 1.4 is based,
\[
(n - 1)(p'(x))^2 - np''(x)p(x) \geq 0.
\]

The inequality (1) is a finite difference version of the classical Laguerre inequality for polynomials. Indeed, let us define
\[
f_n(x, h, p) := (n - 1)[p(x + h) - p(x - h)]^2 - 4np(x)[p(x + h) - 2p(x) + p(x - h)].
\]
Then (1) can be written as \( f_n(x, 1, p) \geq 0 \) \((x \in \mathbb{R})\), and we recover the classical Laguerre inequality for polynomials by taking the following limit:
\[
\lim_{h \to 0} \frac{f_n(x, h, p)}{4h^2} = (n - 1)\left( \lim_{h \to 0} \frac{p(x + h) - p(x - h)}{2h} \right)^2 - np(x)\left( \lim_{h \to 0} \frac{p(x + h) - 2p(x) + p(x - h)}{h^2} \right) = (n - 1)p'(x)^2 - np''(x)p(x).
\]

As I. Krasikov points out, the motivation for inequalities of type (1) is that classical discrete orthogonal polynomials \( p_k(x) \) satisfy a three-term difference equation (see [15, p. 27], [8])
\[
p_k(x + 1) = b_k(x)p_k(x) - c_k(x)p_k(x - 1),
\]
where \( b_k(x) \) and \( c_k(x) \) are continuous over the interval of orthogonality. Many of the classical discrete orthogonal polynomials satisfy the condition that \( c_k(x) > 0 \) on the interval of orthogonality, and this implies that \( \mu(p) \geq 1 \) (see [11]). Therefore, inequalities when \( \mu(p) \geq 1 \) are of interest and may help provide sharp bounds on the loci of zeros of discrete orthogonal polynomials [8, 5, 6]. Indeed, W. H. Foster, I. Krasikov, and A. Zarkh have found bounds on the extreme zeros of many orthogonal polynomials using discrete and continuous Laguerre and new Laguerre type inequalities which they discovered [5, 6, 7, 8, 9, 10, 11].
In this paper, we prove I. Krasikov’s conjecture (see Theorem 2.17), extend it to a class of transcendental entire functions in the Laguerre-Pólya class, and formulate several conjectures (cf. Conjecture 2.19, Conjecture 2.21, Conjecture 2.22, and Conjecture 3.5). In Section 2, we establish several preliminary results about polynomials which satisfy a zero spacing requirement. In Section 3, we establish the existence of a polynomial sequence which satisfies a zero spacing requirement and converges uniformly on compact subsets of \( \mathbb{C} \) to the exponential function. We use this result to extend a version of (1) to transcendental entire functions in the Laguerre-Pólya class up to order \( \rho = 2 \) and minimal type, and conjecture that it is true for all functions in \( \mathcal{L} \cdot \mathcal{P} \).

2. Proof of I. Krasikov’s Conjecture

In this section we develop some discrete analogues of classical inequalities, form some intuition about the effect of imposing a minimal zero spacing requirement on a polynomial in \( \mathcal{L} \cdot \mathcal{P} \), and prove Conjecture 1.4. First, note that one can change the zero spacing requirement in Conjecture 1.4 by simply rescaling \( p(x) \) (in other words “measuring \( x \) in units of \( h \)).

**Conjecture 2.1.** Let \( p(x) \in \mathcal{L} \cdot \mathcal{P}_n \). Suppose that \( \mu(p) \geq h > 0 \). Then for all \( x \in \mathbb{R} \),

\[
(6) \quad f_n(x, h, p) = (n - 1)[p(x + h) - p(x - h)]^2 - 4np(x)[p(x + h) - 2p(x) + p(x - h)] \geq 0.
\]

For the sake of clarity, we will work with (1) directly \((h = 1)\), and keep in mind that we can always make statements about polynomials with an arbitrary positive minimal zero spacing by rescaling \( p(x) \) (in other words “measuring \( x \) in units of \( h \)).

**Lemma 2.2.** A local minimum of a polynomial, \( p(x) \in \mathcal{L} \cdot \mathcal{P}_n \), with only real simple zeros, is negative. Likewise, a local maximum of \( p(x) \) is positive.

**Proof.** Because \( p(x) \) is a polynomial on \( \mathbb{R} \) with simple zeros, at a local minimum \((x_{\text{min}}, p(x_{\text{min}}))\), we have that \( p'(x_{\text{min}}) = 0 \) and \( p''(x_{\text{min}}) > 0 \) (because \( p''(x_{\text{min}}) = 0 \) would imply that \( p' \) has a multiple zero at \( x_{\text{min}} \) which is not possible). The classical Laguerre inequality asserts that if \( p(x) \in \mathcal{L} \cdot \mathcal{P} \), then for all \( x \in \mathbb{R} \), \((p'(x))^2 - p''(x)p(x) \geq 0 \). At a local minimum this expression becomes \(-p''(x_{\text{min}})p(x_{\text{min}}) \geq 0 \). Therefore, at a local minimum we have \( p(x_{\text{min}}) \leq 0 \). Since the zeros of \( p \) are simple, \( p(x_{\text{min}}) \neq 0 \). Thus \( p(x_{\text{min}}) < 0 \). The second statement of the lemma can be proved the same way, or by considering \(-p\) and using the first statement. \( \square \)

A statement similar to Lemma 2.2 is proved by G. Csordas and A. Escassut [4, Theorem 5.1] for a class of functions whose zeros lie in a horizontal strip about the real axis.

**Lemma 2.3.** Let \( p(x) \in \mathcal{L} \cdot \mathcal{P}_n \), \( n \geq 2 \), \( \mu(p) \geq 1 \).

(i) If \( p(x - 1) > p(x) \) and \( p(x + 1) > p(x) \), then \( p(x) < 0 \).
(ii) If \( p(x - 1) < p(x) \) and \( p(x + 1) < p(x) \), then \( p(x) > 0 \).

**Proof.** (i) Fix an \( x_0 \in \mathbb{R} \). Let \( p(x_0 - 1) > p(x_0) \), \( p(x_0 + 1) > p(x_0) \), and assume for a contradiction that \( p(x_0) \geq 0 \). There cannot be any zeros of \( p(x) \) in the interval \([x_0 - 1, x_0]\), for if there were, \( p(x_0)p(x_0 - 1) > 0 \) implies that the number of zeros in \((x_0 - 1, x_0)\) must be even, and this violates the zero spacing \( \mu(p) \geq 1 \). Similarly, there cannot be any zeros of \( p(x) \) in \([x_0, x_0 + 1]\). If \( p(x_0) < p(x_0 - 1) \) and \( p(x_0) < p(x_0 + 1) \) then there is a point in \((x_0 - 1, x_0 + 1)\) where \( p' \) changes sign from negative to positive. This implies \( p \) achieves a non-negative local minimum on \([x_0 - 1, x_0 + 1]\) which contradicts Lemma 2.2.

(ii) The second statement follows by replacing \( p \) with \(-p\) in (i). \( \square \)
Using Lemma 2.3 we can verify that if \( p(x) < \min(p(x + 1), p(x - 1)) \), then \( p(x) < 0 \) and thus the function

\[
f_m(x, 1, p) = (n - 1)[p(x + 1) - p(x - 1)]^2 - 4np(x)[p(x + 1) - 2p(x) + p(x - 1)]
\]

\[
= (n - 1)[p(x + 1) - p(x - 1)]^2 - 4np(x)[p(x + 1) - p(x) + p(x - 1) - p(x)]
\]

(7)

has a non-negative second term and (1) is satisfied. Similarly, (1) is valid when \( p(x) > \max(p(x + 1), p(x + 1)) \). The proof of Conjecture 1.4 is now reduced to the case where \( \min(p(x + 1), p(x - 1)) \leq p(x) \leq \max(p(x + 1), p(x - 1)) \). It is easy to show that if for some \( p(x) \in \mathcal{L}_n \), \( f_m(x, 1, p) \geq 0 \) for all \( x \in \mathbb{R} \), then for all \( m \geq n \), \( f_m(x, 1, p) \geq 0 \) for all \( x \in \mathbb{R} \).

Since the zeros are spaced at least 1 unit apart, the function

\[
evaluating this at a zero of \( p \) yields \( p(\alpha_k + 1) = A_k \prod_{j \neq k}(\alpha_k - \alpha_j) = A_kp'(\alpha_k) \).

Thus,

\[
A_k = \frac{p(\alpha_k + 1)}{p'(\alpha_k)}
\]

and similarly \( B_k = -\frac{p(\alpha_k - 1)}{p'(\alpha_k)} \).

Since the zeros of \( p \) are simple, for some neighborhood of \( \alpha_k, U(\alpha_k) \),

\[
\begin{align*}
x \in U(\alpha_k), x < \alpha_k & \quad \text{implies} \quad p(x)p'(x) < 0 \\
x \in U(\alpha_k), x > \alpha_k & \quad \text{implies} \quad p(x)p'(x) > 0.
\end{align*}
\]

Since the zeros are spaced at least 1 unit apart, \( p(\alpha_k + 1) \) is either 0 or has the same sign as \( p(x) \) for \( x > \alpha_k \) on \( U(\alpha_k) \). So for all \( \varepsilon > 0 \) sufficiently small, \( p(\alpha_k + 1)p'(\alpha_k + \varepsilon) \geq 0 \), and by continuity \( p(\alpha_k + 1)p'(\alpha_k) \geq 0 \). Thus \( A_k = \frac{p(\alpha_k + 1)}{p'(\alpha_k)} \geq 0 \). Note \( p'(\alpha_k) \neq 0 \) since \( \alpha_k \) is simple. Likewise, \( p(\alpha_k - 1) \) is either 0 or has the same sign as \( p'(x) \) for \( x < \alpha_k \) on
Observe that

\[ \sum \]

Let

\[ \text{If } p \]

Lemma 2.10.

\[ \text{F} \]

(Pólya and Szegö [18, vol. II, p. 39])

Lemma 2.9.

This proves the following lemma.

Example 2.7. If the zero spacing requirement in Lemma 2.6 is violated then some \( A_k \) or \( B_k \) may be negative. Indeed, consider \( p(x) = x(x + 1 - \varepsilon) \). Then

\[ \frac{p(x + 1) - p(x)}{p(x)} = \frac{A_1 x + A_2}{x + 1 - \varepsilon}, \]

where

\[ A_1 = \frac{2 - \varepsilon}{1 - \varepsilon}, \quad A_2 = \frac{-\varepsilon}{1 - \varepsilon}. \]

For any positive \( \varepsilon < 1 \), \( \mu(p) = 1 - \varepsilon \), and \( A_2 \) is negative.

Corollary 2.8. For \( p(x) \in \mathcal{L}^{-P_n} \), \( n \geq 2 \), with \( \mu(p) \geq 1 \), the associated functions \( F(x) \) and \( R(x) \) (see Definition 2.4) satisfy \( F'(x) < 0 \) and \( R'(x) < 0 \) on their respective domains.

Proof. This corollary is a direct result of differentiating the partial fraction expressions for \( F \) and \( R \) and applying Lemma 2.6.

Note that the degree of the numerator of \( F(x) \) is \( n - 1 \). If \( \mu(p) \geq 1 \), then \( F(x) \) has \( n - 1 \) real zeros, because \( F(x) \) is strictly decreasing between any two consecutive poles of \( F(x) \). This proves the following lemma.

Lemma 2.9. (Pólya and Szegő [18, vol. II, p. 39]) For \( p(x) \in \mathcal{L}^{-P_n} \), \( n \geq 2 \), with \( \mu(p) \geq 1 \), \( F(x) \) and \( R(x) \) have only real simple zeros.

In the sequel (see Lemma 2.16), we show that if \( \mu(p(x)) \geq 1 \), then \( \mu(p(x + 1) - p(x)) \geq 1 \), and the zeros of \( F(x) \) and \( R(x) \) are spaced at least one unit apart.

Lemma 2.10. If \( p(x) \in \mathcal{L}^{-P_n} \), then the associated sequences \( \{A_k\}_{k=1}^n \) and \( \{B_k\}_{k=1}^n \) satisfy

\[ \sum_{k=1}^n A_k = n \quad \text{and} \quad \sum_{k=1}^n B_k = n. \]

Proof. Let \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathcal{L}^{-P_n} \) and denote the zeros of \( p(x) \) by \( \{\alpha_k\}_{k=1}^n \). Observe that

\[ (10) \quad \lim_{|z| \to \infty} zF(z) = \lim_{|z| \to \infty} z \left( \frac{p(z + 1) - p(z)}{p(z)} \right) = \lim_{|z| \to \infty} z \left( \sum_{k=1}^n \frac{A_k}{z - \alpha_k} \right) = \sum_{k=1}^n A_k. \]

Then (10) and

\[ p(z + 1) - p(z) = a_n(z + 1)^n + a_{n-1}(z + 1)^{n-1} + \cdots + a_0 - [a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0] \]

imply that

\[ \sum_{k=1}^n A_k = \lim_{|z| \to \infty} zF(z) = \lim_{|z| \to \infty} z \left( \frac{p(z + 1) - p(z)}{p(z)} \right) = \lim_{|z| \to \infty} z \left( \frac{n a_n z^{n-1} + O(z^{n-2})}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0} \right) = n. \]

A similar argument shows that \( \sum_{k=1}^n B_k = n. \)

Lemma 2.11. Given \( p(x) \in \mathcal{L}^{-P_n} \), \( n \geq 2 \), with \( \mu(p) \geq 1 \), the associated functions \( F(x) \) and \( R(x) \) satisfy \( (F(x))^2 \leq -nF'(x) \) and \( (R(x))^2 \leq -nR'(x) \), for all \( x \in \mathbb{R} \), where \( p(x) \neq 0 \).

Proof. From Definition 2.4, \( F(x) = \sum_{k=1}^n \frac{A_k}{x - \alpha_k} \) and therefore \( F'(x) = \sum_{k=1}^n \frac{A_k}{(x - \alpha_k)^2} \). By Lemma 2.6, \( \mu(p) \geq 1 \) implies the constants \( A_k \geq 0 \). Using the Cauchy-Schwarz inequality,

\[ (F(x))^2 = \left( \sum_{k=1}^n \frac{A_k}{x - \alpha_k} \right)^2 \leq \left( \sum_{k=1}^n A_k \right) \sum_{k=1}^n \frac{A_k}{(x - \alpha_k)^2} = -nF'(x), \]
Then if the interval both sides of this inequality by \( n \) consider 2.12

**Remark** Let \( p \) Lemma 2.13.

\[
(\text{where Lemma 2.10 has been used in the last equality. An identical argument shows} \quad \text{Remark} \quad \text{Lemma} 2.13.
\]

\[
\frac{1}{n} F(y) R(y) \leq \frac{(p(y))^2 - p(y + 1)p(y - 1)}{(p(y))^2}.
\]

**Proof.** If no \( \beta_k \) in \( [y - 1, y) \), then \( F(x) = \frac{(p(x))^2}{(p(x))^2} \) can be extended to be continuous and bounded on \([y - 1, y)\). By Lemma 2.11 \( F(x) \leq -nF(x) \). Dividing both sides of this inequality by \( n(F(x))^2 \) and integrating from \( y - 1 \) to \( y \) we have

\[
\frac{1}{n} \leq \frac{1}{F(y)} - \frac{1}{F(y - 1)} = \frac{p(y)}{p(y + 1) - p(y)} - \frac{p(y - 1)}{p(y) - p(y - 1)}.
\]

Using \( \min[p(y + 1), p(y)] < p(y) < \max[p(y + 1), p(y - 1)] \), we have that either \( p(y - 1) < p(y) < p(y + 1) \) or \( p(y + 1) < p(y) < p(y - 1) \). In both cases, \( (p(y + 1) - p(y))(p(y) - p(y - 1)) > 0 \) and therefore

\[
\frac{1}{n} (p(y + 1) - p(y))(p(y) - p(y - 1)) \leq p(y)(p(y) - p(y - 1)) - p(y - 1)(p(y + 1) - p(y)) \leq (p(y))^2 - p(y + 1)p(y - 1).
\]

Dividing both sides by \( (p(y))^2 \) gives the result. \( \square \)

**Lemma 2.14.** For \( p(x) \in \mathcal{L} - \mathcal{P}_n \), the associated functions \( F(x) \) and \( R(x) \) from Definition 2.4 satisfy

\[
F(x) R(x) = (F(x) - R(x)) + \frac{(p(x))^2 - p(x + 1)p(x - 1)}{(p(x))^2}
\]

for all \( x \in \mathbb{R} \), where \( p(x) \neq 0 \).

**Proof.** This lemma is verified by direct calculation using the definitions of \( F(x) \) and \( R(x) \) in terms of \( p(x) \). \( \square \)

**Lemma 2.15.** Let \( p(x) \in \mathcal{L} - \mathcal{P}_n \), \( n \geq 2 \), with \( \mu(p) \geq 1 \).

1. If \( p(\beta) = p(\beta + 1) > 0 \), then for all \( x \in (\beta, \beta + 1) \), \( p(x) > p(\beta) \) and \( p(x) > \max[p(\beta) + 1], p(x - 1)] \).
2. If \( p(\beta) = p(\beta + 1) < 0 \), then for all \( x \in (\beta, \beta + 1) \), \( p(x) < p(\beta) \) and \( p(x) < \min[p(\beta) + 1], p(x - 1)] \).
3. If \( p(\beta) = p(\beta + 1) = 0 \), then for all \( x \in (\beta, \beta + 1) \), either \( p(x) > \max[p(\beta) + 1], p(x - 1)] \) or \( p(x) < \min[p(\beta) + 1], p(x - 1)] \).

**Proof.** Note that by Lemma 2.9, any \( \beta \) which satisfies \( p(\beta) = p(\beta + 1) \) under the hypotheses stated in Lemma 2.15 must be real and simple since \( \beta \) is a zero of \( F(x) \).

For case (i), assume for a contradiction that there exists \( x_0 \in (\beta, \beta + 1) \) such that \( p(x_0) \leq p(\beta) \). There can not be any zeros of \( p \) on \( (\beta, \beta + 1) \), if there were, \( p(\beta)p(\beta + 1) > 0 \) implies that \( p(x) \) must have at least two zeros on \( (\beta, \beta + 1) \), which contradicts \( \mu(p) \geq 1 \). Thus, for all \( x \in (\beta, \beta + 1) \), \( p(x) > 0 \). Specifically \( p(x_0) > 0 \).
Since \( p(x) \) does not change sign on \((\beta, \beta + 1)\), the interval \((\beta, \beta + 1)\) must lie between two neighboring zeros of \( p(x) \), call them \( \alpha_1 \) and \( \alpha_2 \), such that \((\beta, \beta + 1) \subset (\alpha_1, \alpha_2)\). By the mean value theorem there exists \( a \in (\beta, \beta + 1) \) with \( p'(a) = 0 \). The zeros of \( p(x) \) and \( p'(x) \) interlace, and in order to preserve the interlacing \( a \) must be the only zero of \( p'(x) \) in \((\alpha_1, \alpha_2)\), hence \( p'(\beta), p'(\beta + 1) \neq 0 \). Because the zeros are simple, for some \( \varepsilon > 0 \), for all \( x \in (\alpha_1, \alpha_1 + \varepsilon) \), \( p'(x)p(x) > 0 \), and for all \( x \in (\alpha_2 - \varepsilon, \alpha_2) \), \( p'(x)p(x) < 0 \). Since \( p' \) and \( p \) do not change sign on \((\alpha_1, \beta) \) or \((\beta + 1, \alpha_2)\), this gives us that \( p'(\beta) > 0 \) and \( p'(\beta + 1) < 0 \). Then if \( p(x_0) \leq p(\beta) \), \( p' \) must change signs at least twice on \((\alpha_1, \alpha_2)\) (actually three times), at least once on \((\beta, x_0)\) and at least once on \((x_0, \beta + 1)\), and this contradicts the uniqueness of \( a \). Thus for all \( x \in (\beta, \beta + 1) \) we have \( p(x) > p(\beta) \).

To show \( p(x) > p(\beta) \) implies \( p(x) > \max\{p(x + 1), p(x - 1)\} \) for all \( x \in (\beta, \beta + 1) \), notice that since \( p'(y) < 0 \) for all \( y \in (\beta + 1, \alpha_2) \), \( p(\beta + 1) > p(y) \) for all \( y \in (\beta + 1, \alpha_2) \), and due to the zero spacing \( p \leq 0 \) on \((\alpha_2, \alpha_2 + 1) \), hence \( p(\beta + 1) > p(x + 1) \) for all \( x \in (\beta, \alpha_2) \). Thus, for all \( x \in (\beta, \beta + 1) \), \( p(x) > p(\beta + 1) > p(x + 1) \). In the same way, \( p'(y) > 0 \) for \( y \in (\alpha_1, \beta) \) and \( p \leq 0 \) on \((\alpha_1 - 1, \beta) \) imply that \( p(\beta) > p(x) \) for all \( x \in (\alpha_1 - 1, \beta) \) and therefore \( p(x) > p(x - 1) \) for all \( x \in (\beta, \beta + 1) \). Hence, for all \( x \in (\beta, \beta + 1) \), \( p(x) > p(x - 1) \) and \( p(x) > p(x + 1) \), therefore \( p(x) \geq \max\{p(x + 1), p(x - 1)\} \).

Consider case (iii). If \( p(\beta) = p(\beta + 1) = 0 \), then \( p \) does not change sign on \((\beta, \beta + 1)\) since \( \mu(p) \geq 1 \). It suffices to consider the case when \( p \) is positive on \((\beta, \beta + 1) \). Then for all \( x \in (\beta, \beta + 1) \), \( p(x) > 0 = p(\beta) \). The conclusion \( p(x) > \max\{p(x + 1), p(x - 1)\} \) \((p(x) < \min\{p(x + 1), p(x - 1)\})\) is a consequence of \( p(x) > p(\beta) \) \((p(x) < p(\beta))\) by the same argument given in the proof of case (i).

To prove (ii), let \( g(x) = -p(x) \) and apply (i).

\[ \square \]

**Lemma 2.16.** If \( p(x) \in \mathcal{L:\mathcal{P}}, n \geq 2, \mu(p) \geq 1, \) and \( g(x) = p(x + 1) - p(x) \), then \( \mu(g) \geq 1 \).

**Proof.** (Reductio ad Absurdum) If \( \mu(g) < 1 \), then there exist \( \beta_1, \beta_2 \in \mathbb{R} \) such that \( 0 < \beta_2 - \beta_1 < 1 \) and \( g(\beta_1) = g(\beta_2) = 0 \). In the proof of Lemma 2.15 we have shown that \( p(x) \) does not change sign on \((\beta_1, \beta_1 + 1)\). Without loss of generality assume that \( p \) is positive on \((\beta_1, \beta_1 + 1)\). Observe that \( \beta_2 \in (\beta_1, \beta_1 + 1) \), and thus by Lemma 2.15, \( p(\beta_2) > \max\{p(\beta_2 + 1), p(\beta_2 - 1)\} \geq p(\beta_2 + 1) \). But this yields \( p(\beta_2 + 1) - p(\beta_2) < 0 \), and therefore \( g(\beta_2) < 0 \) contradicting \( g(\beta_2) = 0 \). \( \square \)

Note that Lemma 2.16 is equivalent to the statement that if \( p(x) \in \mathcal{L:\mathcal{P}}, n \geq 2, \mu(p) \geq 1 \), then the associated functions \( F(x) \) and \( R(x) \) also have zeros spaced at least 1 unit apart. Preliminaries aside, we prove Conjecture 1.4 of I. Krasikov.

**Theorem 2.17.** If \( p(x) \in \mathcal{L:\mathcal{P}} \) and \( \mu(p) \geq 1 \), then

\[ (11) \quad f_n(x, 1, p) = (n - 1)[p(x + 1) - p(x - 1)]^2 - 4np(x)[p(x + 1) - 2p(x) + p(x - 1)] \geq 0 \]

holds for all \( x \in \mathbb{R} \).

**Proof.** Since (11) is true when \( \deg(p(x)) \) is 1 or 2, we assume \( n \geq 2 \). Fix \( x = x_0 \in \mathbb{R} \). If \( p(x_0 - 1) = p(x_0) = p(x_0 + 1) \), or if \( p(x_0) = 0 \), then \( f_n(x, 1, p) \geq 0 \). Thus, we may assume \( p(x_0) \neq 0 \). If \( p(x_0) < \min\{p(x_0 + 1), p(x_0 - 1)\} \), or if \( p(x_0) > \max\{p(x_0 + 1), p(x_0 - 1)\} \), then \( f_n(x_0, 1, p) \geq 0 \) (use (7) and Lemma 2.3).

We next consider the case when

\[ (12) \quad \min\{p(x_0 - 1), p(x_0 + 1)\} < p(x_0) < \max\{p(x_0 - 1), p(x_0 + 1)\} \]
(thus \( x_0 \neq \beta \) or \( \beta + 1 \), where \( p(\beta + 1) = p(\beta) \)), and show
\[
\frac{f_n(x_0, 1, p)}{(p(x_0))^2} = (n - 1)(F(x_0) + R(x_0))^2 - 4n(F(x_0) - R(x_0)) \geq 0,
\]
where \( F(x) \) and \( R(x) \) are defined by (8) and (9) respectively. By Lemma 2.14,
\[
\frac{f_n(x_0, 1, p)}{(p(x_0))^2} = (n - 1)(F(x_0) - R(x_0))^2 - 4n \left( \frac{1}{n} F(x_0) R(x_0) - \frac{p(x_0)^2 - p(x_0 + 1) p(x_0 - 1)}{(p(x_0))^2} \right).
\]

By Lemma 2.16, \( \mu(p(x + 1) - p(x)) \geq 1 \), and thus the zeros \( \{\beta_k\}_{k=1}^n \) of \( F(x) \) \( (p(\beta_k + 1) = p(\beta_k)) \) are spaced at least one unit apart. If \([x_0 - 1, x_0]\) does not contain any \( \beta_k \), \( \frac{f_n(x_0, 1, p)}{(p(x_0))^2} \geq 0 \) holds by Lemma 2.13 (see (13)). If, on the other hand, \( \beta_j \in (x_0 - 1, x_0) \) (recall \( \beta_j \neq x_0, x_0 - 1 \), then \( x_0 \in (\beta_j, \beta_j + 1) \) and by Lemma 2.15 either \( p(x_0) > \max\{p(x_0 - 1), p(x_0 + 1)\} \) or \( p(x_0) < \min\{p(x_0 - 1), p(x_0 + 1)\} \), and both of these cases contradict our assumption (see (12)). We have now shown \( f_n(x_0, 1, p) \geq 0 \) for all \( x_0 \in \mathbb{R} \), except for the isolated points where \( x_0 = \beta_j \) or \( x_0 = \beta_j + 1 \) for some \( j \), but by continuity of \( f_n(x, 1, p) \), (11) will hold.

\[\square\]

The converse of Theorem 2.17 is false in general. Indeed, the following example shows that there are polynomials with arbitrary minimal zero spacing that still satisfy \( f_n(x, 1, p) \geq 0 \) for all \( x \in \mathbb{R} \).

**Example 2.18.** Let \( p(x) = (x + n + a) \prod_{k=1}^{n-1} (x + k) \) with \( n \geq 2, a \in \mathbb{R} \). Using a symbolic manipulator (we used Maple)
\[
f_n(x, 1, p) = C(x, n, a) \prod_{k=2}^{n-2} (x + k)^2
\]
where
\[
C(x, n, a) := (n - 1)(-2n^3 - 4na + 4a^2 + n^2 + n^4)x^2
+ (n - 1)(6n^2a + 4n^4 - 8n^3a + 8a^2 - 12na + 4an^2 - 8n^3 + 2n^4a + 4n^2)x
+ (n - 1)(-8na - 4na^2 + 4a^2 + 4an^2 - 8n^3 + 4n^4 + 4n^2 + 12n^2a
+ n^4a^2 + 13n^2a^2 - 16n^3a - 6n^3a^2).
\]
\( C(x, n, a) \) is quadratic in \( x \) and its discriminant is \( D = -16na^2(n - 1)^2(n - 2)^2(a - n)^2 \leq 0 \). Therefore \( C(x, n, a) \) does not change sign and is always positive (this is verified by showing that the coefficient of \( x^2 \) is positive when considered as a quadratic in \( a \)), whence \( f_n(x, 1, p) \geq 0 \) for all \( x \in \mathbb{R} \).

In general, a polynomial \( p \) may satisfy \( f_n(p, 1, x) \geq 0 \) for all \( x \in \mathbb{R} \), even if \( p \) has multiple zeros. If \( p(x) = x^2(x + 1) \), which has \( \mu(p) = 0 \), then \( f_3(x, 1, p) = 56x^2 + 32x + 8 \) is non-negative for all \( x \in \mathbb{R} \). A polynomial \( p \) with non-real zeros may also satisfy \( f_n(p, 1, x) \geq 0 \) for all \( x \in \mathbb{R} \). For example, let \( p(x) = (x^2 + 1)(x + 1) \), then \( f_3(x, 1, p) = 32x^2 - 32x + 8 \geq 0 \) for all \( x \in \mathbb{R} \).

It is known that a polynomial \( p(x) \in \mathcal{P}_n \) with only real zeros satisfies \( \mu(p) \leq \mu(p') \); that is, \( p'(x) \) will have a minimal zero spacing which is larger than that of \( p(x) \) (N. Obreschkoff [16, p. 13, Satz 5.3], P. Walker [19]). In light of Lemma 2.16, the aforementioned result suggests the following conjecture.
Conjecture 2.19. If \( p(x) \in \mathcal{L} \cdot \mathcal{P}_n \), \( n \geq 2 \), \( \mu(p) \geq d \geq 1 \), and \( g(x) = p(x + 1) - p(x) \), then \( \mu(g) \geq d \).

The derivation of the classical Laguerre inequality relies on properties of the logarithmic derivative of a polynomial. In the same way, Conjecture 1.4 was proved using a discrete version of the logarithmic derivative. The analogy between the discrete and continuous logarithmic derivatives motivates the following conjectures, based on Theorem 2.20 and its converse (B. Muranaka [14]).

**Theorem 2.20.** (P. B. Borwein and T. Erdélyi [1, p. 345]) If \( p \in \mathcal{L} \cdot \mathcal{P}_n \), then

\[
m \left\{ x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda \right\} = \frac{n}{\lambda} \quad \text{for all } \lambda > 0,
\]

where \( m \) denotes Lebesgue measure.

**Conjecture 2.21.** If \( p \in \mathcal{L} \cdot \mathcal{P}_n \), \( n \geq 2 \), \( \mu(p) \geq 1 \), then

\[
m \left\{ x \in \mathbb{R} : \frac{p(x + 1) - p(x)}{p(x)} \geq \lambda \right\} = \frac{n}{\lambda} \quad \text{for all } \lambda > 0,
\]

where \( m \) denotes Lebesgue measure.

**Conjecture 2.22.** If \( p(x) \) is a real polynomial of degree \( n \geq 2 \), and if

\[
m \left\{ x \in \mathbb{R} : \frac{p(x + 1) - p(x)}{p(x)} \geq \lambda \right\} = \frac{n}{\lambda} \quad \text{for all } \lambda > 0,
\]

where \( m \) denotes Lebesgue measure, then \( p \in \mathcal{L} \cdot \mathcal{P}_n \) with \( \mu(p) \geq 1 \).

### 3. Extension to a Class of Transcendental Entire Functions

In analogy with (5) we define, for a real entire function \( \varphi \),

\[
f_\omega(x, h, \varphi) := \varphi(x + h) - \varphi(x - h)]^2 - 4\varphi(x)[\varphi(x + h) - 2\varphi(x) + \varphi(x - h)].
\]

For \( \varphi \in \mathcal{L} \cdot \mathcal{P} \), with zeros \( \{\alpha_i\}_{i=1}^{\infty} \), \( \omega \leq \infty \), we introduce the mesh size

\[
\mu_\omega(\varphi) := \inf_{i \neq j} |\alpha_i - \alpha_j|.
\]

We remark that if \( \psi \notin \mathcal{L} \cdot \mathcal{P} \), then \( \psi \) need not satisfy \( f_\omega(x, h, \psi) \geq 0 \) for all \( x \in \mathbb{R} \). A calculation shows that if \( \psi(x) = e^x \), then \( f_\omega(0, 1, \psi) = -8(e - 1) < 0 \). When \( \varphi \in \mathcal{L} \cdot \mathcal{P}_n \), \( f_\omega(x, h, \varphi) \geq 0 \) for all \( x \in \mathbb{R} \) by Theorem 2.17. In order to extend Theorem 2.17 to transcendental entire functions, we require the following preparatory result to ensure that the approximating polynomials we use will satisfy a zero spacing condition.

**Lemma 3.1.** For any \( a \in \mathbb{R} \), \( n \in \mathbb{N} \), \( n \geq 2 \),

\[
\lim_{n \to \infty} \sum_{k=1}^{\kappa^n} \frac{1}{n \ln(n)(k + n) + a} = 1.
\]

**Proof.** Fix \( a \in \mathbb{R} \). Since the terms \( \frac{1}{n \ln(n)(k + n) + a} \) are decreasing with \( k \) for \( n \) sufficiently large, we obtain

\[
\int_1^{\kappa^n+1} \frac{1}{n \ln(n)(k + n) + a} \, dk \leq \sum_{k=1}^{\kappa^n} \frac{1}{n \ln(n)(k + n) + a} \leq \int_0^{\kappa^n} \frac{1}{n \ln(n)(k + n) + a} \, dk,
\]
for $n$ sufficiently large, by considering the approximating Riemann sums for the integrals. Thus

\[
\frac{1}{n \ln(n)} \ln \left( \frac{n^n + 1 + \frac{a}{n \ln(n)}}{n + 1 + \frac{a}{n \ln(n)}} \right) \leq \sum_{k=1}^{n} \frac{1}{n \ln(n)(k + n)} + a \leq \frac{1}{n \ln(n)} \ln \left( \frac{n^n + \frac{a}{n \ln(n)}}{n + \frac{a}{n \ln(n)}} \right).
\]

As $n \to \infty$, both the left and right sides of (17) approach 1, and whence the sum in the middle approaches 1.

\[\square\]

**Lemma 3.2.** The set of polynomials \( \{q_n(x) = \prod_{k=1}^{p} \left( 1 + \frac{x}{n \ln(n)(k + n)} \right) : n \in \mathbb{N}, n \geq 2 \} \) forms a normal family on \( \mathbb{C} \). There is a subsequence of \( \{q_n(x)\}_{n=2}^{\infty} \) which converges uniformly on compact subsets of \( \mathbb{C} \) to \( e^x \).

**Proof.** Let \( K \subset \mathbb{C} \) be any compact set and let \( R = \sup_{z \in K} |z| \). Recall the inequality

\[
\frac{1}{2} |z| \leq |\ln(1 + z)| \leq \frac{3}{2} |z| \quad \text{for } |z| < \frac{1}{2}
\]

[2, p. 165]. Then for $n > 2R$, \( \left| \frac{z}{n \ln(n)(k + n)} \right| < \frac{1}{2} \), hence, for $k \geq 1$ and $z \in K$

\[
\frac{1}{2} \frac{|z|}{n \ln(n)(k + n)} \leq \left| \ln \left( 1 + \frac{z}{n \ln(n)(k + n)} \right) \right| \leq \frac{3}{2} \frac{|z|}{n \ln(n)(k + n)},
\]

and therefore

\[
\frac{1}{2} \sum_{k=1}^{n} \frac{|z|}{n \ln(n)(k + n)} \leq \sum_{k=1}^{n} \left| \ln \left( 1 + \frac{z}{n \ln(n)(k + n)} \right) \right| \leq \frac{3}{2} \sum_{k=1}^{n} \frac{|z|}{n \ln(n)(k + n)}.
\]

As $n \to \infty$ the sums on the left and right sides of the inequality converge by Lemma 3.1 to \( \frac{1}{2}|z| \) and \( \frac{3}{2}|z| \) respectively. In particular, for some \( \varepsilon > 0 \) and \( N > 2R \) sufficiently large, for all $n \geq N$ and for all $z \in K$,

\[
\sum_{k=1}^{n} \left| \ln \left( 1 + \frac{z}{n \ln(n)(k + n)} \right) \right| \leq \frac{3}{2} R + \varepsilon.
\]

Then for all $n \geq N$, for all $z \in K$,

\[
|q_n(z)| \leq e^{\frac{3}{2} R + \varepsilon} \leq e^{\frac{3}{2} R + \varepsilon}.
\]

So for $n > N$ sufficiently large, the sequence \( \{q_n(z)\}_{n=2}^{\infty} \) is uniformly bounded on compact subsets \( K \subset \mathbb{C} \) and thus form a normal family by Montel’s theorem [2, p. 153]. Thus, there is a subsequence of \( \{q_n(z)\}_{n=2}^{\infty} \) which converges uniformly on compact subsets of \( \mathbb{C} \) to a function $f$, and therefore satisfies

\[
f'(x) = \lim_{n \to \infty} \frac{q_n'(x)}{q_n(x)} = \lim_{n \to \infty} \frac{1}{n \ln(n)(k + n)} = 1,
\]

for a fixed $x \in \mathbb{R}$, where the last equality is by Lemma 3.1. Equation (18) and $f(0) = 1$, imply $f(x) = e^x$ on \( \mathbb{R} \), and thus $f$ is the exponential function. \( \square \)

**Lemma 3.3.** If $\phi(x) = p(x)e^{bx}$, $b \in \mathbb{R}$, $p \in L^p_n$, $n \geq 2$, and $\mu(p) \geq 1$, then $f_{\infty}(x, 1, \phi) \geq 0$ for all $x \in \mathbb{R}$. 

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Proof. By Lemma 3.2, there is a subsequence of \( \left\{ q_j(x) = \prod_{k=1}^{2} \left( 1 + \frac{x}{a_k} \right) \right\}_{j=2}^{\infty} \) call it \( \{ q_{j_n}(x) \}_{n=1}^{\infty} \), such that \( q_{j_n}(x) \to e^x \) uniformly on compact subsets of \( \mathbb{C} \), as \( m \to \infty \). Let \( \{a_k\}_{k=1}^{\infty} \) be the zeros of \( p(x) \), and \( R = \max |a_k| \). The zero of least magnitude of \( q_{j_n}(bx), z_{j_n} \), satisfies \( |z_{j_n}| = \frac{k_{\infty} \ln M + bx}{b}, b \neq 0 \). Both \( \mu(q_{j_n}(bx)) \to \infty \) as \( m \to \infty \) and \( |z_{j_n}| \to \infty \) as \( m \to \infty \). Thus, there is an \( M \) such that for all \( m > M \), \( |z_{j_n}| \geq R + 1 \), and the sequence of polynomials \( h_m(x) = p(x)q_{j_n,\ell}(bx) \), \( m \geq 1 \), is in \( L^{-\mathcal{P}_n} \) for some \( \ell \), and satisfies \( \mu(h_m) \geq 1 \). By Theorem 2.17, \( f_{\gamma}(x, 1, h_m) \geq 0 \) for all \( x \in \mathbb{R} \), for all \( m \). Since \( h_m \to p(x)e^{bx} \) by construction, \( \lim_{m \to \infty} f_{\gamma}(x, 1, h_m) = f_{\gamma}(x, 1, p(x)e^{bx}) \geq 0 \). \( \Box \)

Theorem 3.4. If \( \varphi \in L^{-\mathcal{P}} \) has order \( \rho < 2 \), or if \( \varphi \) is of minimal type of order \( \rho = 2 \), then \( f_{\gamma}(x, 1, \varphi) \geq 0 \) for all \( x \in \mathbb{R} \).

Proof. By the Hadamard factorization theorem, \( \varphi \) has the representation

\[
\varphi(x) = cx^m e^{bx} \prod_{k=1}^{m} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{\gamma}} \quad (\omega \leq \infty),
\]

where \( a_k, c \in \mathbb{R} \), \( m \) is a non-negative integer, \( a_k \neq 0 \), and \( \sum_{k=1}^{m} \frac{1}{a_k} < \infty \). Let

\[
g_n(x) = cx^{n} e^{bx} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right) e^{-\frac{x}{\gamma}}.
\]

Then, \( g_n(x) = ce^{bx - \sum_{k=1}^{n} \frac{x}{a_k}} x^{n} \prod_{k=1}^{n} \left( 1 + \frac{x}{a_k} \right) \) has the form \( p(x)e^{\gamma x}, \gamma \in \mathbb{R}, p \in L^{-\mathcal{P}_n} \), and thus by Lemma 3.3, \( f_{\gamma}(x, 1, g_n) \geq 0 \) for all \( x \in \mathbb{R} \), and for all \( n \). Since we also have \( g_n \to \varphi \) by construction, \( \lim_{n \to \infty} f_{\gamma}(x, 1, g_n) = f_{\gamma}(x, 1, \varphi) \geq 0 \) for all \( x \in \mathbb{R} \). \( \Box \)

In light of Theorem 3.4, we make the following conjecture.

Conjecture 3.5. If \( \varphi \in L^{-\mathcal{P}} \) and \( \mu_{\omega}(\varphi) \geq 1 \) then \( f_{\gamma}(x, 1, \varphi) \geq 0 \) for all \( x \in \mathbb{R} \).

REFERENCES


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