Team Decision Theory and Information Structures in Optimal Control Problems

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**Problem Setup**

- **Team:** A set of decision makers which have a common goal, \( I = \{1, 2, \cdots, 3\} \).

- The member \( i \in I \) receives information \( z_i \) and makes a decision \( u_i = \gamma_i(z_i) \), where \( \gamma_i \in \Gamma_i \).

- A common pay-off: \( J(\gamma_1, \gamma_2, \cdots, \gamma_N) \).

- **Problem 1:** Find \( \gamma^*_i \in \Gamma_i \) for all \( i \) such that \( J \) is minimized.

- Static vs Dynamic Teams.
Let $\xi \in \mathbb{R}^n$ represent uncertainties of the external world, with pdf $\xi \sim \mathcal{N}(0, X)$ known to all members.

The information variable $z_i = H_i \xi + \sum_j D_{ij} u_j$.

Interested in causal systems: $D_{ij} \neq 0 \Rightarrow D_{ji} = 0$.

**Definition 1:** We say $j$ is related to $i$, $jRi$, if $D_{ij} \neq 0$.

**Definition 2:** We say $j$ is precedent of $i$, $j\{i$, iff (a) $jRi$ or (b) there exist distinct $r, s, \cdots, k \in I$ such that $jRr$ and $rRs, \cdots, kRi$.

Precedence diagram to graphical represent the idea of precedence.
Example 1:

\[ D_{ij} = 0, \quad \forall i \text{ and } j \]
\[ z_i = H_i \xi, \quad \forall i \quad (1) \]

- Static or dynamic team?
- Precedence diagram.
- No causal precedence relation among DMs, therefore actual time instants of observation and decisions are not important.
Example 2 (Classical multi-stage stochastic control): A dynamic LTI system (organization)

\[ x_{i+1} = Fx_i + Gu_i + w_i, \quad i = 1, 2, \cdots, N, \]  
\[ y_i = Hx_i + v_i, \quad i = 1, 2, \cdots, N, \]  

where \( x_i \) is the state, \( u_i \) the control, and \( w_i, v_i \) independent sequences.

Perfect memory system—Each DM remembers its own past observations and decisions.

Express information at the \( i - th \) DM in the following form:
\[ z_i = H_i \xi + \sum_j D_{ij} u_j. \]

Draw the precedence diagram with.
Problem P1: The common goal for all members is to minimize

\[ J(\gamma_1, \cdots, \gamma_N) = \mathbb{E}[J] = \mathbb{E}[u^T Qu + u^T S\xi + u^T c], \]

where \( u = [u_1 u_2 \cdots u_N]^T, u_i = \gamma_i(z_i), \gamma_i \in \Gamma_i, i \in \{1, 2, \cdots, N\} \).

Problem P2 (Member-by-member optimality): Find \( \gamma_i^* \in \Gamma_i \) for all \( i \) such that

\[ J(\gamma_1, \cdots, \gamma_i^* - 1, \gamma_i^*, \gamma_i^* + 1, \cdots, \gamma_N^*) \leq J(\gamma_1, \cdots, \gamma_i^* - 1, \gamma_i, \gamma_i^* + 1, \cdots, \gamma_N^*) \]

for all \( \gamma_i \in \Gamma_i \) and for all \( i \).

Problem P3: For fixed \( \gamma_j^*(z_j), j \neq i \), and any \( z_i \)

\[ \min_{u_i} \mathbb{E}[u^T Qu + u^T S\xi + u^T c] =: \min_{u_i} J_i \] \hspace{1cm} (4)
A linear information structure: $z_i = H_i \xi$ for all $i$.

**Theorem** (due to Radner): The optimal control law for each DM is unique and affine in its observation variable. That is of the form, 
$$u_i = \gamma_i(z_i) + b_i.$$
Dynamic Teams with Partially Nested Information Structure

- An information structure is called **partially nested** if \( j \{ i \) implies \( Z_j \subset Z_i \) for all \( i, j \), and \( \gamma \).

- **Theorem 1**: In a dynamic team with partially nested information structure,

\[
    z_i = H_i \xi + \sum_{j \{ i} D_{ij} u_j
\]

is equivalent to an information structure in static form for any fixed set of control laws:

\[
    \hat{z}_i = \{ H_i \xi | j \{ i \text{ or } j = i \} \}.
\]
Example:

\[ z_1 = H_1 \xi \]
\[ z_2 = H_2 \xi \]
\[ z_3 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}' \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{32}' \end{bmatrix} u_2 \]
\[ z_4 = \begin{bmatrix} H_1 \\ H_4' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ D_{41}' \end{bmatrix} u_1 \]
\[ z_5 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \\ H_5' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}' \\ D_{51}' \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{32}' \\ D_{52}' \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u_3 \]
\[ z_6 = H_6 \xi \]
\[ z_7 = \begin{bmatrix} H_6 \\ H_7' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ D_{76}' \end{bmatrix} u_6 \]
\[ z_8 = H_8 \xi. \]
Theorem 2: In a dynamic team with partially nested information structure, the optimal control for each member exists, is unique and linear. How to prove?

Application 1: LQG Control Problem.
Two coupled linear discrete-time dynamic systems controlled by $u_1(t)$ and $u_2(t)$, $t = 1, 2, \cdots, N$

The two controllers observe:

$$z_1(t) = \{y_1(\tau), y_2(k) | \tau = 1, 2, \cdots, t; k = 1, 2, \cdots, t - 1\}$$

$$z_2(t) = \{y_2(\tau), y_1(k) | \tau = 1, 2, \cdots, t; k = 1, 2, \cdots, t - 1\}$$
Further applications: Interconnected/Hierarchical System
Take-home message

In a decentralized decision-making environment, if a DM’s action effects our information, then knowing what he knows will yield linear optimal solutions.

Thank you for your attention!