Consensus Algorithms

Lecture 8
Principles of Wireless Sensor Networks

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Today's Lecture

- Previous lectures
  - the basic theory for distributed computation
  - important aspects of the physical layer, medium access control and routing

- Today we study an application: the consensus
  - Non expansive mappings
  - Agreement algorithm
  - Consensus Algorithm
  - Consensus with packet losses
The Agreement Algorithm

- By the agreement algorithm, a set of nodes try to reach agreement on a common scalar value by exchanging tentative values and summing them by a convex combination.

- The algorithm has applications for the:
  - invariant distribution of Markov chains
  - consensus algorithm
  - distributed estimation
  - synchronization
  - ...

- First, we need to recall the partial asynchronism and introduce non expansive mappings.
Partial asynchronism

Consider the mapping $x := f(x)$

Let $T^i$ be the set of times when $x_i(t)$ is updated.

Let $\tau^i_j(t)$ be the time when the j-th component of $x_i(t)$ is updated.

**Assumption** (Partial asynchronism) There is a positive integer $B$ such that

1. for every $i$ and $t > 0$, at least one of the elements of the set \{t, t - 1, \ldots t - B + 1\} belongs to $T^i$.

2. there holds $t - B < \tau^i_j(t) \leq t$ for all $i$ and $j$ and all $t \geq 0$ belonging to $T^i$.

3. there holds $\tau^i_i(t) = t$ for all $i$ and $t \in T^i$. 
**Definition** Consider the partially asynchronous iteration $x := f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $X^* = \{x \in \mathbb{R}^n | f(x) = x\}$ be the set of fixed points of $f$.

- The set $X^*$ is nonempty.
- The function $f$ is continuous.
- The function $f$ is nonexpansive if
  \[\|f(x) - x^*\|_\infty \leq \|x - x^*\|_\infty, \quad \forall x \in \mathbb{R}^n, \forall x^* \in X^*.\]

Non expansive mappings differs from contractive mappings because the modulus can be 1.
Example: weakly diagonally dominant system of equations

Consider a system $Ax = b$ of linear equations, with $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n}$.

**Assumption**

- $|1 - a_{ii}| + \sum_{j \neq i} |a_{ij}| \leq 1 \quad \forall i$.

- The set $X^*$ of solutions of the equation $Ax = b$ is nonempty.

- The matrix $A$ is irreducible, i.e. $(I + |A|)^n > 0$.

Note that $\|I - A\|_{\infty} \leq 1$

How to compute the solution to $Ax = b$ by a distributed algorithm?
Consider the partially asynchronous iteration

$$x := f(x) = x - \gamma(Ax - b) \quad 0 < \gamma < 1$$

Note that

$$\|I - \gamma A\|_\infty \leq (1 - \gamma) + \gamma\|I - A\|_\infty \leq 1 - \gamma + \gamma = 1$$

It follows that $\|I - \gamma A\|_\infty$ could be 1...

Convergence of the mapping?

**Proposition** The partial asynchronous iteration $x := f(x)$ converges to some $x^*$ such that $Ax^* = b$. 
The Agreement Algorithm

- Consider a set of \( N=\{1,\ldots,n\} \) nodes
- Assume that the i-th node has a scalar \( x_i(0) \) stored initially in its memory
- Each node exchange messages to agree eventually on a value
  \[ \min_{i \in N} x_i(0) \leq y \leq \max_{i \in N} x_i(0) \]
- There is no central coordination in the network
- Node i builds a combination
  \[ x_i := \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n \]
  \[ \sum_{j=1}^{n} a_{ij} = 1 \quad \forall i \]
  \( n \) nodes
The Agreement Algorithm

\[ x_i(t + 1) = x_i(t), \quad t \in T^i \]

\[ x_i(t + 1) = \sum_{j=1}^{n} a_{ij} x_j(\tau_j^i(t)), \quad t \in T^i \]

- In matrix form \( f(x) = Ax \)

- Note that any vector \( x \) whose components are all equal is a fixed point of \( f \) (remember \( \sum_{j=1}^{n} a_{ij} = 1 \forall i \)).

- If \( A \) were irreducible or the diagonal elements of \( A \) were all strictly positive and weakly dominant, then the proposition about weakly diagonally dominant system of equations can be used and convergence of the iteration \( x := f(x) \) is guaranteed.

- However, we can give up to diagonal dominance and convergence can be guaranteed under more general assumptions
Distinguished nodes

**Definition** Let $G = (N, E)$ be a directed graph describing the communication network, where $N = \{1, \ldots, n\}$ and $E = \{(i, j)| i \neq j, a_{ji} \neq 0\}$.

**Assumption** There exists a non empty set $D \subseteq N$ of “distinguished” nodes such that

1. for every $i \in D$, $a_{ii} > 0$.

2. for every $i \in D$ and every $j \in N$, there exists a positive path from $i$ to $j$ in the graph $G$.

- The first assumption ensures that a node does not forget its initial value.
- The second assumption (e.g. clusterheads in a IEEE 802.15.4 network) ensures that a node has influence on the value of any other node in the network.
**Proposition** Consider the agreement algorithm \( x := Ax \) with partial asynchronous iterations. Let \( \alpha > 0 \) be the smallest of the nonzero entries of \( A \). Then, there exists constants \( \eta > 0, C > 0, \rho \in (0, 1) \) that depends only on the number of nodes, on \( \alpha \) and on the maximum delay \( B \), such that for any initial values \( x_i(t), t \leq 0 \), then

1. The sequence \( \{x_i(t)\} \) converges and its limit is the same for each node \( i \).

2. \( \max_i x_i(t) - \min_i x_i(t) \leq C \rho^t (\max_i \max_{-B+1 \leq \tau \leq 0} x_i(t) - \min_i \min_{-B+1 \leq \tau \leq 0} x_i(t)) \)

3. If \( x_i(\tau) \geq 0 \) for every \( i \) and \( \tau \), and if \( k \in D \), then \( y \geq \eta x_k(0) \).
The key parts of the proof are the definition of the functions

\[
M(t) = \max_i \max_{t-B+1 \leq \tau \leq t} x_i(\tau)
\]

\[
m(t) = \min_i \min_{t-B+1 \leq \tau \leq t} x_i(\tau)
\]

Then, it is easy to show that

- \( \forall t \geq 0, m(t+1) \geq m(t) \) and \( M(t+1) \leq M(t) \)
- \( \forall t \geq 0 \) and \( t' \geq t - B + 1 \), \( m(t) \leq x_i(t') \leq M(t) \)
Markov chains are widely used to model and analyze communication protocols (e.g., IEEE 802.15.4, IEEE 802.11,...).

Parallel computation allows one to compute the steady-state probability distribution of a Markov chain.

Let $P$ be the transition probability of a discrete time homogeneous $n$-state Markov chain. Then,

$$P \geq 0$$

$$\sum_{j=1}^{n} p_{ij} = 1.$$

**Proposition** Let $P$ be a stochastic matrix. Then

1. The spectral radius $\rho(P)$ of $P$ is 1

2. Let $\pi \in \mathbb{R}^n$, $\pi \geq 0$. If $1^T \pi = 1$, then $1^T P \pi = 1$. 

The algorithm for the computation of the invariant distribution is
\[ \pi(t + 1) = P\pi(t) \]
or
\[ \pi(t) = P^t\pi(0), \quad t > 0 \]

**Proposition** Let P be a primitive stochastic matrix. Then

1. There exists a unique row vector \( \pi^* \) such that \( \pi^* = P\pi^* \) and \( 1^T\pi^* = 1 \)

2. \( \lim_{t \to \infty} P^t \) exists and is the matrix with all rows equal to \( \pi^* \). If \( 1^T\pi(0) = 1 \), then the iteration \( \pi(t + 1) = P\pi(t) \) converges to \( \pi^* \).
Asynchronous Algorithm for the Invariant Distribution of a Markov Chain

\[
\pi_i(t + 1) = \pi_i(t), \quad t \in \mathcal{T}^i
\]

\[
\pi_i(t + 1) = \sum_{j=1}^{n} a_{ij} \pi_j(\tau_j^i(t)), \quad t \in \mathcal{T}^i
\]

**Proposition** Let \( P \) be a irreducible stochastic matrix. Suppose that there exists \( i^* \) such that \( p_{i^*i^*} > 0 \). Let the asynchronous iteration be initialized with positive values. Then, for any partially asynchronous iteration there exists a positive constant \( c \) such that \( \lim_{t \to \infty} \pi(t) = c\pi^* \). Convergence takes place at the rate of a geometric progression.

- The constant \( c \) is due to the asynchronous iteration.
The consensus is the problem of finding linear iterations that achieve the average of some initial values given at the nodes.

**Definition** Let $G = (N, E)$ be a connected graph that describes a network.

- $N = \{1, \ldots, n\}$ is a set of nodes.
- $E$ is a set of edges, where an edge $\{i, j\} \in E$ in an unordered pair of distinct nodes.
- $N_i = \{j | \{i, j\} \in E\}$.

- Each node has an initial real scalar $x_i(0)$.
- How to compute the average of these initial values by a distributed algorithm, where nodes communicate only with their neighbors?
Consensus via distributed linear iterations

- Consensus via distributed linear iterations
  \[ x_i(t+1) = W_{ii}x_i(t) + \sum_{j \in N_i} W_{ij}x_j(t) \quad i = 1, \ldots, n, \quad t = 0, 1, 2, \ldots \]

- In matrix form
  \[ x(t + 1) = W x(t) \quad W \in \mathcal{S} = \{W \in \mathbb{R}^{n \times n} | W_{ij} = 0 \text{ if } \{i, j\} \ni \mathcal{E} \text{ and } i \neq j \} \]

- How to choose the weight matrix \( W \) so that for any initial value \( x(0) \), \( x(t) \) converges to the average vector
  \[ \bar{x} = \left(1^T x(0) / n\right) 1 = \left(11^T / n\right) x(0) \]
  \[ \lim_{t \to \infty} x(t) = \lim_{t \to \infty} W^t x(0) = \frac{11^T}{n} x(0) \]
Convergence speed measures

- **Asymptotic convergence factor**
  \[ r_{\text{asym}}(W) = \sup_{x(0) \neq \bar{x}} \lim_{t \to \infty} \left( \frac{\|x(t) - \bar{x}\|_2}{\|x(0) - \bar{x}\|_2} \right)^{1/t} \]

- **Convergence time** (asymptotic number of steps for the error to decrease by a factor 1/e)
  \[ \tau_{\text{asym}}(W) = \frac{1}{\log(1/r_{\text{asym}})} \]

- **Per-step convergence factor**
  \[ r_{\text{step}}(W) = \sup_{x(0) \neq \bar{x}} \lim_{t \to \infty} \left( \frac{\|x(t + 1) - \bar{x}\|_2}{\|x(t) - \bar{x}\|_2} \right)^{1/t} \]
Theorem $\lim_{t \to \infty} W^t = 11^T/n$ if and only if

\[
1^T W = 1^T, \\
W1 = 1, \\
\rho(W - 11^T/n) < 1,
\]

where $\rho(\cdot)$ denotes the spectral radius of a matrix, and

\[
\begin{align*}
\rho_{\text{sym}}(W) & = \rho(W - 11^T/n) \\
\rho_{\text{step}}(W) & = \|W - 11^T/n\|_2,
\end{align*}
\]

where $\|\cdot\|_2$ is the spectral norm, or maximum singular value.
The first condition says that $1$ is a left eigenvector of $W$ with eigenvalue $1$, and the sum of the vector of nodes value is preserved.

$$1^T x(t + 1) = 1^T x(t)$$

The second condition says that $1$ is also a right eigenvector of $W$ with eigenvalue $1$, hence $1$ is a fixed point of the iteration $W1 = 1$.

The first three conditions say that $1$ is a simple eigenvalue of $W$, and all others eigenvalues have magnitude less than one.

If the elements of $W$ are nonegative, then the first two conditions say that $W$ is doubly stochastic and the last condition says that the associated Markov chain is primitive.
Some popular weights based on the Laplacian

- There are simple heuristics to choose weights $W$ that give convergence.

- Suppose that the communication graph has $m$ edges labeled from 1 to $m$. Suppose we assign a +1 or -1 direction to the edge.

- The incidence matrix of the graph

$$M \in \mathbb{R}^{n \times m}, M = \begin{cases} 1 & \text{if edge } l \text{ starts from node } i \\ -1 & \text{if edge } l \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

- The Laplacian matrix of the graph is $L = MM^T$. 

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Some popular heuristic weights based on the Laplacian

- **Constant edge weights**
  \[ W = I - \alpha MM^T = I - \alpha L \]
  \[ W_{ij} = \begin{cases} 
    \alpha & \text{if } \{i, j\} \in \mathcal{E} \\
    1 - d_i \alpha & \text{if } i=j \\
    0 & \text{otherwise} 
  \end{cases} \]
  \(d_i\) is degree of node \(i\), i.e., the number of neighbours of node \(i\).

- **Maximum degree weights**
  \[ W_{ij} = \begin{cases} 
    1/n & \text{if } \{i, j\} \in \mathcal{E} \\
    1 - d_i/n & \text{if } i=j \\
    0 & \text{otherwise} 
  \end{cases} \]

- **Metropolis weights**
  \[ W_{ij} = \begin{cases} 
    \frac{1}{1+\max\{d_i,d_j\}} & \text{if } \{i, j\} \in \mathcal{E} \\
    1 - \sum_{\{i,k\}\in \mathcal{E}} W_{ik} & \text{if } i=j \\
    0 & \text{otherwise} 
  \end{cases} \]
Summary

- We studied the agreement algorithm among the nodes of a network and its application to the computation of the
  - invariant distribution of a Markov chain (simple stochastic matrixes are used)
  - consensus for static networks (double stochastic matrixes are used).

- References:
  - D. Bertsekas, J. Tsitsiklis, Parallel and Distributed Computation, 1997
Next Lecture

- We consider another application over WSNs: distributed estimation