How to Select the OOK Detection Threshold in Wireless Ad Hoc and Sensor Networks

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Abstract—On-off keying (OOK) is an attractive modulation technique to reducing energy consumption of power-constrained wireless networks. The OOK detection threshold, however, must be carefully selected to minimize the bit error probability (BER). This is a challenging task to accomplish on resource-limited nodes or on networks with high-mobility. In this paper, an efficient algorithm to compute the optimal threshold is proposed. The system scenario considers nodes simultaneously transmitting over same frequencies in Rayleigh-log-normal or Rice-log-normal fading conditions. It is shown that by using the Stirling expansion for the BER, and a log-normal approximation, a quick contraction-mapping can be build to achieve the threshold numerically. The mapping is simple to implement and converges quickly. Numerical simulations verify the validity of the theoretical analysis, and show that the new algorithm performs quite well in scenarios of practical interest.

Keywords: Wireless Sensor Network (WSNs), On-Off Keying (OOK) modulation, CDMA.

I. INTRODUCTION

Energy efficiency is a fundamental requirement for the correct deployment of wireless sensor networks and power-constrained ad-hoc networks. On-off keying (OOK) modulation has been proposed as an effective technique for these networks [1], [2]. To minimize the BER at the receiver, OOK requires an optimal real-time computation of the detection threshold. However, such an operation is difficult or impossible to perform on resource-limited nodes, because the instantaneous channels attenuation of both the useful signal and the interfering signals must be accurately estimated. The difficulty is exacerbated when nodes have a high mobility, which is a typical situation, for instance, in factory automation. In these environments, networks of nodes are used in severe fading conditions, due to fast moving metal objects or fast movements of nodes themselves. Vehicular applications are also a typical example where nodes or obstacles mobility give short coherence times with the consequence of quick and deep fading fluctuations. In all these situations, the minimization of the average BER is of relevant interest [3].

The challenging task of this optimization is that no-closed form expression of the BER is available for Rayleigh-log-normal or Rice-log-normal wireless channels. In this paper we approximate the average BER by an explicit expression that gives sufficient accuracy. Then, we develop an efficient numerical algorithm that minimizes such a probability. The algorithm runs locally in the nodes, provides them a detection threshold quickly, and requires little computational complexity. We provide a method to minimize the number of iterations of the algorithm, thereby achieving quick convergence. The computational complexity is reduced as well because we show that the detection threshold depends only on the average channel coefficient of the useful signal, and on the average and variance of the sum of all the interfering signals.

The remainder of this paper is as follows. In Section II, the system model is introduced. The average BER is analyzed in Section III. In Section IV, the algorithm for the computation of the detection threshold is proposed. In Section V, the algorithm is applied in some wireless scenarios of practical interest. Numerical results are reported in Section VI. Finally, conclusions and future perspectives are given in Section VII.

II. SYSTEM MODEL

Consider a scenario wherein $K$ sender-receiver pairs of nodes are communicating (see Fig. 1). The bits of sender $i$, for $i = 1, \ldots, K$, are transmitted by an OOK modulation: only bits having value one are processed and transmitted over the wireless channel, while no signal is transmitted when the bit is a zero. Assume that the transmitted bits are processed by a Direct Sequence Code Division Multi Access (DS-CDMA). Since perfect synchronization among distributed nodes is impossible, an asynchronous DS-CDMA is considered. The same fixed bandwidth $W$ is allocated to each transmitter-receiver pair. Let $G = T_b/T_c$ be the processing gain, where $T_b$ is the bit interval, and $T_c$ is the chip interval. Let $h_j(t)$ be the wireless channel coefficient associated to the path from the transmitter of link $j$ to the receiver of link $i$.

The transmitted signal, after being attenuated by the wireless channel, is received corrupted by an additive Gaussian noise and Multiple Access Interference (MAI) caused by the other $K - 1$ transmitting nodes. The output of the coherent correlation receiver of link $i$, for $i = 1, \ldots, K$, is

$$ Z_i(t) = D_i(t) + I_i(t) + N_g(t) , \quad (1) $$

where $D_i(t)$ is the signal bearing the information for the pair $i$, $I_i(t)$ is the interference due to the presence of multiple
transmitting nodes (causing MAI) and \( N_p(t) \) is the AWGN noise, which is modelled as a Gaussian random variable with zero mean and variance \( N_0 T_b / 4 \). It can be shown that

\[
A_i(t) \nu_i(t) \triangleq D_i(t) = \sqrt{\frac{h_{i,i}(t)p_i}{2}} T_b \nu_i(t) , \tag{2}
\]

and that the variance of the MAI plus the AWGN, on a bit time scale and conditioned to the distribution of the high bits and the wireless channel, is

\[
V_i^{-2}(t) \triangleq \frac{T_b^2}{6G} \sum_{j \neq i}^K \nu_j(t) h_{j,i}(t) p_j + \frac{T_b}{4} N_0 . \tag{3}
\]

In (2) and (3), \( p_i, i = 1, \ldots, K \), denotes the radio power of the sender node in link \( i \). Let \( p = [p_1, \ldots, p_i, \ldots, p_K]^T \).

The term \( \nu_j(t) \) is a binary random variable abstracting the transmission of the OOK bit \( (\nu_j(t) = 1) \) or no-transmission \( (\nu_j(t) = 0) \), with probability mass function \( \Pr[\nu_j(t) = 1] = \alpha_j \) and \( \Pr[\nu_j(t) = 0] = 1 - \alpha_j \), respectively. Let \( \nu(t) = [\nu_1(t), \ldots, \nu_K(t)] \). Let \( h_i(t) = [h_{1,i}(t), \ldots, h_{i,i}(t), \ldots, h_{K,i}(t)]^T \) be the channel coefficients seen by the receiver of the pair \( i \).

In the next section, we investigate the BER.

### III. BIT ERROR PROBABILITY

The derivation of the BER distinguishes two cases of error: the decision variable \((1)\) is decoded as a zero bit, when a one bit was transmitted; or \((1)\) is decoded as a one bit when a no bit was transmitted. We denote these probabilities with \( P_{i|0,h_i(t),\nu(t)} \) and \( P_{i|1,h_i(t),\nu(t)} \), respectively, where

\[
P_{i|0,h_i(t),\nu(t)} = \Pr[Z_i(t) \geq \delta_i] P_i(t), h_i(t), \nu(t)] = Q(\delta_i V_i(t)) , \tag{4}
\]

\[
P_{i|1,h_i(t),\nu(t)} = \Pr[Z_i(t) < \delta_i] P_i(t), h_i(t), \nu(t)] = Q((A_i(t) - \delta_i) V_i(t)) . \tag{5}
\]

where \( \delta_i \) is the decision threshold for \( Z_i(t) \), \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-t^2/2) \, dt \) is the complementary standard Gaussian distribution, and \( A_i(t) \) and \( V_i(t) \) are defined in (2) and (3), respectively. The probabilities in (4) and (5) are computed adopting the usual standard Gaussian approximation [4], where \( Z_i(t) \) is modeled by a Gaussian random variable conditioned to the distribution of the channel coefficients and OOK activity. Specifically, it is assumed that \( Z_i(t) \sim \mathcal{N}(\nu_i(t) A_i(t), V_i(t)) \).

The bit error probability, conditioned to the channel coefficients and OOK activity, is

\[
P_{i|h_i(t),\nu(t)}(\delta_i) = \Pr[\nu_i(t) = 0] P_{i|0,h_i(t),\nu(t)}(\delta_i) + \Pr[\nu_i(t) = 1] P_{i|1,h_i(t),\nu(t)}(\delta_i) = (1 - \alpha_i) Q(\delta_i V_i(t)) + \alpha_i Q((A_i(t) - \delta_i) V_i(t)) . \tag{6}
\]

This expression could be minimized with respect to \( \delta_i \). However, each receiver node should be equipped with a channel estimator that provides the vector \( \hat{h}_i(t) \) at each bit time instant (i.e. accurate wireless channel estimation for each interfering node is required). The implementation of these tasks on resource-constrained nodes is prohibitive, for they have reduced computing capabilities. Furthermore, when nodes have a high mobility, or there are quickly moving metal obstacles, the average of the BER with respect to the channel distribution and nodes activity is more meaningful. Therefore, an alternative approach to the instantaneous minimization of the BER is based on taking the average of the BER with respect to the channel coefficient and OOK activity, and then minimizing the resulting expression. With this approach, the optimal threshold depends only on the averages of the MAI, which is simple to compute, as we see below.

By averaging (6) with respect to \( h_i(t) \) and \( \nu(t) \) we obtain

\[
P_i(\delta_i) = (1 - \alpha_i) E\{Q(\delta_i V_i(t))\} + \alpha_i E\{Q((A_i(t) - \delta_i) V_i(t))\} . \tag{7}
\]

The minimization of (7) with respect to \( \delta_i \) is difficult, because the function is non linear and no closed-form is available for the expectations. Hence, we resort to the Stirling approximation proposed by Holtzman [5]. With this goal in mind, let us define the random variables

\[
\zeta_{i0}(\delta_i) = \delta_i V_i(t) , \tag{8}
\]

\[
\zeta_{i1}(\delta_i) = (A_i(t) - \delta_i) V_i(t) . \tag{9}
\]

Then [5], [4]

\[
E\{Q(\zeta_{i0})\} = f_0(\delta_i) \triangleq \frac{2}{3} Q(\mu_{\zeta_{i0}}(\delta_i)) + \frac{1}{6} Q(\mu_{\zeta_{i0}}(\delta_i) + \sqrt{3}\sigma_{\zeta_{i0}}(\delta_i)) + \frac{1}{6} Q(\mu_{\zeta_{i0}}(\delta_i) - \sqrt{3}\sigma_{\zeta_{i0}}(\delta_i)) . \tag{12}
\]

Eq. (12) can be used to compute an approximated expression of the BER (7):

\[
P_i(\delta_i) \approx (1 - \alpha_i) f_0(\delta_i) + \alpha_i f_1(\delta_i) . \tag{13}
\]

We see in the next Section how to select the optimal value of the detection threshold that minimizes (13).

### IV. OPTIMAL DETECTION THRESHOLD

The optimal value of \( \delta_i \), which we denote by \( \delta_i^* \), is given by the minimization of (13). In the following, we restrict ourselves to the case of wireless channel and interference for which \( \delta_i^* \in (0, \mu_{A_i}) \). If \( \delta_i^* \geq \mu_{A_i} \), then \( Q(\zeta_{i0}) \geq 0.5 \) and the BER would be too high no matter how the detection threshold is chosen. Clearly, this situation is of no practical interest.

We have the following result:

**Proposition 1:** Suppose \( \mu_{A_i} > \mu_{V_i} \). If \( \delta_i^* \geq \mu_{A_i} \), then \( Q(\zeta_{i0}) \geq 0.5 \) and the BER would be too high no matter how the detection threshold is chosen. Clearly, this situation is of no practical interest.

We have the following result:
Then, there is a unique minimum of the bit error probability \((7)\), which is attained when
\[
(1 - \alpha_i) \frac{df_0(\delta_i)}{d\delta_i} + \alpha_i \frac{df_1(\delta_i)}{d\delta_i} = 0,  \tag{14}
\]

**Proof:** See Appendix B.

**Remark 1:** The assumptions of the proposition are satisfied in all natural situations of the wireless channel and interference, as we will see in the numerical results section. These assumptions are motivated by the fact that they give important technical conditions for the validity of the proposition. This proposition allows us to develop an efficient algorithm for the computation of the optimal value of the detection threshold, as we see next.

### A. Quick Numerical Computation

From Proposition 1 it follows that the minimum of the bit error probability is given by solving an equation. Unfortunately, the derivative of the functions \(f_0(\delta_i)\) and \(f_1(\delta_i)\) are highly non-linear and a closed-form solution cannot be achieved. Numerical techniques have to be used. Simple algorithms such as the steepest-descent or the bisection [6] can be applied to compute the solution of \((14)\). These algorithms, however, are not optimized for our function at hand. In the following, we propose an algorithm for a quick numerical computation of the optimal detection threshold.

Proposition 1 suggests us to build the following iterative algorithm
\[
\delta_i(k+1) = \delta_i(k) - \beta(k) \left[ (1 - \alpha_i) \frac{df_0(\delta_i)}{d\delta_i} + \alpha_i \frac{df_1(\delta_i)}{d\delta_i} \right],  \tag{15}
\]
where \(k\) is the iteration step, and \(\beta(k)\) is any scalar. When \(\beta(k)\) is such that the mapping is contractive, then it is easy to show that the fixed point of the mapping gives the optimal detection threshold [7, Pag.191]. The reason is that if \(\delta_i(k) > \delta^*_i\), then the difference within the squared parenthesis will be negative, and \(\delta_i(k+1) \leq \delta_i(k)\). By the same argument, if \(\delta_i(k) < \delta^*_i\), then \(\delta_i(k+1) \geq \delta_i(k)\). The important question is how to choose \(\beta(k)\) so that the convergence of the algorithm is quick.

We have the following result:

**Proposition 2:** Let \(d^2 f_0(\delta_i)/d\delta_i^2\) and \(d^2 f_1(\delta_i)/d\delta_i^2\) be given by \((23)\) and \((24)\) in appendix A, respectively. Let
\[
\beta^*(k) = \frac{1}{(1 - \alpha_i) \frac{df_0(\delta_i)}{d\delta_i} + \alpha_i \frac{df_1(\delta_i)}{d\delta_i}}.  \tag{16}
\]
Then the convergence speed of Algorithm \((15)\) is maximized.

**Proof:** See Appendix C.

### V. APPLICATION TO A WIRELESS SCENARIO

Let the wireless channel coefficient be \(h_{ji} = r_{ji}(t)\) \(\exp(\xi_{ji}(t))\). The Nakagami distribution for the fast fading is considered, so \(r_{ji}(t)\) has a gamma distribution having average \(\mu_{r_{ji}}\) and correlation \(\rho_{r_{ji}}\), \([4]\), \(\exp(\xi_{ji}(t))\) is the shadow fading component, with \(\xi_{ji}(t)\) being a Gaussian random variable having zero average and standard deviation \(\sigma_{\xi_{ji}}\). By the Nakagami distribution, we are able to embrace both the Rice and Rayleigh cases. The SINR is then defined as \(A_i^2(t)V_i^{-2}(t)\). Given the fading scenario we are considering, the SINR distribution is unknown. We approximate it in two steps, as described next.

First, we note that the numerator of the SINR is a product of a gamma random variable, \(z_{ii}(t)\), and a log-normal one, \(\exp(\xi_{ii}(t))\). Then, as proposed in \([4, pag. 92]\), this product can be well approximated with a log-normal random variable:
\[
A_i(t)^2 = \frac{T_i^2}{2} p_i l_i r_{ii}(t) \exp(\xi_{ii}(t)) \approx \exp(X_i(t)),  \tag{17}
\]
where \(X_i(t)\) is a Gaussian random variable having average and standard deviation, respectively
\[
\mu_{X_i} = \ln(p_i l_i T_i^2) - \ln 2 + \psi(1),  \tag{18}
\]
\[
\sigma_{X_i}^2 = \zeta(2,1) + \sigma_{\xi_{ii}}^2,  \tag{19}
\]
where \(\psi(1)\) is Euler’s psi function, and \(\zeta(2,1)\) is Riemann’s zeta function. It follows that
\[
\mu_{A_i(t)} = \exp(0.5\mu_{X_i} + 0.25\sigma_{X_i}^2),  \tag{20}
\]
\[
\sigma_{A_i(t)}^2 = \exp(\mu_{X_i} + 0.5\sigma_{X_i}^2).  \tag{21}
\]

Second, the denominator of the SINR is sum of log-normal random variables weighted by gamma and binary random variables. Therefore, following the same approach as in \([8]\), a moment-matching method can be applied and the denominator is well approximated with a log-normal random variable \(V_i^{-2}(t) \approx \exp(Y_i(t))\), where \(Y_i(t)\) is a Gaussian random variable having average and standard deviation obtained by matching the first and second order moments of \(V_i^{-2}(t)\):
\[
\mu_{Y_i} = 2 \ln E[V_i^{-2}(t)] - \frac{1}{2} \ln E[V_i^{-4}(t)],  \tag{22}
\]
\[
\sigma_{Y_i}^2 = \ln E[V_i^{-4}(t)] - 2 \ln E[V_i^{-2}(t)].  \tag{23}
\]

The expressions of \(E[V_i^{-2}(t)]\) and \(E[V_i^{-4}(t)]\) are provided in the appendix. By computing the average and standard deviation of a log-normal random variable from the natural logarithm of the variable, we have:
\[
\mu_{V_i(t)} = \exp(-0.5\mu_{Y_i} + 0.25\sigma_{Y_i}^2),  \tag{24}
\]
\[
\sigma_{V_i(t)}^2 = \exp(-\mu_{Y_i} + 0.5\sigma_{Y_i}^2).  \tag{25}
\]

We can use \((18)–(21)\), in \((8)–(11)\), and then we are able to apply algorithm \((15)\) with \((16)\) to find the optimal detection threshold in the Rayleigh-log-normal faring and Rice-log-normal fading scenarios.

### Table I

**System parameters used for the simulations.**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_{ji}) (j = 1, \ldots, K)</td>
<td>0 dBm</td>
</tr>
<tr>
<td>(\alpha_{ji}) (i = 1, \ldots, K)</td>
<td>0.5</td>
</tr>
<tr>
<td>(l_{ji}) (j = 1, \ldots, K)</td>
<td>-70 dB</td>
</tr>
<tr>
<td>(T_i)</td>
<td>1E-6 s</td>
</tr>
<tr>
<td>(N_0)</td>
<td>-170 dBm/Hz</td>
</tr>
<tr>
<td>(G)</td>
<td>11</td>
</tr>
</tbody>
</table>
δ has been evaluated as a function of the detection threshold (\(\delta\)) varying the number of Tx-Rx pairs and the standard deviation \(\sigma\).

The optimal threshold computed by our algorithm provides a noisy, approximated BER. For example, since the two BER profiles in simulations have minimum values at different detection thresholds, we observe that the actual BER, 1.26E-12 for the approximated BER, which is the optimal detection threshold. Simulations show that the standard deviation for the interference term is known to work better with a high number of interferers.

We conclude that the accuracy of the BER is fair, which is a positive result when considering the approximations we have adopted to develop a computationally affordable detection optimization.

**VI. Numerical Results**

In this section, we present system level Monte Carlo simulations to validate the theoretical analysis and discuss performance of the optimization of the detection threshold. Tab. 1 reports the system parameters used in the simulations.

In Fig. 2, we report an example of the iteration steps needed to our iterative numerical algorithm for reaching convergence. Our algorithm searches for the detection threshold that minimize the value of the approximated BER, and assumes it as the optimal detection threshold. Simulations show that the algorithm actually achieves the minimum value, and that few iterations are needed to converge. In general, less than 10 iterations are enough.

In Fig. 3 we report the BER for a pair of Tx-Rx nodes in the system with \(K = 12\) nodes and \(\sigma_\xi = 0.5\). The BER has been evaluated as a function of the detection threshold (\(\delta\)) through the proposed approximation (13) and by Monte Carlo simulations. The two BER profiles in Fig. 3 have minimum values at different detection thresholds (6.92E-13 mV for the actual BER, 1.26E-12 for the approximated BER), which shows that our algorithm depends on the accuracy of the BER approximation. For example, since the two BER profiles in Fig.3 have minimum values at different detection thresholds, the optimal threshold computed by our algorithm provides a BER of 0.19, while the system would be able to achieve a BER of 0.15. We investigate these differences in Fig.4, which shows the actual minimum of the BER and the ones given by our algorithm for different system configurations, as obtained by varying the number of Tx-Rx pairs and the standard deviation of the shadow fading. We notice that the gap between the approximated BER and the actual one is quite small in general. The accuracy increases for scenarios with a large number of users and low standard deviation. This is due to the adoption of the standard gaussian approximation for the interference term, which is known to work better with a high number of interferers.

We conclude that the accuracy of the BER is fair, which is a positive result when considering the approximations we have adopted to develop a computationally affordable detection optimization.

**VII. Conclusions**

In this paper, we proposed an algorithm for the optimization of the detection threshold for OOK modulation used in wireless sensor networks. The algorithm is based on a simple approximation of the bit error probability, which provides a computationally affordable solution. Monte Carlo simulation assessed the validity of our method and performance of the detection algorithm.

Future work includes an improvement of the bit error rate approximation and the application of the detection algorithm to several cases of wireless propagation scenarios.

**Appendix**

**A. Derivatives**

Here we provide the expressions of the derivatives used in Propositions 1 and 2.

\[
\frac{df_i(\delta_i)}{d\delta_i} = \frac{2}{3}Q'(\mu_{\zeta_{ib}}(\delta_i) + \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) + \frac{1}{6}Q'\left(\mu_{\zeta_{ib}}(\delta_i) + \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) + \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) + \frac{1}{3}\sqrt{3}\sigma'_{\zeta_{ib}}(\delta_i)\right) \\
\times \left[\mu'_{\zeta_{ib}}(\delta_i) + \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) + \frac{1}{6}Q'\left(\mu_{\zeta_{ib}}(\delta_i) - \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) - \frac{1}{\sqrt{2}}\sigma'_{\zeta_{ib}}(\delta_i) - \frac{1}{3}\sqrt{3}\sigma'_{\zeta_{ib}}(\delta_i)\right)\right],
\]

(22)

\[
\mu'_{\zeta_{ib}}(\delta_i) = \mu_{V_i(t)} \\
\sigma'_{\zeta_{ib}}(\delta_i) = \sigma_{V_i(t)}
\]
\[
\sigma''_{\zeta_1}(\delta_i) = \frac{\sigma^2_{V_1(t)}(\delta_i) - \sigma_{\zeta_1}(\delta_i)\left[\sigma^2_{V_1(t)}(\delta_i) - \mu_{A_1(t)}\sigma^2_{V_1(t)}(\delta_i)\right]}{\sigma^2_{\zeta_1}(\delta_i)}. 
\]

Consider the first function, \(Q(\mu_{\zeta_1}(\delta_i))\). For \(\delta_i = \mu_{A_1(t)}\) it follows that \(\mu_{\zeta_1}(\delta_i) = 0\) and \(Q(\mu_{\zeta_1}(\delta_i)) = 0.5\). For \(\delta_i \leq \delta_i\), it follows that \(Q(\mu_{\zeta_1}(\delta_i))\) is a strictly increasing convex function.

The function \(Q(\mu_{\zeta_1} + \sqrt{3}\sigma_{\zeta_1})\) is increasing if its argument is decreasing with \(\delta_i\). Simple algebra gives that the derivative of the argument of this \(Q\) function is negative when
\[
\delta_i \leq \mu_{A_1(t)} + \frac{\mu_{V_1(t)}\sigma_{\zeta_1}(\delta_i)}{\sqrt{3}\sigma^2_{V_1(t)}}. 
\]
This inequality is always satisfied, since \(\delta_i \leq \mu_{A_1(t)}\).

The function \(Q(\mu_{\zeta_1} - \sqrt{3}\sigma_{\zeta_1})\) is increasing if its argument is decreasing with \(\delta_i\). Simple algebra gives that the derivative of the argument of this \(Q\) function is negative when
\[
\delta_i \geq \mu_{A_1(t)} - \frac{\mu_{V_1(t)}\sigma_{\zeta_1}(\delta_i)}{\sqrt{3}\sigma^2_{V_1(t)}}. 
\]
this inequality holds if the right hand-side is negative, which is equivalent to check
\[
\mu_{A_1(t)}\sqrt{3}\sigma^2_{V_1(t)} \geq \mu_{V_1(t)}\sigma_{\zeta_1}(0) \geq \mu_{V_1(t)}\sigma_{\zeta_1}(\delta_i). 
\]
Previous inequality is always satisfied by the assumptions of the Lemma.

By putting together the previous two lemmata, it follows that the BER (7) is given by the sum of a strictly decreasing function plus a strictly increasing function in the region \([0, \mu_{A_1})\). Therefore, there is a unique minimum in \([0, \mu_{A_1})\), which is attained when the first derivative of the BER is zero. This concludes the proof.

C. Proof of Proposition 16

Proposition 1.10 in [7, Pag.193] gives a sufficient condition to establish that Algorithm (15) is a contraction mapping. If
\[
1 > \rho_i(k) = \left| 1 - \beta_i(k) \left[ (1 - \alpha_i) \frac{d^2f_1(\delta_i)}{d^2\delta_i} + \alpha_i \frac{d^2f_0(\delta_i)}{d^2\delta_i} \right] \right|, 
\]
then (15) is contractive. The scalar \(\rho_i(k)\) determines the converge speed of the mapping, so that the lower is \(\rho_i(k)\) the faster is the convergence. \(\rho_i(k)\) is minimized by (16), for which \(\rho_i(k) < 1\). This concludes the proof.

REFERENCES