An Approximation of the Outage Probability in Rayleigh-lognormal Fading Scenarios

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Abstract—In this report we propose an approximation of the outage probability of the Signal to Interference plus Noise Ratio for a CDMA system in a Rayleigh-lognormal fading environment. Specifically, the SINR is approximated as an overall lognormal random variable. The approximation moves from the one proposed in [1] for the product of Rayleigh-lognormal random variables, and from the method presented in [2] for the approximation of the sum of weighted lognormal random variables. Numerical results show that our approach performs quite well for any situation of practical interest.

Index Terms—Outage probability, CDMA, lognormal random variables, Rayleigh random variables.

I. INTRODUCTION

The problem of providing good approximations of the Signal to Noise plus Interference Ratio (SINR) is a classic one in communication theory for many cases of mixed fast and slow fading propagation conditions. Characterization of the SINR is relevant for the computation of the outage probability of wireless communication systems, where such a probability is often used as quality of service for real-time communications [3]. The availability of accurate analytic expression is of paramount importance for system analysis and optimization purposes. A typical example is found in the radio power control problems, where the powers must be minimized under outage probability constraints [4].

While the outage probability is known for the case of Rayleigh fading, up to day, no closed-form expression is known for Rayleigh-lognormal fading scenarios and, in general, for the Rice or Nakagami-lognormal fading. An exact numerical analysis of the outage probability can be found, e.g., in [5]. Numerical approximations, in mixed fading conditions, can be found in [6]. These analysis, however, do not provide closed form expressions, which makes difficult to use them efficiently for optimization purposes.

Owing to the lack of analytical expressions for the statistics of the SINR, many contributions can be found in the literature dealing with useful analytical approximations. Bounds and approximation to the outage probability have been proposed in [7]–[10]. In particular, in [7], the author considers only lognormal fading, and the outage probability is approximated with the well-known Wilkinson approach [1]. In [4], the authors consider the Rayleigh fading and neglect the thermal noise in the SINR. Indeed, as they write, the presence of the thermal noise “...greatly complicates the analysis and resulting optimization problems...”. In [8] the authors extend the approach presented in [4] to the case of lognormal fading. Since the outage probability cannot be computed in closed form, it is relaxed by a Gaussian approximation. In [9], [10], the authors investigate the problem of radio power minimization under outage constraints considering a single type of wireless channel. Specifically, in [9] Rayleigh fading is assumed, and the outage probability is computed as in [4]. The approach in [9] is then extended in [10], where the Rayleigh and Nakagami distributions for the wireless channel are considered (in separate cases). The outage probability is upper bounded (and hence relaxed) through the Jensen’s inequality in the Rayleigh case, and for a particular case of the Nakagami distribution.

However, none of the contributions mentioned above has dealt with an accurate and practical modeling of the wireless channel taking into account both fast and slow fading, and source activity. In this report, we present a lognormal approximation for the SINR in the context of Rayleigh-lognormal fading. Specifically, by observing that the numerator of the SINR can be approximated with a lognormal random variable, as proposed in [1, pag. 92], and considering that the denominator of the SINR can be well approximated by a lognormal random variable [2], an overall lognormal approximation is proposed. We perform extensive Monte Carlo simulations to check the validity of this novel approximation.

We present the system model in Section II. In Section III we describe the approximation of the SINR and corresponding outage probability. Numerical results are presented and discussed in Section IV.

II. PROBLEM FORMULATION

We consider a system scenario where $K$ mobile users transmit to a Base Station (BS). Each user $i=1,\ldots,K$ is associated to a traffic source type (voice, video, data, etc.), employs the same chip time $T_c$, and transmits with power level $P_i$. The channel gain experienced by the signals of user $i$ is denoted with $h_i$. We adopt the Lee and Yeh multiplicative model [1, pag. 91]: $h_i = l_i z_i \Omega_i$, where $l_i$ is the path loss, $z_i$ is the power of the fast fading, and $\Omega_i$ is the power of the shadowing. We consider the Rayleigh distribution for the fast
fading [1], so \( z_i \) has a Gamma distribution. For any \( z_i \) and \( z_j, i \neq j \), are statistically independent. The shadow fading is \( \Omega_i = \exp (\xi_i) \), where \( \xi_i \) is a Gaussian random variable having zero average and standard deviation \( \sigma_{\xi_i} \). Let the binary random variable \( \nu_i \) be the activity status at the source. The probability mass function is such that \( \Pr[\nu_i = 1] = \alpha_i \) and \( \Pr[\nu_i = 0] = 1 - \alpha_i \), where \( \alpha_i \) is the activity factor of source \( i \).

Let \( h = [h_1, \ldots, h_K]^T \) and \( \nu = [\nu_1, \ldots, \nu_K]^T \). Independence is assumed between any pair of processes of the vectors \( h \) and \( \nu \). The model of the physical layer for the up-link of a single-cell asynchronous binary phase shift keying DS/CDMA system is summarized by the following expression of the SINR
\[
\text{SINR}_i(h, \nu) = \frac{P_i h_i}{\frac{N_0}{2T_s} + \Psi_i(h, \nu)},
\]
where
\[
\Psi_i(h, \nu) = \sum_{j=1}^{K} \frac{P_j h_j \nu_j}{G_i}
\]

We model the rate of each user as \( R_i = R_{00} n_i \), where \( R_{00} = 1/T_{00} \) is the basic rate of user \( i \), with basic bit time \( T_{00} \), and \( n_i \) is an integer denoting the assigned rate. The rate is a power of two due to the spreading code structure [1]. Consequently, the spreading factors are expressed as \( G_i = G_{00} n_i \), where \( G_{00} = T_{00}/T_c \) corresponds to the basic rate \( R_{00} \). We assume that the users’ power is expressed as \( p_i = p_i n_i \), where \( p_i \) is the power at the basic rate \( R_{00} \) (i.e., when \( n_i = 1 \)). We use the notation \( p = [p_1, \ldots, p_K]^T \) and \( p^{-i} = [p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_K]^T \), \( n = [n_1, n_2, \ldots, n_K]^T \), and \( n^{-i} = [n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_K]^T \).

The outage probability is defined as
\[
\Pr[\text{SINR}_i(h, \nu) < \gamma_i] \quad \forall i = 1, \ldots, K \quad (3)
\]
where the outage probability of the SINR for user \( i \) is defined with respect to the threshold \( \gamma_i \).

III. SINR APPROXIMATION

The probability in (3) is expressed by the probability distribution function (pdf) of the SINR. However, the expression of the SINR pdf is in general unknown. The uncertainty concerns the statistics of the MAI, which are a mixture of the on-off activity of the sources and the fadings. Consequently, they have to be approximated. We adopt a log-normal approximation of the SINR pdf.

Let us rewrite the SINR in (1) as follows:
\[
\text{SINR}_i(h, \nu) = \frac{z_i}{L_i(h, \nu)},
\]
where
\[
L_i(h, \nu) = \frac{N_0}{2p_i G_{00} T_c} l_i^{-1} \exp (-\xi_i) + \sum_{j=1}^{K} \frac{p_j n_j}{G_{00} p_i} l_j^{-1} j_j \nu_j \exp (\xi_j - \xi_i).
\]

We resort to an accurate approximation of the SINR in two steps. First, we note that (5) is a combination of log-normal random variables, weighted by one-sided random variables. Thus, we can use the extended Wilkinson Moment matching method [11, 13, 14] to model (5) with a log-normal random variable, so that \( L_i(h, \nu) \approx \exp (-X_i) \), where \( X_i \) is a Gaussian random variable with average \( \mu_{X_i}(n_{-i}, p) \) and standard deviation \( \sigma_{X_i}(n_{-i}, p) \) we derive in the following. Notice that this method provides a tight approximation. The resulting SINR is given by the product of a gamma random variable, \( z_i \), times a log-normal one, \( \exp (-X_i) \). Then, this product can be well approximated with an overall log-normal random variable, as proposed in [1, pag. 92]. In summary, we have that
\[
\text{SINR}_i(h, \nu) \approx \exp (Y_i),
\]
where \( Y_i \) is a Gaussian random variable with average and standard deviation, respectively, as
\[
\mu_{Y_i}(n_{-i}, p) = \psi(1) - \mu_{X_i}(n_{-i}, p),
\]
\[
\sigma_{Y_i}^2(n_{-i}, p) = \zeta(2, 1) + \sigma_{X_i}^2(n_{-i}, p)
\]
where \( \psi(1) \) is Euler’s psi function, and \( \zeta(2, 1) \) is Riemann’s zeta function, as defined in [1, pag.107]. The extended Wilkinson moment matching approximation in [2] computes the average and variance of \( X_i \) as
\[
\mu_{X_i}(n_{-i}, p) = 2 \ln M_i^{(1)}(n_{-i}, p) - \frac{1}{2} \ln M_i^{(2)}(n_{-i}, p)
\]
\[
\sigma_{X_i}^2(n_{-i}, p) = \ln M_i^{(2)}(n_{-i}, p) - 2 \ln M_i^{(1)}(n_{-i}, p),
\]
where
\[
M_i^{(1)}(n_{-i}, p) \triangleq \mathbb{E}_{h, \nu} \{L_i(h, \nu)\},
\]
\[
M_i^{(2)}(n_{-i}, p) \triangleq \mathbb{E}_{h, \nu} \{L_i^2(h, \nu)\},
\]
where we have denoted with \( \mathbb{E}_{h, \nu} \{ \cdot \} \) the expectation w.r.t. the distribution of \( h \) and \( \nu \). The expressions of (7) and (8) can be derived applying the statistical expectation operator to (5), recalling its linear properties, and that the random vectors \( h \) and \( \nu \) are independent:
\[
M_i^{(1)}(n_{-i}, p) = \frac{N_0}{2p_i G_{00} T_c} \sum_{j=1}^{K} p_j n_j l_j^{-1} \mu_{L_i} \exp \left( \frac{1}{2} \sigma_{L_i}^2 \right) + \sum_{j=1}^{K} \frac{p_j n_j}{G_{00} p_i} l_j^{-1} j_j \nu_j \exp \left( \frac{1}{2} \sigma_{L_i}^2 \right).
\]
In previous expressions, we have denoted with $\mu_{z_j}$ and $\rho_{z_j}$ the expectation and the correlation of $z_j$, respectively, which easily follow from the Nakagami distribution of $z_j$.

With the approximation (6), the SINR in logarithmic units is a Gaussian random variable. Hence the derivation of the standard Gaussian distribution.

\[
M_i^{(2)}(n_{-i}, \mathbf{p}) = \sum_{j=1}^{K} \sum_{k=1 \atop k \neq i}^{K} \frac{p_j p_k n_j n_k}{G_{ij}^0 p_i^2} I_i^{-2} I_j I_{k} \mu_{z_j} \mu_{z_k} \alpha_j \alpha_k \exp \left(2\sigma_{z_j}^2 + \frac{1}{2} \sigma_{\xi_j}^2 \right) + \sum_{j=1 \atop j \neq i}^{K} \frac{p_j n_j}{G_{ij}^0 p_i} I_i^{-2} I_j \rho_{z_j} \alpha_j \exp \left(2\sigma_{z_j}^2 + 2\sigma_{\xi_j}^2 \right) + \sum_{j=1 \atop j \neq i}^{K} \frac{N_0 p_j n_j}{G_{ij}^0 p_i^2 T_c} I_i^{-2} I_j \mu_{z_j} \alpha_j \exp \left(2\sigma_{z_j}^2 + \frac{1}{2} \sigma_{\xi_j}^2 \right) + \frac{N_0^2}{4p_i^2 G_{ij}^0 T_c^2} I_i^{-2} \exp \left(2\sigma_{\xi_j}^2 \right)
\]

In all the cases considered is the one typically adopted for the definition of outage events [1].

In Fig. 2 the ccdf percentage error of the outage approximation in a scenario with $K = 8$ users is reported, whereas in Fig. 3 a scenario with $\sigma_{\xi_j} = 0.5$, $i = 1, \ldots, K$ and different number of users ($K$) is considered. The figures are associated to a generic user of the $K$ ones present in the system. The figures have been obtained setting the same activity factor $\alpha_i = 0.5$ and a homogeneous propagation environment for the path loss $l_i$, which is set to 90 dB for each user. As it can be observed, the approximation achieves a good accuracy for any value of the SINR threshold $\gamma$, with an error that is below 1.8% in all the cases. The error shows a higher sensitivity with respect to the number of users. Even in the case of $K = 12$, however, the error is still below 1.8%. These results are general, and it can be shown that the conclusions drawn for Fig. 2 and 3 apply to all other users in the system.

In Fig. 4 the ccdf percentage error of the outage approximation for a scenario with $K = 8$ users is reported for the case of mixed activity factor. Specifically, $\alpha_i$, $i = 1, \ldots, K$, is randomly selected in $[0.2, 0.7]$. The figure is related to one of the users. A good accuracy of the approximation is achieved. It can be shown that the same conclusion applies to all other users of the system.

In Fig. 5 the ccdf percentage error of the outage approximation for a scenario with $K = 8$ users is reported for the case of mixed path loss. Specifically, $l_i$, $i = 1, \ldots, K$, is randomly selected in $[-110, -90]$ dB. The figure is related to one of the users. Once again, a good accuracy of the approximation can be recognized clearly, and it can be shown that the same conclusion applies to all other users of the system.

V. Conclusion

In this report we investigated an approximation of the outage probability of the Signal to Interference + Noise Ratio of CDMA systems. Numerical results show that our approximation performs well, with errors less than 1.8% in all the cases of practical interest.
Fig. 2. Complementary cumulative distribution function percentage error of the outage approximation for a scenario with $K = 8$ users. Each curve refers to a different standard deviation of the shadowing ($\sigma_{\xi_i}$). The curves are associated to a generic user.

Fig. 3. Complementary cumulative distribution function percentage error of the outage approximation for a scenario with $\sigma_{\xi_i} = 0.5$, $i = 1, \ldots, K$. Each curve refers to a system with a different number of users ($K$). The curves are associated to a generic user.

Fig. 4. Complementary cumulative distribution function percentage error of the outage approximation for a scenario with $K = 8$ users. Each user transmits with a different activity factor $\alpha_i$, $i = 1, \ldots, K$, uniformly selected in $[0.2, 0.7]$. The curve is associated to a generic user.

Fig. 5. Complementary cumulative distribution function percentage error of the outage approximation for a scenario with $K = 8$ users. Each user experiences a different path loss $l_i$, $i = 1, \ldots, K$, uniformly selected in $[-110, -90]$ dB. The curve is associated to a generic user.

REFERENCES


