Non-evasiveness, collapsibility and explicit knotted triangulations
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Collapsibility is a combinatorial strengthening of the topological notion of contractibility. In the Sixties, Bing and his student Goodrick proved with knot-theoretic techniques that not all triangulated 3-balls are collapsible [4, 7]. Building on a construction by Lutz [10], we announce the finding of a first explicit example:

**Theorem 1.** There exists a non-collapsible simplicial 3-ball $B_{15,66}$ with 15 vertices and 66 tetrahedra.

Non-evasiveness is a further strengthening of collapsibility, emerged in theoretical computer science and later studied by Kahn, Saks and Sturtevant [9] and Welker [11]. A 0-dimensional complex is non-evasive if and only if it consists of a single point. Recursively, a $d$-dimensional simplicial complex ($d > 0$) is non-evasive if and only if there is some vertex $v$ whose link and deletion are both non-evasive.

Every non-evasive complex is collapsible. The converse is false: Collapsibility is not maintained under taking links. In fact, there are elementary examples of collapsible 2-complexes all of whose vertex links are non-contractible. A first such example with only six vertices was found by Björner; for another example, see Barmak–Minian [1, Figure 7]. However, the difference between collapsibility and non-evasiveness does not simply depend on vertex links. In fact, we show that even a manifold can be collapsible and evasive:

**Theorem 2.** There exists a collapsible and evasive simplicial 3-ball $B_{12,38}$ with 12 vertices and 38 tetrahedra.

This triangulation $B_{12,38}$ is obtained via knot theory: It contains a trefoil knot in its 1-skeleton, realized with one interior edge plus four boundary edges. Note that the link of every boundary vertex of a 3-ball is a 2-ball and hence non-evasive.

For $d$-manifolds, there exists also a combinatorial strengthening of the topological notion of simply-connectedness, known as local constructibility. This property was introduced by Durhuus and Jonsson [5] and later studied by the speaker and Ziegler [3]. A 3-sphere is locally constructible if and only if it can be obtained from a “tree of tetrahedra” (i.e. a simplicial 3-ball whose dual graph is a tree) by repeatedly identifying two adjacent boundary triangles. With knot-theoretic arguments one can show that not all 3-spheres are locally constructible [3]. More precisely, a 3-sphere is locally constructible if and only if the removal of any tetrahedron would turn it into a collapsible ball [3 Corollary 2.11]. Here we present a first explicit example:
Theorem 3. There exists a non-locally-constructible simplicial 3-sphere $S_{18,125}$ with 18 vertices and 125 tetrahedra.

In the language of discrete Morse theory, $B_{15,66}$ and $S_{18,125}$ are triangulations on which no discrete Morse function is sharp in bounding the Betti numbers from above. We believe that these first, explicit examples at the level of manifolds can help in developing and testing algorithms to find “good” Morse matchings. (Compare Engström [6] and Joswig–Pfetsch [8].)

Details will appear in [2].

References