CORRIGENDUM TO 'TIME-INCONSISTENT MEAN-FIELD OPTIMAL STOPPING: A LIMIT APPROACH'

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A CORRECTION OF THE PROOF OF THEOREM 3.3

We provide a correction of proof of Theorem 3.3 in [DM23]. This result plays a key role in the proof of the main result of [DM23], Theorem 3.4. In the proof of Theorem 3.3, we used a well known law of large numbers whose statement in Eq. (3.24) is unfortunately wrong.

First, we recall that the family of interacting Snell envelopes $\{Y^{i,n}\}_{i=1}^n$ and the family of finite horizon stopping problems $\{Y^i\}_{i\geq 1}$ are defined by (2.4) and (2.5) in [DM23]. The function h and sequence $\{\xi^i\}_{i\geq 1}$ satisfy Assumption 2.1 in [DM23] (in what follows we will simply say that Assumption 2.1 holds). For the new proof, we need to introduce a (fixed) sequence of random variables $\{\alpha_j\}_{j\geq 1}$ which are all independent of $\{\mathbb{F}^i\}_{i\geq 1}$ and for which

$$\mathbb{E}[\alpha_j] = 1 - 2^{-j}, \;\; \mathrm{Var}(\alpha_j) \leq a^j, \;\; j \geq 1, \quad |\mathbb{E}[\alpha_j\alpha_k]| \leq a^{|j-k|}, \quad j,k \geq 1, \tag{1.1}$$

for a given $a \in (0,1)$. We note that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n (\mathbb{E}[\alpha_j]-1)^2, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \mathrm{Var}(\alpha_j) = 0, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n a^j = 0. \tag{1.2}$$

The next lemma is the main ingredient in the new proof of Theorem 3.3.

Lemma 1.1. *Let Assumption 2.1 hold. Then the following law of large numbers (LLN) holds*

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n}\sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] = 0. \tag{1.3}$$

Moreover,

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\tfrac{1}{n}\textstyle\sum_{j=1}^n (1-\alpha_j) Y_\tau^j \right)^2 \right] = 0. \tag{1.4}$$

Proof. Due to (1.1), the limit (1.4) is straightforward. To show (1.3), we note that since

$$\frac{1}{n}\sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) = \frac{1}{n}(\alpha_i Y_\tau^i - \mathbb{E}[\alpha_i Y_\tau^i]) + \frac{n-1}{n}\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]),$$

by Dominated Convergence it suffices to show

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] = 0. \tag{1.5}$$

By the properties of the essential supremum, for each $n \geq 2$, there exists a sequence $\{\tau_m^n\}_{m\geq 1}$ from $\mathcal{T}_0^i$ such that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 = \lim_{m\to\infty} \left(\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_{\tau_m^n}^j - \mathbb{E}[\alpha_j Y_{\tau_m^n}^j]) \right)^2 \quad \text{a.s.}$$

and by Dominated Convergence, we have

$$\begin{aligned} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right] &= \lim_{m\to\infty} \mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n (\alpha_j Y_{\tau_m^n}^j - \mathbb{E}[\alpha_j Y_{\tau_m^n}^j]) \right)^2 \right] \\ &\leq \sup_{\tau\in\mathcal{T}_0^i} \mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) \right)^2 \right]. \end{aligned}$$

Now, by direct calculations it holds that, for every $\tau \in \mathcal{T}_0^i$ and every $\ell, j \neq i$,

$$\mathbb{E}[Y_\tau^\ell] = \mathbb{E}[Y_\tau^j] = \mathbb{E}[\mathbb{E}[Y_s^1]\big|_{s=\tau}], \quad \mathbb{E}[Y_\tau^\ell Y_\tau^j] = \mathbb{E}[(\mathbb{E}[Y_s^1])^2\big|_{s=\tau}] \geq 0,$$
$$\mathrm{cov}(\alpha_j Y_\tau^j, \alpha_k Y_\tau^k) = \mathrm{cov}(\alpha_j, \alpha_k)\mathbb{E}[Y_\tau^j Y_\tau^k] + \mathbb{E}[\alpha_j]\mathbb{E}[\alpha_k]\mathrm{cov}(Y_\tau^j, Y_\tau^k).$$

Thus, the independence of (α_j, α_k) from (Y_τ^j, Y_τ^k) entails

$$\begin{aligned}\mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j=1, j\neq i}^n(\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right] &= \tfrac{1}{(n-1)^2}\textstyle\sum_{j,k,j,k\neq i}^n \mathrm{cov}(\alpha_j Y_\tau^j, \alpha_k Y_\tau^k)\\ &\leq \left(\tfrac{1}{n-1} + \tfrac{n^2}{(n-1)^2}\tfrac{2}{n}\textstyle\sum_{m=1}^n a^m\right)\mathbb{E}[\sup_{t\in[0,T]}|Y_t^1|^2].\end{aligned}$$

This bound, being uniform in τ and in $i, 1 \leq i \leq n$, yields (1.3) due to (1.2). □

We will now state a new and correct version of [DM23, Theorem 3.3] and sketch its proof. To this end we need to substitute the smallness condition $\gamma_1^2 + \gamma_2^2 < \frac{1}{16}$ with a new one.

Theorem 1.2. *Let Assumptions 2.1 hold and let us assume that γ_1 and γ_2 satisfy the new condition*

$$\gamma_1^2 + 3\gamma_2^2 < \frac{1}{28}. \tag{1.6}$$

Then, it holds that

$$\lim_{n\to\infty}\sup_{1\leq i\leq n}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{i,n} - Y_t^i|^2\right] = 0. \tag{1.7}$$

Proof. As in the proof of [DM23, Theorem 3.3], we have that for any $t \leq T$,

$$\begin{aligned}|Y_t^{i,n} - Y_t^i| \leq \mathbb{E}\Big[&\gamma_1 \sup\nolimits_{s\in[0,T]}|Y_s^{i,n} - Y_s^i| + \tfrac{\gamma_2}{n}\textstyle\sum_{j=1}^n \sup_{s\in[0,T]}|Y_s^{j,n} - Y_s^j|\\ &+\tfrac{\gamma_2}{n}\textstyle\sum_{j=1}^n \mathbb{E}[|\alpha_j|]\mathbb{E}[\sup_{s\in[0,T]}|Y_s^{j,n} - Y_s^j|] + \tfrac{\gamma_2}{n}\sum_{j=1}^n \mathbb{E}[|\alpha_j|]\mathbb{E}[\sup_{s\in[0,T]}|Y_s^{j,n} - Y_s^i|]\,|\,\mathcal{F}_t^i\Big]\\ &+\gamma_2\mathbb{E}[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}|\tfrac{1}{n}\textstyle\sum_{j=1}^n(\alpha_j - 1)Y_\tau^j|\,|\,\mathcal{F}_t^i] + \gamma_2\mathbb{E}[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}|\tfrac{1}{n}\sum_{j=1}^n(\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])|\,|\,\mathcal{F}_t^i]\\ &+\gamma_2|\tfrac{1}{n}\textstyle\sum_{j=1}^n(\mathbb{E}[\alpha_j] - 1)|\mathbb{E}[\sup_{s\in[0,T]}|Y_t^1|].\end{aligned} \tag{1.8}$$

Set

$$\begin{aligned}\Lambda_n := {}& 28\gamma_2^2 \sup_{1\leq i\leq n}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n(\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right]\\ &+28\gamma_2^2\sup_{1\leq i\leq n}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n(\alpha_j - 1)Y_\tau^j\right)^2\right] + 28\gamma_2^2\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n(\mathbb{E}[\alpha_j] - 1)\right|^2\mathbb{E}\left[\sup_{s\in[0,T]}|Y_t^1|\right]^2.\end{aligned}$$

Since $\mathbb{E}[\alpha_i^2] \leq 1$, in view of the exchangeability of $\{Y^{j,n}, Y^j\}_{j=1}^n$, the Cauchy-Schwarz inequality and Doob's inequality, if we set $C := (1 - 28(\gamma_1^2 + 3\gamma_2^2))^{-1}$, by (1.1) and Lemma 1.1 we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{i,n} - Y_t^i|^2] \leq C\lim_{n\to\infty}\Lambda_n = 0. \tag{1.9}$$

But, again due to the exchangeability of $\{Y^{j,n}, Y^j\}_{j=1}^n$, it holds that

$$\sup_{1\leq i\leq n}\frac{1}{n}\sum_{j=1}^n \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{j,n} - Y_t^i|^2] = \frac{1}{n}\sum_{j=1}^n \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{j,n} - Y_t^j|^2] \to 0, \quad \text{as } n\to\infty, \tag{1.10}$$

where the limit follows from (1.9). Now, from (1.8), we get

$$\begin{aligned}\tfrac{1}{28\gamma_2^2}(1 - 28\gamma_1^2)\sup_{1\leq i\leq n}\mathbb{E}[\sup\nolimits_{t\in[0,T]}|Y_t^{i,n} - Y_t^i|^2] &\leq \sup_{1\leq i\leq n}\tfrac{1}{n}\textstyle\sum_{j=1}^n \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{j,n} - Y_t^i|^2]\\ &\quad+2\tfrac{1}{n}\textstyle\sum_{j=1}^n \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{j,n} - Y_t^j|^2] + \tfrac{1}{28\gamma_2^2}\Lambda_n.\end{aligned} \tag{1.11}$$

Finally, since (1.6) entails $28\gamma_1^2 < 1$ and in view of (1.3), (1.9) and (1.10), we obtain

$$\lim_{n\to\infty} \sup_{1\le i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2\right] = 0. \tag{1.12}$$

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References

[DM23] B. Djehiche and M. Martini, *Time-inconsistent mean-field optimal stopping: A limit approach*, Journal of Mathematical Analysis and Applications **528** (2023), no. 1, 127582.

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