Assignment 1

This assignment is on Friday April 26 in class.

1. Suppose X and Y are independent Gaussian random variables with zero mean and unit variance. Derive the probability density functions for $\min(X, Y)$ and $\max(X, Y)$.

2. Suppose a random variable U is uniformly distributed on the interval [0,1]. Derive the density function of the random variables

   - $Y = -\ln(1 - U)$
   - $Y = U^n$ for $n \geq 1$.

3. Simulate in Matlab the realization of 6 state Markov chain using the acceptance rejection method. You should pick any aperiodic irreducible transition probability matrix where all the 36 elements are different. Give your Matlab code – we **do not** require a printout of the sample path of the Markov chain.

4. Given the above Markov chain realization, devise a method for estimating the transition probabilities of the Markov chain. Also devise a method for estimating the stationary distribution of the Markov chain.

   Verify the correctness of your estimator, by directly computing the stationary distribution from the transition probability matrix.

5. Suppose we wish to simulate a (non iid) Markov chain whose stationary probability vector is $\mu = [0.3\ 0.2\ 0.1\ 0.22\ 0.18]'$. Devise an algorithm for doing so.

6. Suppose a 2 state Markov chain with states -1 and 1 and transition probability matrix
$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$
   is input to a FIR channel with transfer function
$$H(z) = 1 + 2z^{-1} + 3z^{-2}$$
   Simulate the system and plot the output of the channel. Show that the output of the channel is a vector valued Markov chain. Compute its transition probabilities.

7. Use the inverse transform method to generate a random variable X with density $f_X(x) = 2e^{-2x}$.
   Suppose now that we wish to generate a random variable with density $f_X(x) = 3e^{-3x}$ from the random variable X generated above, without conducting an additional simulation. Explain a procedure for doing so. More generally suppose you have generated via simulation a rv $X \sim \theta e^{-\theta x}$. Without doing an additional simulation, how would you generate a random variable from the density $(\theta + \Delta\theta)e^{-(\theta+\Delta\theta)x}$? Use this to show how given the random sample X you can also compute the derivative $dX/d\theta$ without doing any additional simulation.

8. Use the composition method to generate rvs with the following distribution functions:

$$(i) \qquad F(x) = \frac{x + x^3 + x^5}{3} \qquad 0 < x < 1$$

$$(ii) \qquad F(x) = \sum_{i=1}^{n} \alpha_i x^i. \qquad 0 < x < 1, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$$

   Simulate the above in Matlab. Then using the `hist` command (suitably scaled) plot the empirical density function. Compare with the actual distribution.

9. Give an algorithm to generate a random variable with distribution function

$$F(x) = \int_0^\infty x^y e^{-y} dy \quad 0 \leq x \leq 1.$$

10. **Advanced question** Farkas lemma states: Let M be an $m \times n$ matrix and b an m-dimensional vector. Then only one of the following statements is true:

    (a) There exists a vector $x \in \mathbb{R}^n$ such that $Mx = b$ and $x \geq 0$.

    (b) There exists a vector $y \in \mathbb{R}^m$ such that $M'y \geq 0$ and $b'y < 0$.

    Here $x \geq 0$ means that all components of the vector x are non-negative.

    Use Farkas lemma to prove that every transition matrix P has a stationary distribution. That is, for any $X \times X$ stochastic matrix P, there exists a probability vector π such that $P'\pi = \pi$. (Recall a probability vector π satisfies $\pi(i) \geq 0$, $\sum_i \pi(i) = 1$).

Hint: Write alternative (a) of Farkas lemma as

$$\begin{bmatrix}(P-I)' \\ \mathbf{1}'\end{bmatrix}\pi = \begin{bmatrix}0_X \\ 1\end{bmatrix}, \quad \pi > 0$$

Show that this has a solution by demonstrating that alternative (b) does not have a solution.