Online Edge Coloring is (Nearly) as Easy as Offline

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TTIC online seminar

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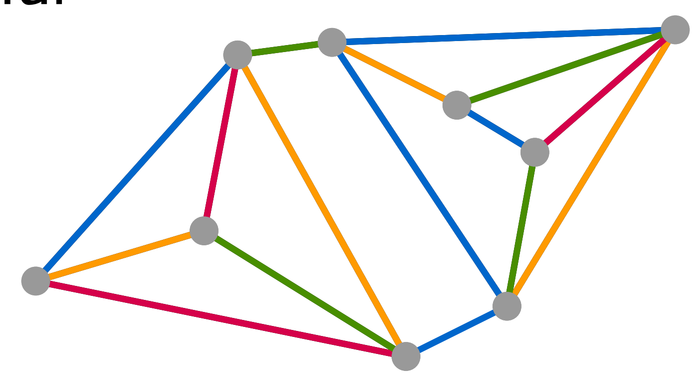
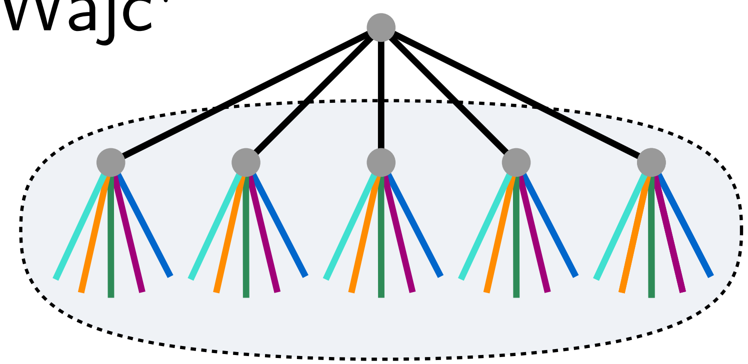
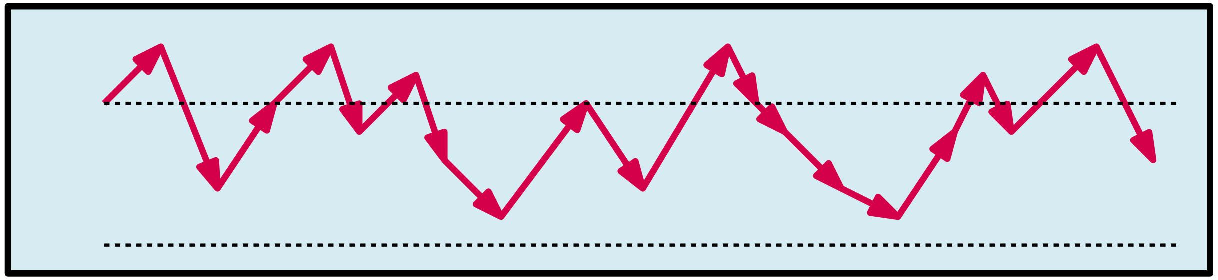
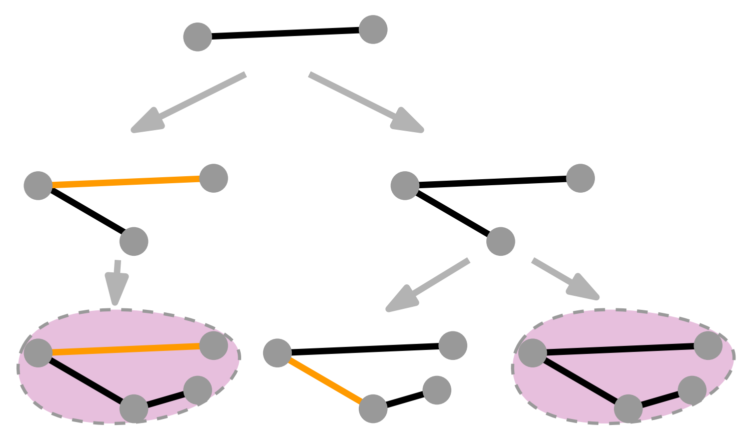
‡Technion, Israel









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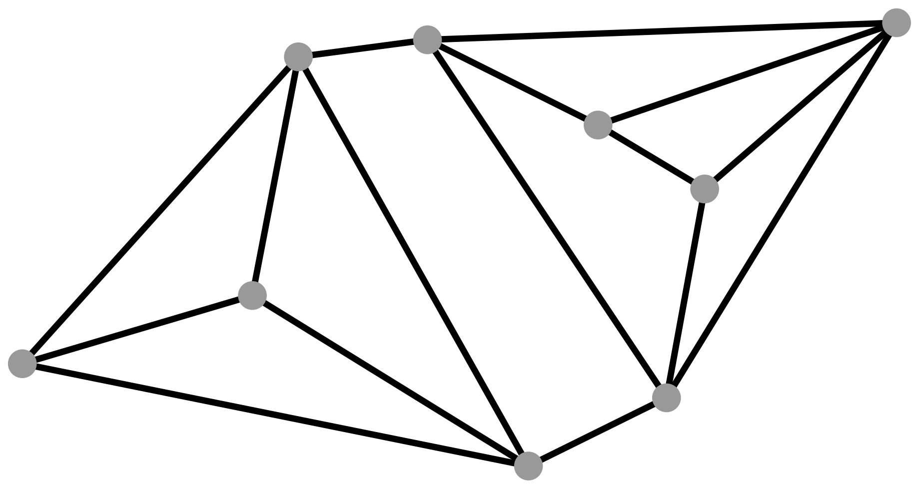
Given: Graph $G = (V, E)$
Goal: Color *edges* with few colors
Constraint: No two incident edges get the same color



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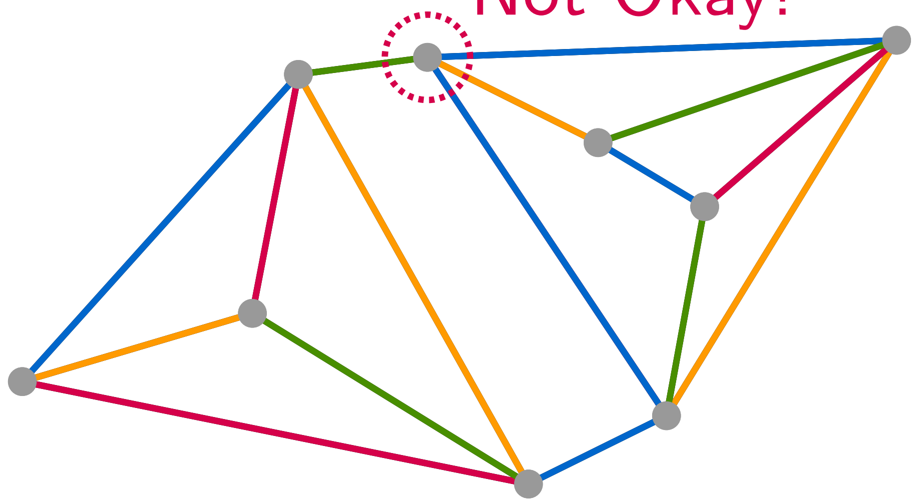

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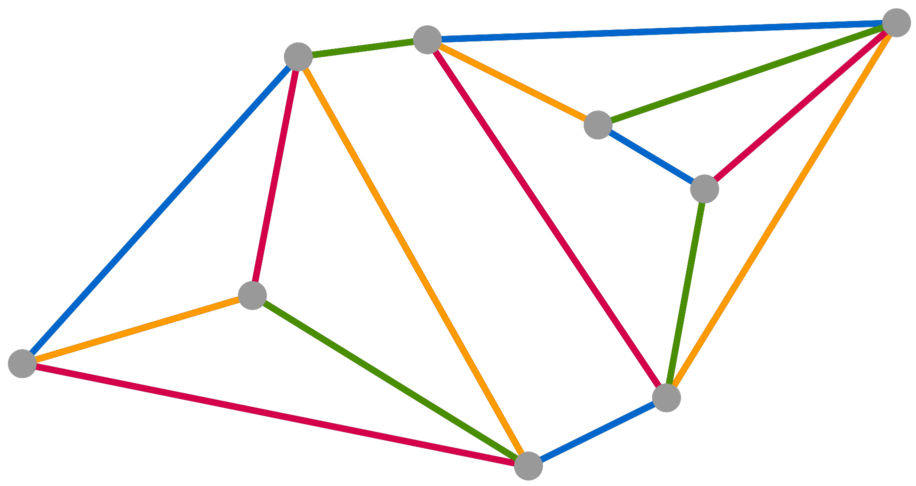

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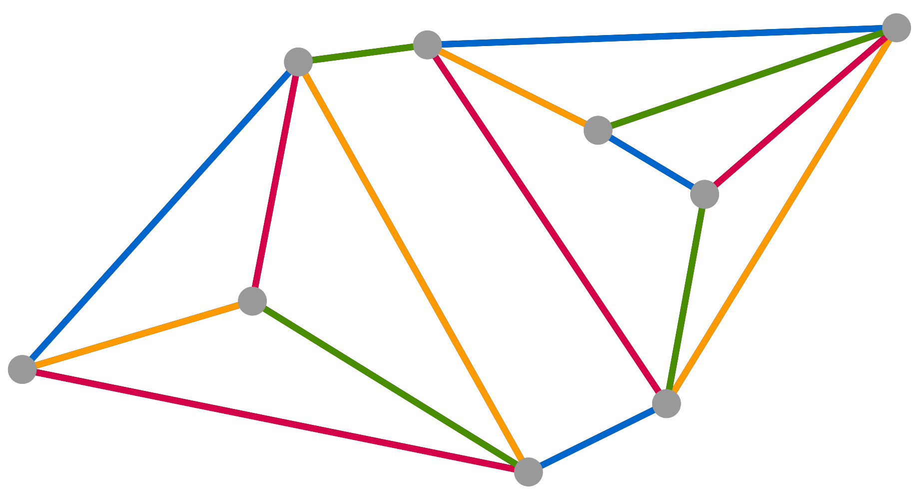

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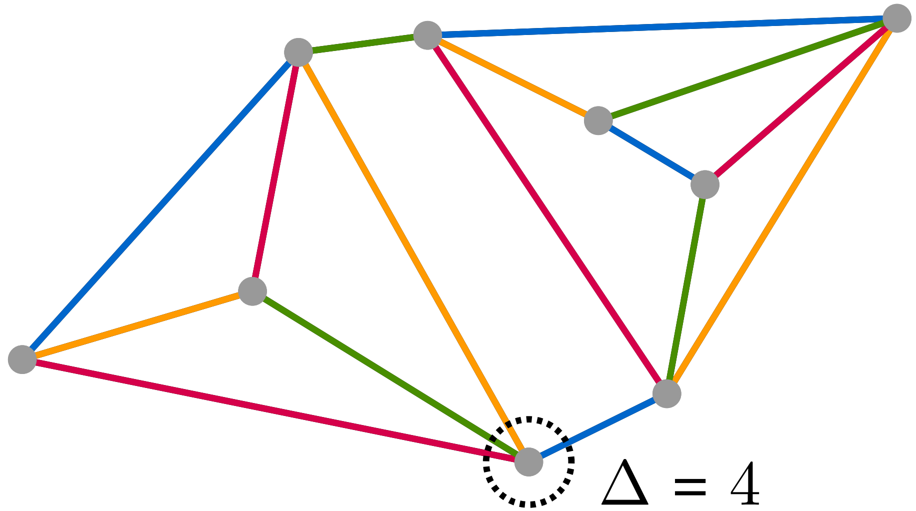

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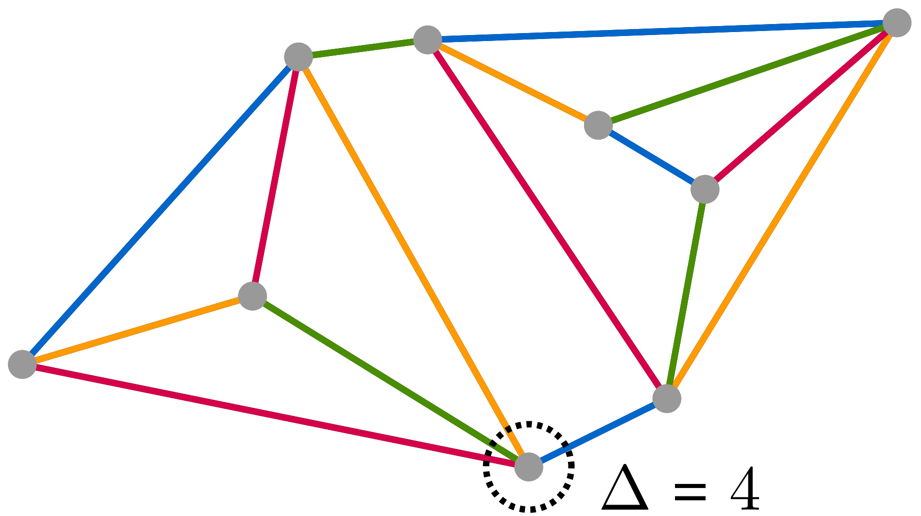

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Claim: $\#\text{Colors} \geq \Delta$

Theorem: $\#\text{Colors} \leq \Delta + 1$

[Vizing 1964]

Edge Coloring Algorithms

- Many algorithms computing $(\Delta + 1)$-edge-colorings
[Vizing'64, Gabow/Nishizeki/Kariv/Leven/Osmau'85,Misra/Gries'92,...]
- NP-Hard to Δ-edge-color.
[Holyer'81]
- Many algorithms computing Δ-edge-color in *bipartite graphs*
[Cole/Hopcroft'82,Cole/Ost/Schirra'01,Alon'03,Goel/Kapralov/Khanna'13,...]
- Studied in various computational models:

  Distributed [PanconesiSrinivasan'97,DubhashiGrablePanconessi'98,...]
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This Talk: Online

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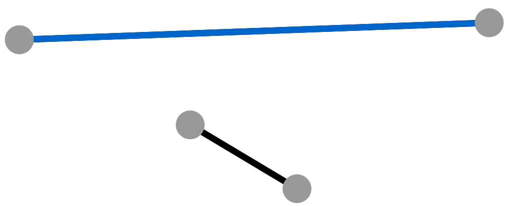

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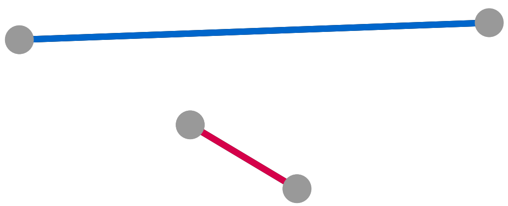

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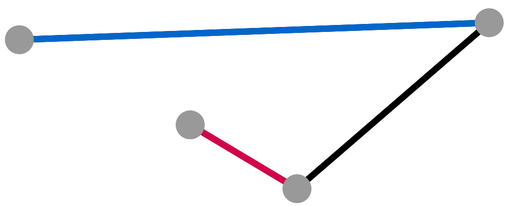

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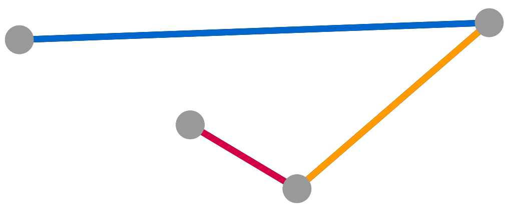

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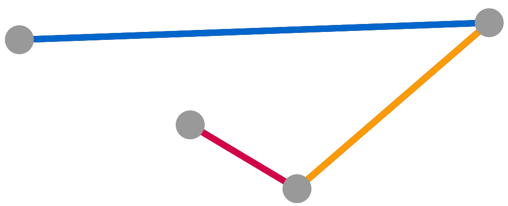

Variants:
Edge or Vertex arrivals
Adversarial or Random order
Deterministic or Oblivious or Adaptive
General or Bipartite graphs

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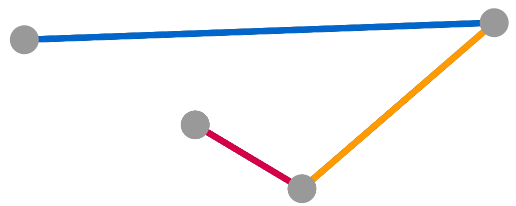

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How many colors do we need? Still $\approx \Delta$?

Warm-up: Greedy Algorithm

Greedy: Color edge with "lowest" avaliable color.

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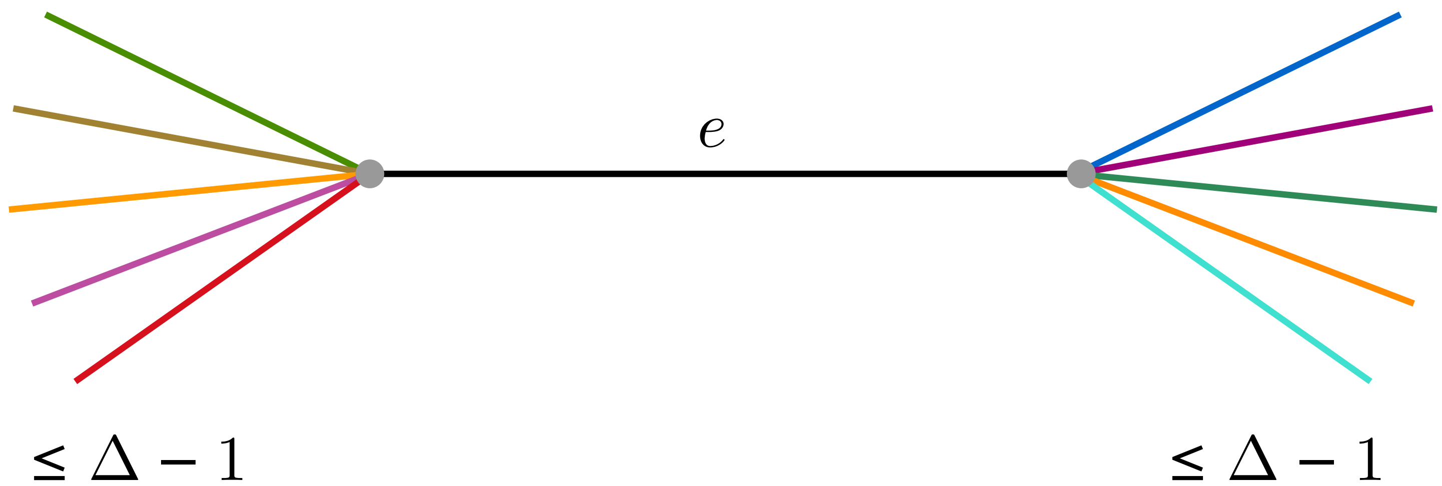

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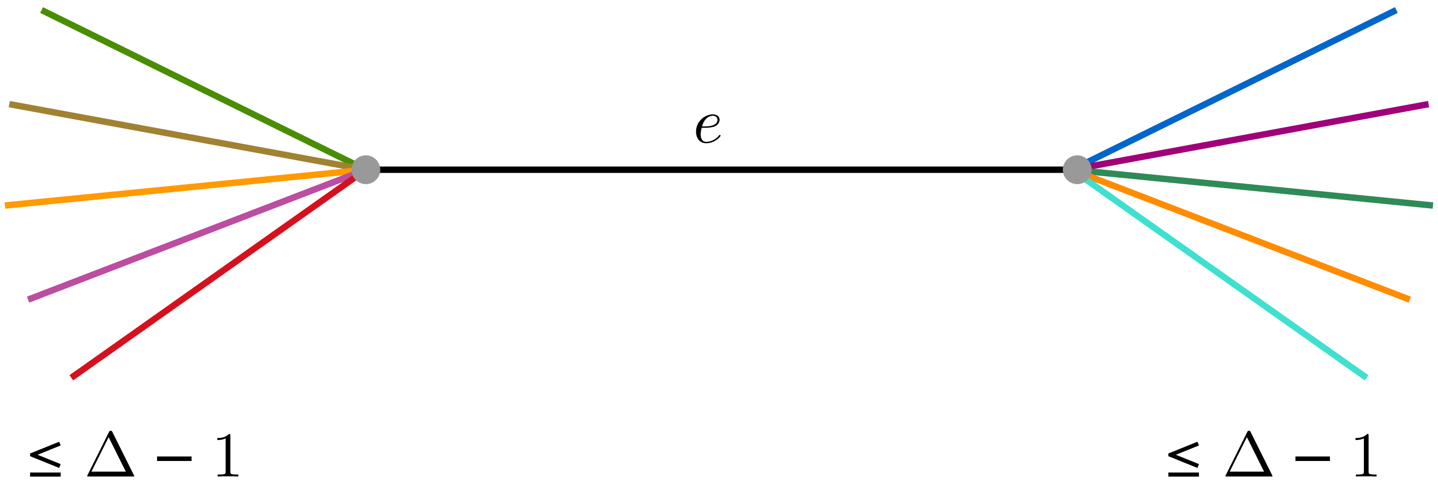

Claim: $\leq 2(\Delta - 1)$ blocked colors

Claim: Greedy uses $\leq 2\Delta - 1$ colors

Can we do better?

Can we beat $2\Delta - 1$ colors?

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NO!

Lower Bound

Theorem: No online algorithm can $(2\Delta - 2)$-edge-color every graph.

[Bar-Noy/Motwani/Naor 1992]

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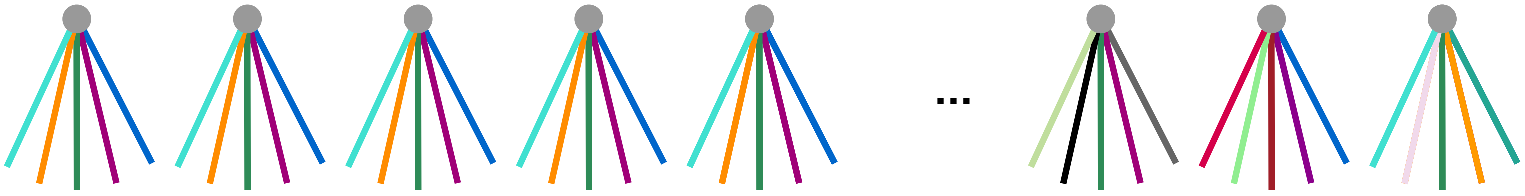

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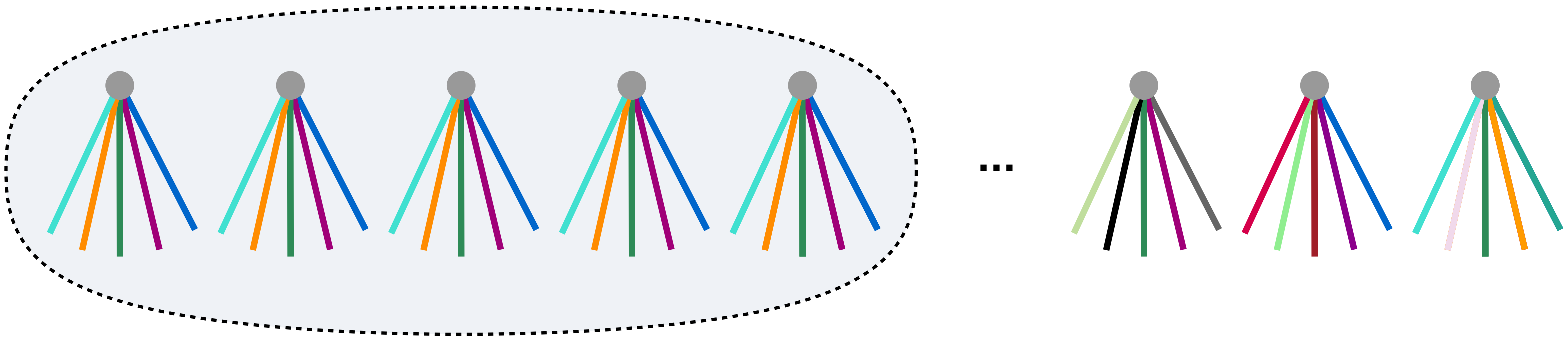
Eventually have Δ stars colored the same (pigeonhole principle)



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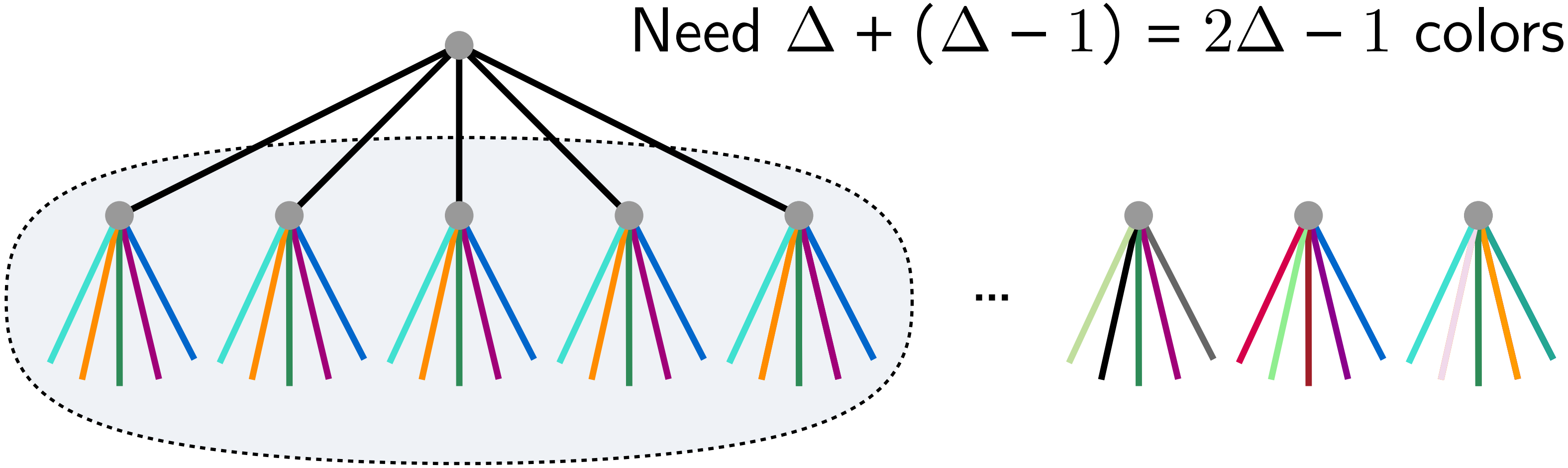

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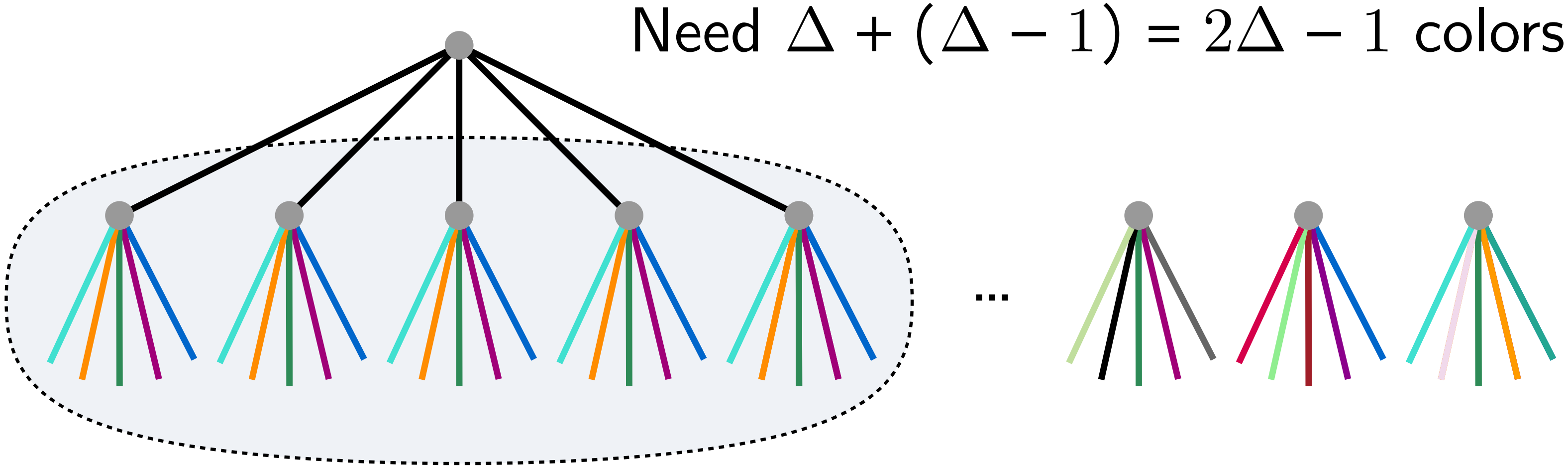

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Can we do better when $\Delta = \omega(\log n)$?

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YES $\approx \Delta$ colors :)

Progress

Conjecture: $(1 + o(1))\Delta$-colors sufficent when $\Delta = \omega(\log n)$.

[Bar-Noy/Motwani/Naor 1992]

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- **Random order** edge arrivals:
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  - [Bahmani/Mehta/Motwani'10]: $\mathbf{1.27\Delta}$-coloring if $\Delta = \omega(\log n)$
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- Adversarial vertex arrivals:
  - [Cohen/Peng/Wajc'19] (simplified [B./Svensson/Vintan/Wajc'24]:
    $\approx \Delta$-coloring **bipartite graphs**

    For unknown Δ: $\approx \frac{e}{e-1}\Delta$-coloring **bipartite graphs (optimal)**
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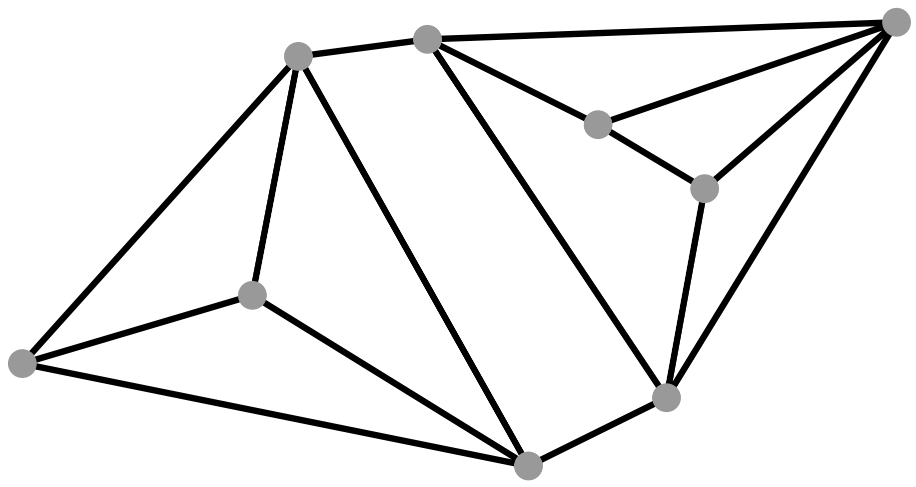
Techniques

Technical Part — Outline

- Edge Coloring $\iff$ Fair Matchings
  - Reduction
- Online Fair Matching Algorithm
  - First Attempt
  - New Algorithm
  - Analysis: Martingales

Fair Matching Problem

Given: Graph $G = (V, E)$
Goal: Find a matching M
α-**Fairness:** $\Pr[e \in M] \geq \frac{1}{\alpha\Delta}$ for each edge $e \in E$



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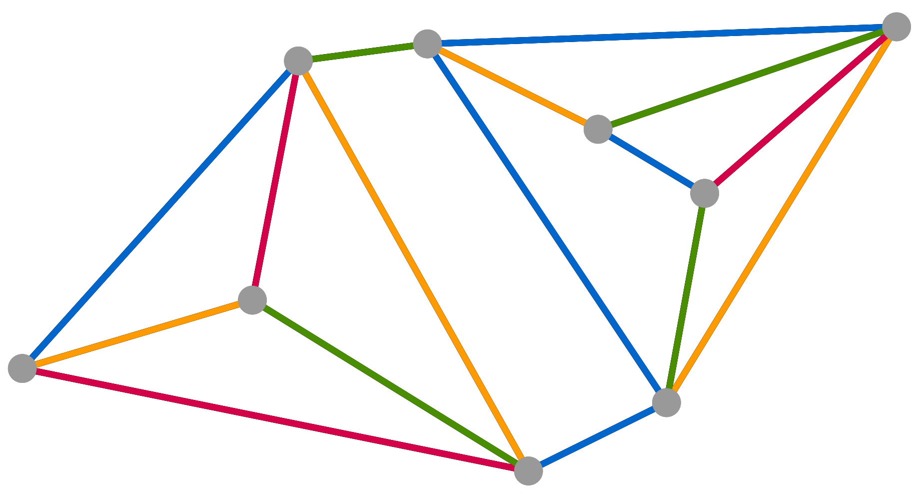
Proof: Pick random color as matching



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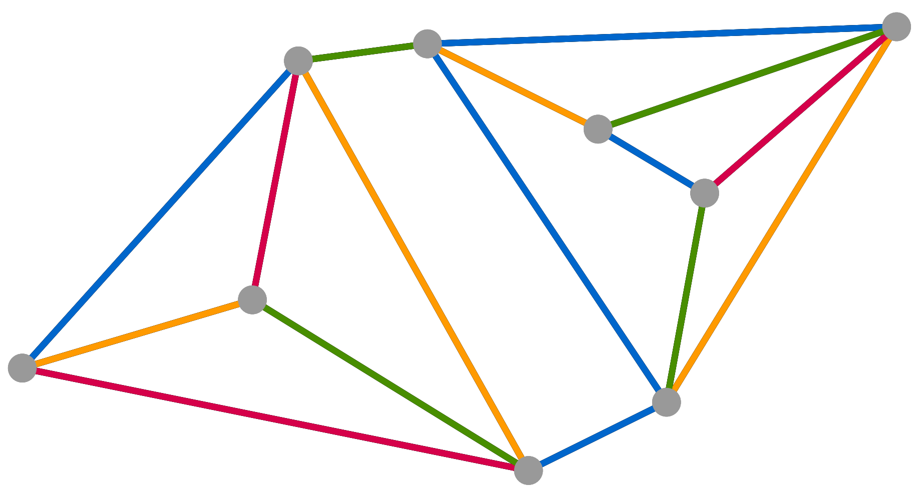

Lemma: α-fair matching $\implies$ $(1 + o(1))\alpha\Delta$-edge-coloring

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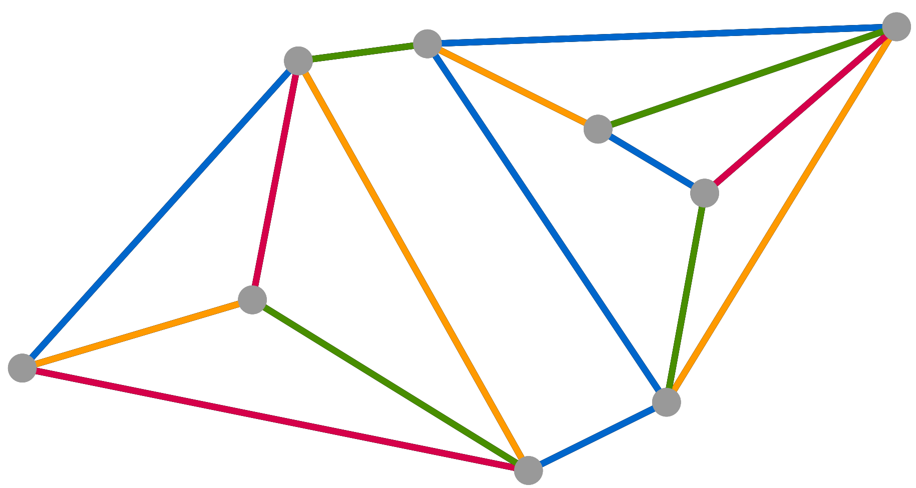

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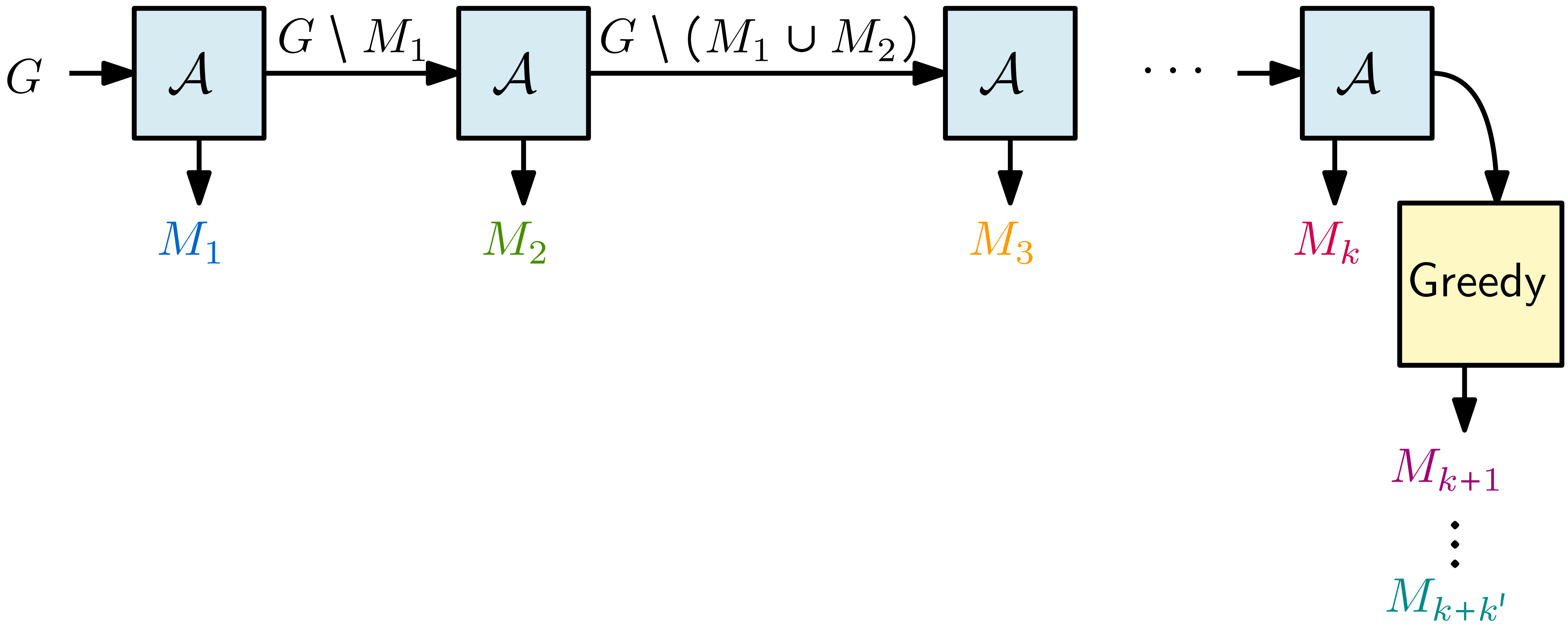
New Objective: $(1 + o(1))$-**fair matching algorithm**

From Fair Matchings to Edge Coloring [Cohen/Peng/Wajc'19]

$\mathcal{A}$: $(1 + o(1))$-fair matching algorithm



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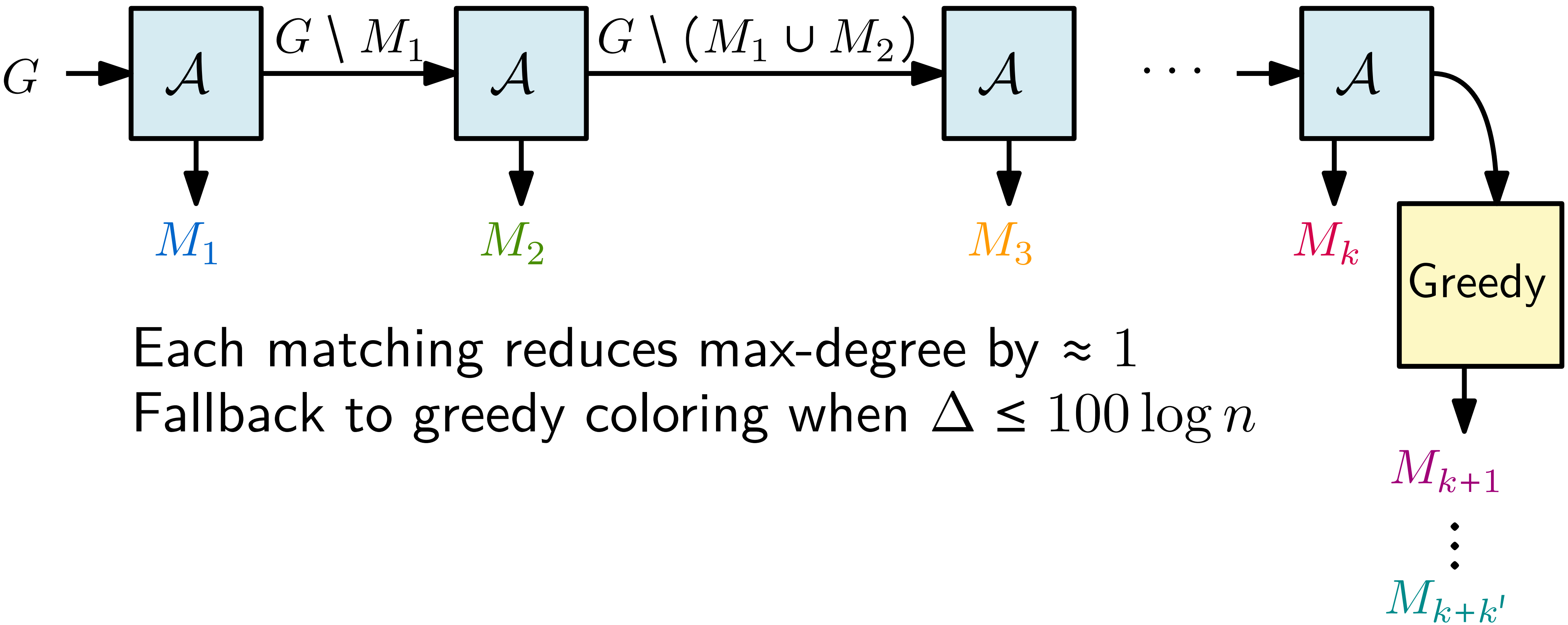

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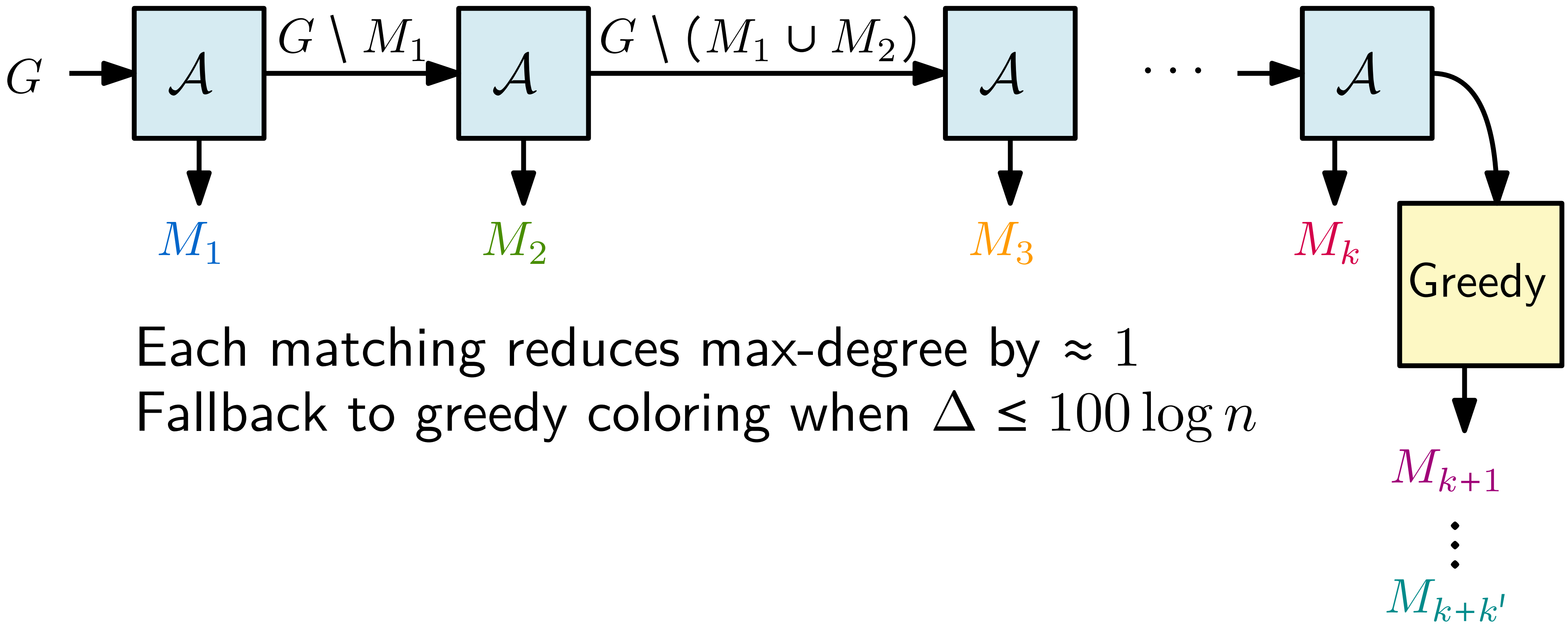

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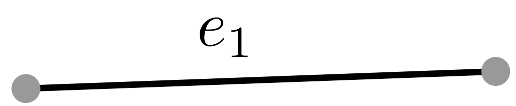

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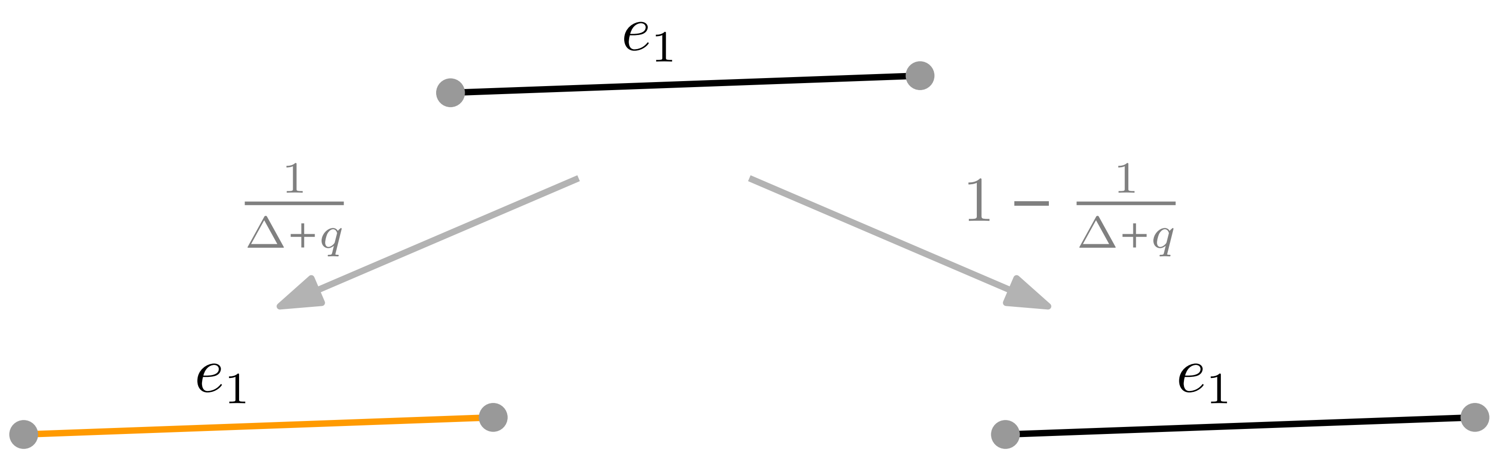

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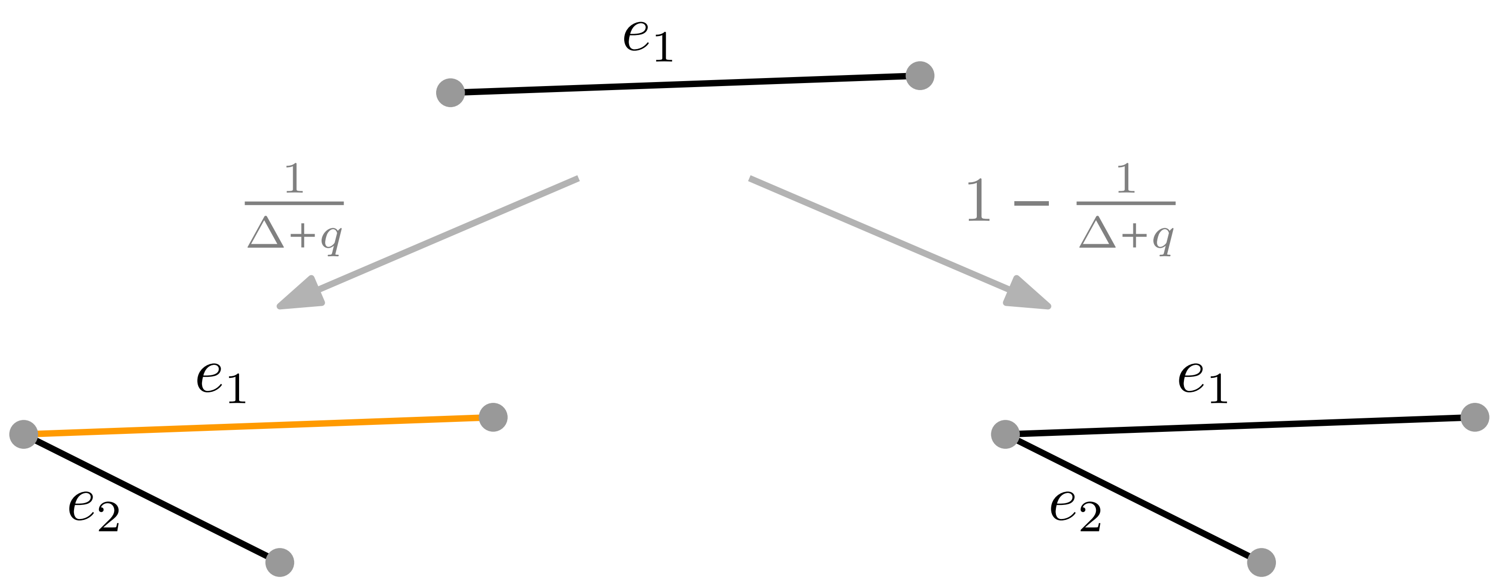

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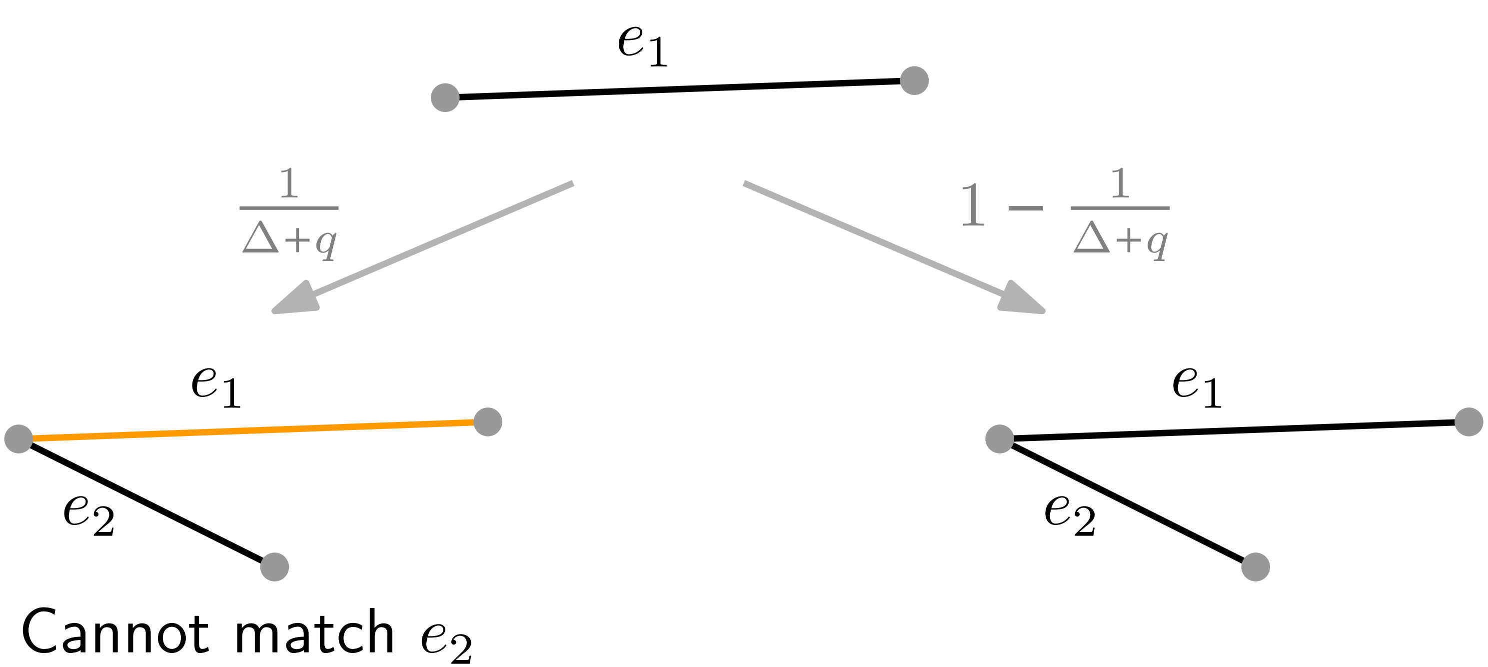

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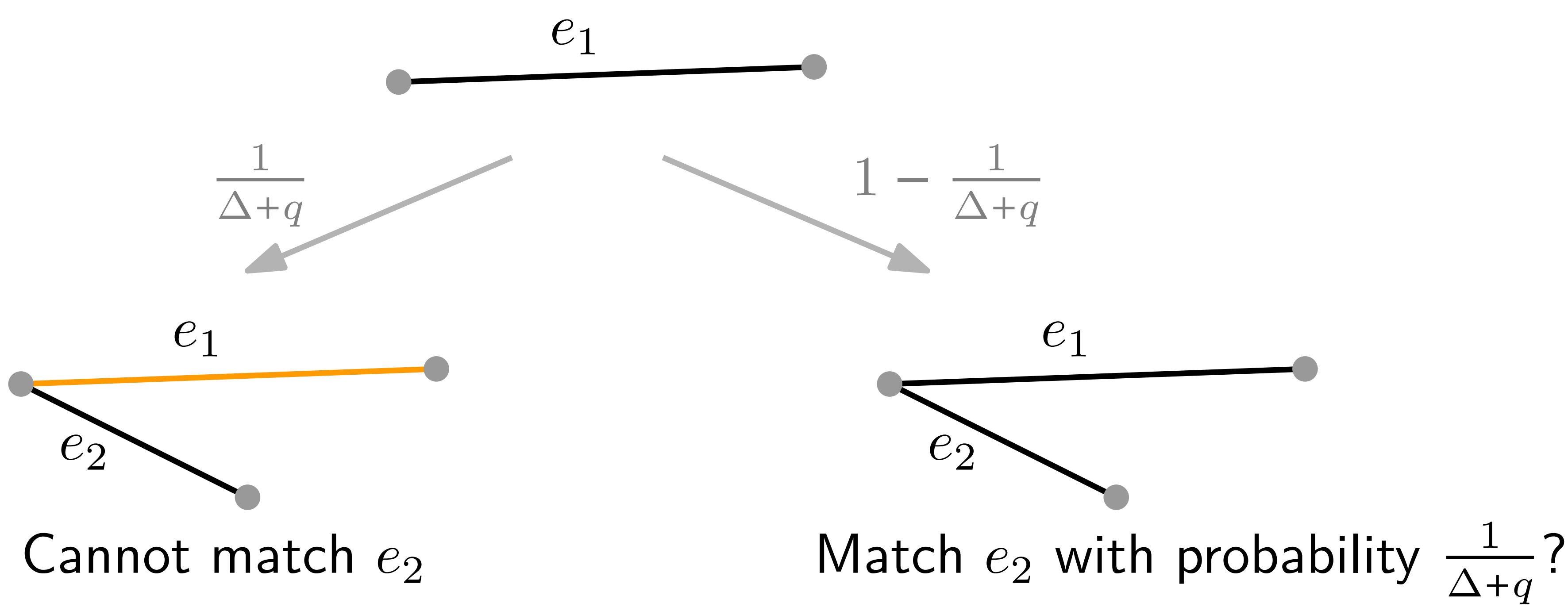

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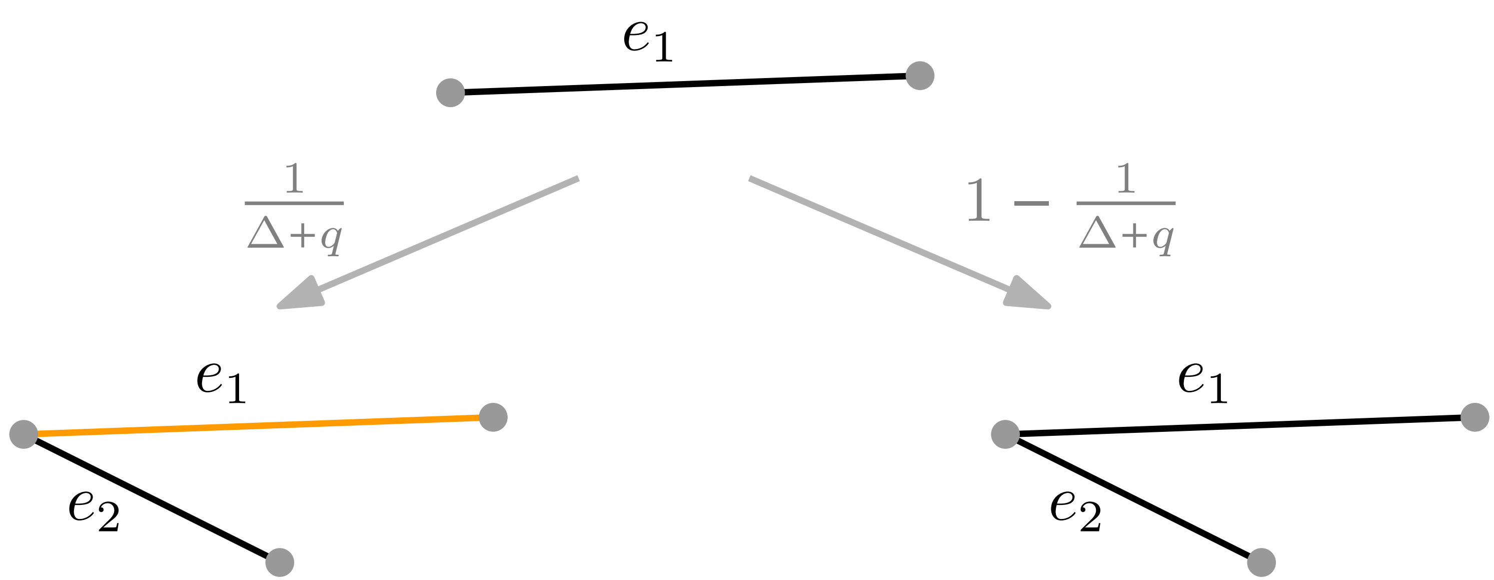
Goal: Match each edge with probability $\Pr[e \in M] \approx \frac{1}{\Delta}$

$$\Pr[e \in M] = \frac{1}{\Delta+q} \qquad q := \Theta(\Delta^{3/4}\sqrt{\log \Delta}) = o(\Delta)$$



Cannot match e_2

Match e_2 with probability $\frac{1}{\Delta+q}$?

Must scale up: $\frac{1}{\Delta+q} / (1 - \frac{1}{\Delta+q})$

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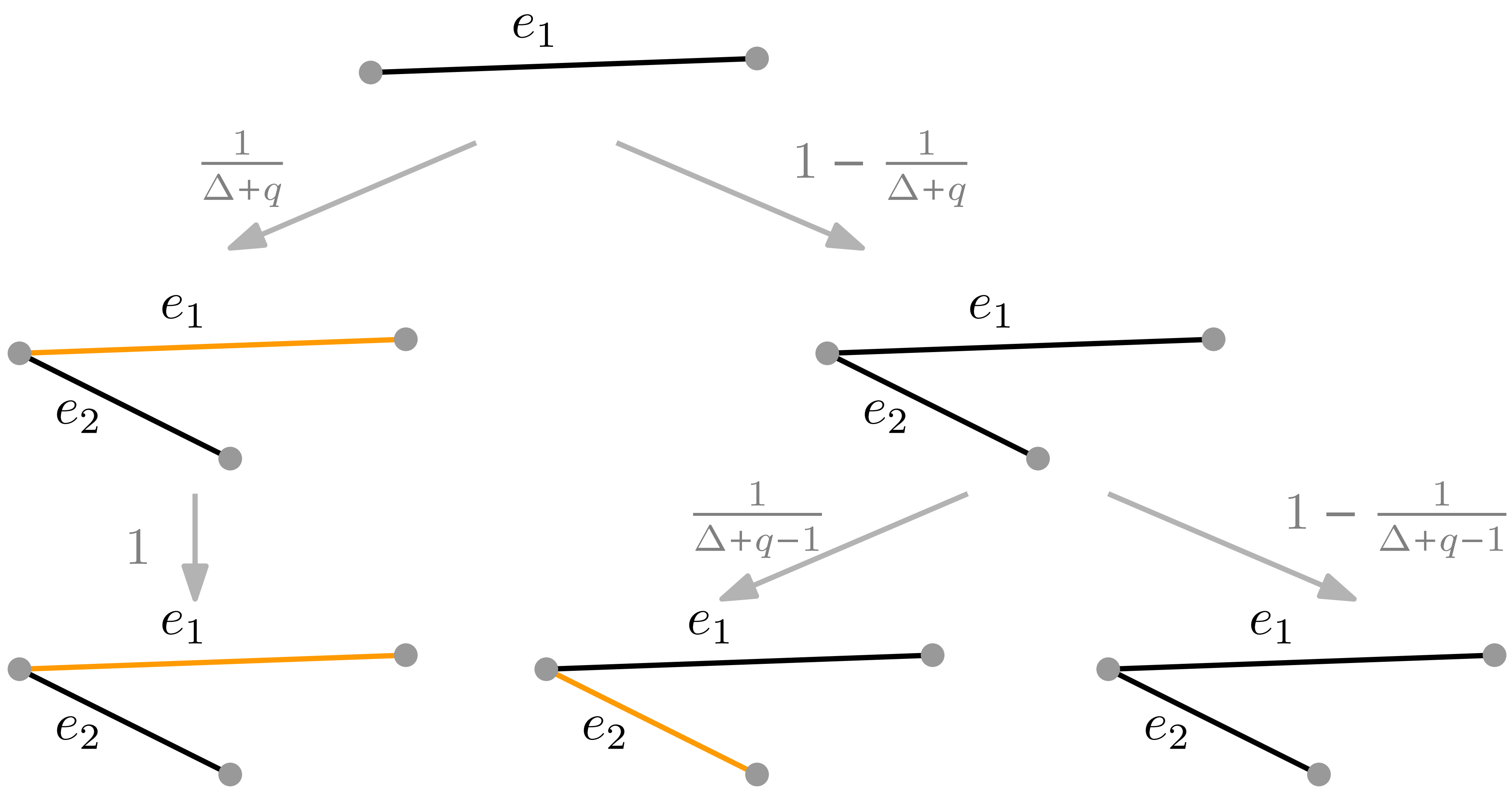

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Example: G is a tree

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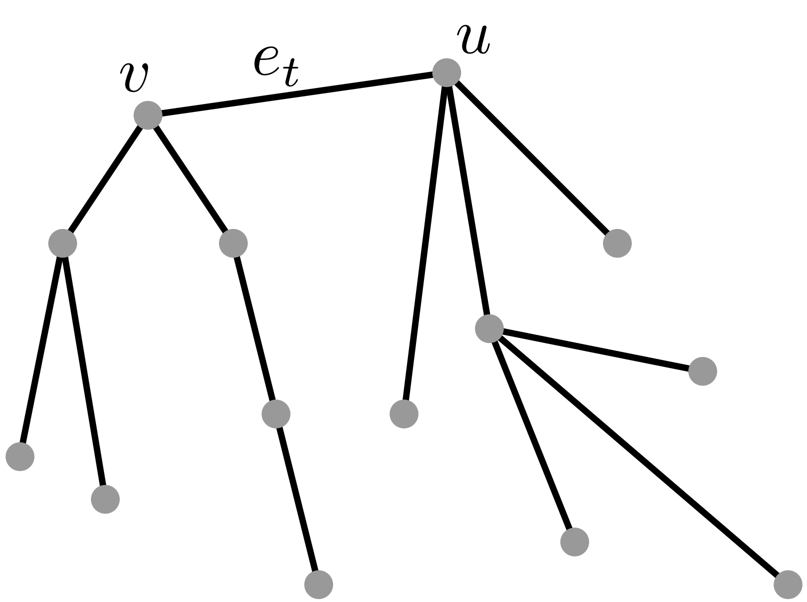

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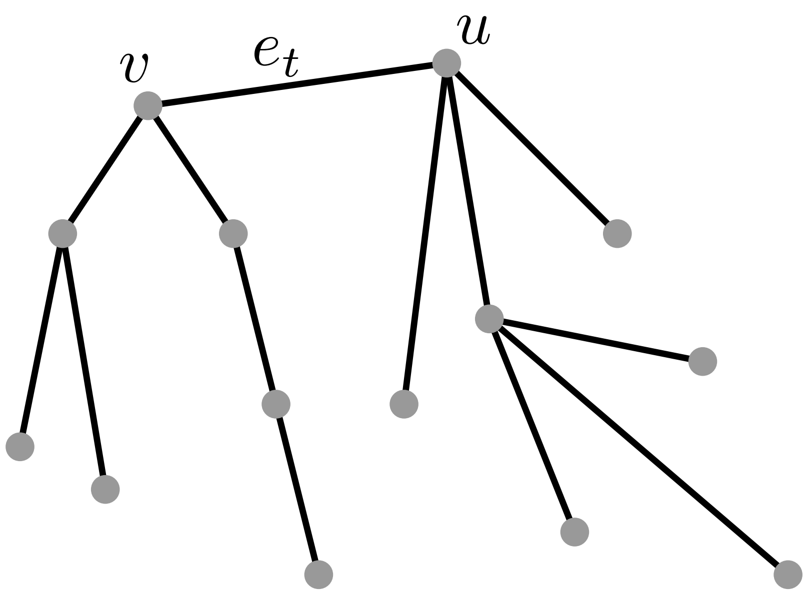

(tree)

$\Pr[\text{both } u, v \text{ free}] = \Pr[u \text{ free}] \cdot \Pr[v \text{ free}]$

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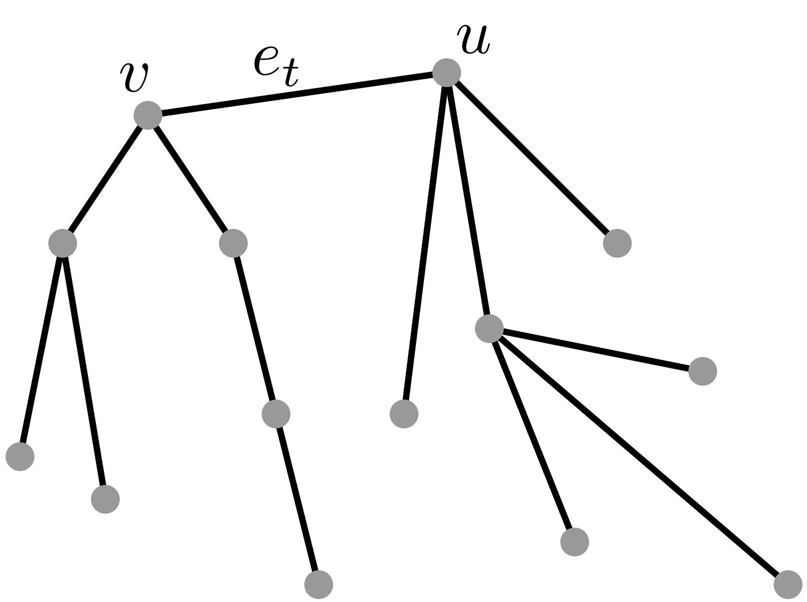

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$$\Pr[u \text{ free}] = 1 - \frac{\deg(u)}{\Delta+q} \geq \frac{q}{2\Delta}$$

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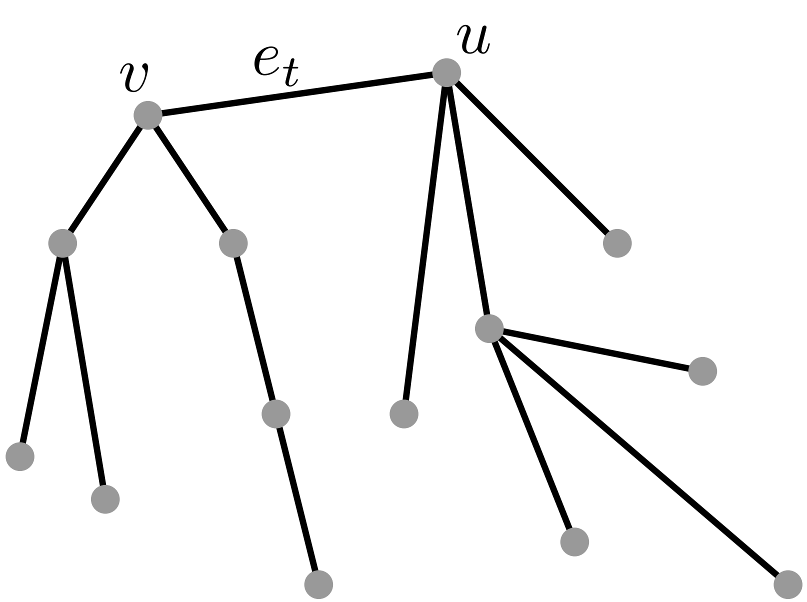

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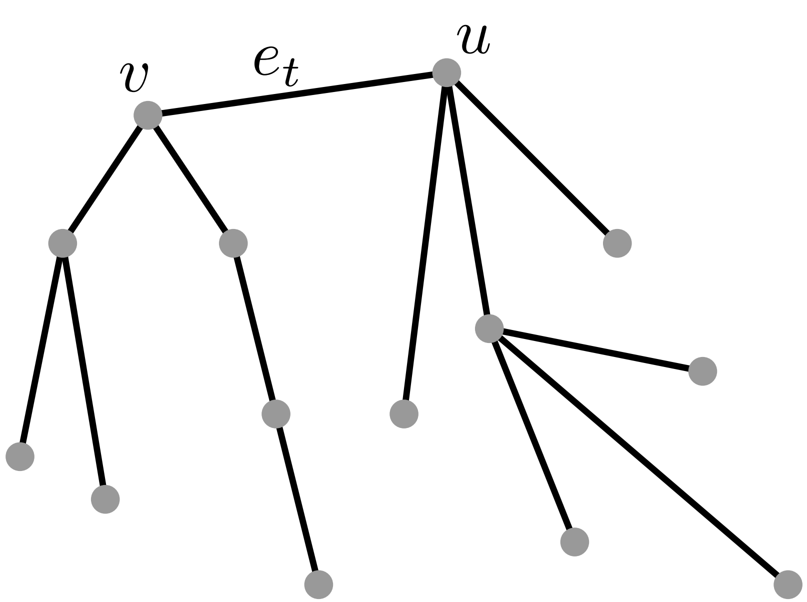

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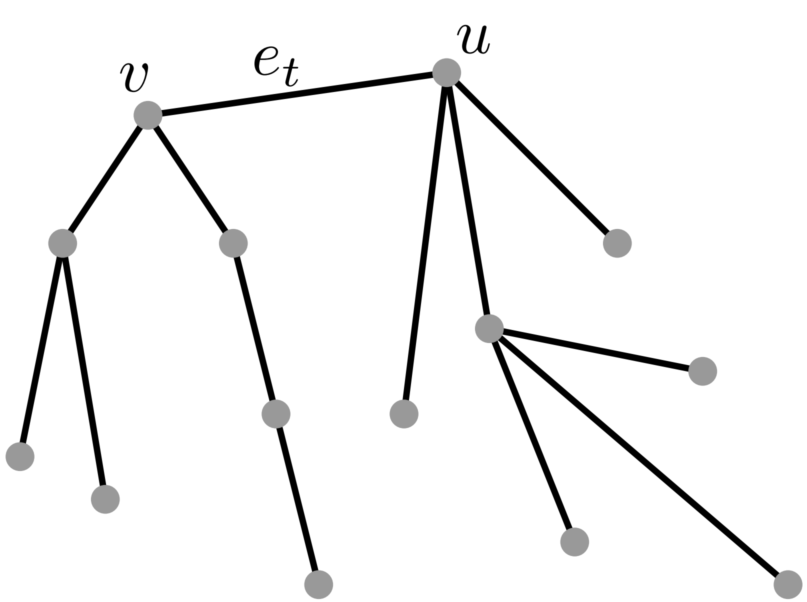

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[Kulkarni/Liu/Sah/Sawhney/Tarnawski'22]

$(\frac{e}{e-1} + o(1))\Delta$-coloring subsampling locally tree-like graphs

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Open Problem: Does this "Natural Algorithm" work in general?

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Alternative Natural Algorithm

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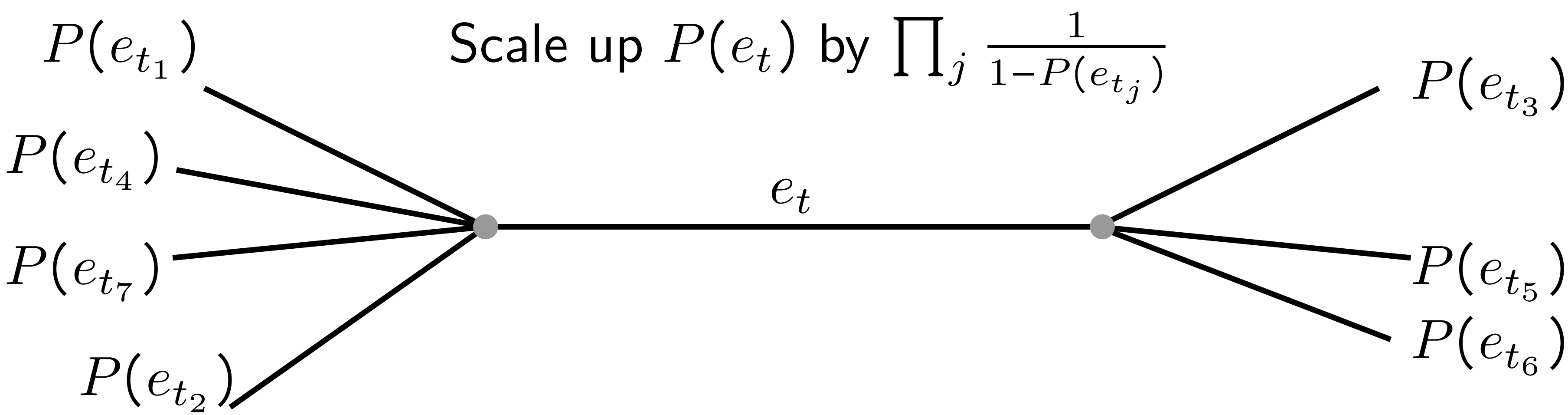
Algorithm 1 (NaturalMatchingAlgorithm).

When an edge $e_t = (u, v)$ *arrives, match it with probability*

$$P(e_t) \leftarrow \begin{cases} \frac{1}{\Delta+q} \cdot \frac{1}{\prod_{j=1}^{k}(1-P(e_{t_j}))} & \text{if } u \text{ and } v \text{ are still unmatched,} \\ 0 & \text{otherwise,} \end{cases}$$

where $e_{t_1}, \ldots, e_{t_k}$ *are those previously-arrived edges incident to the endpoints of* e_t.

Scale up $P(e_t)$ by $\prod_j \frac{1}{1-P(e_{t_j})}$



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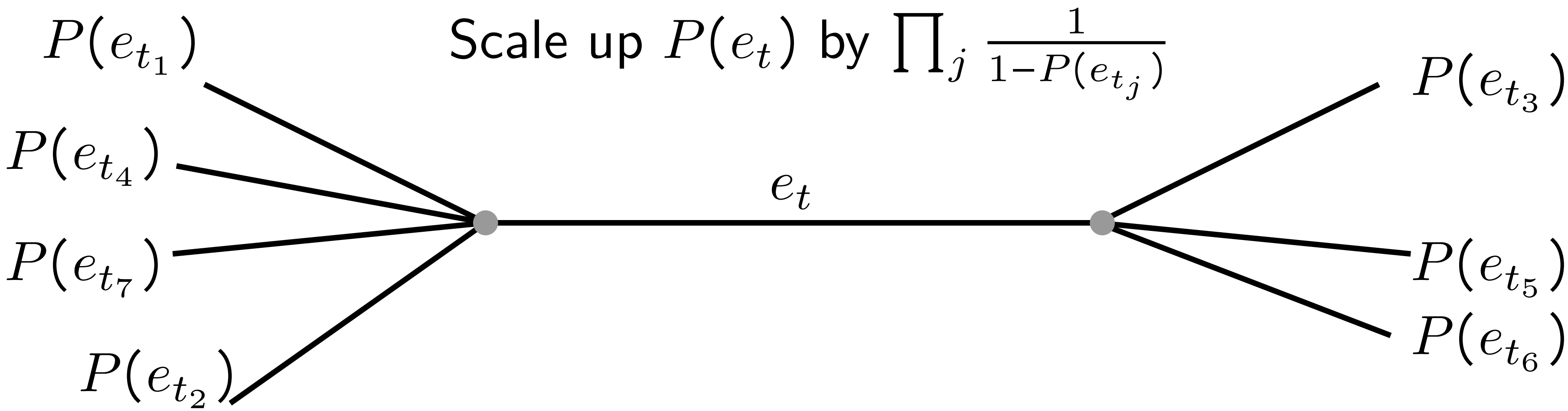

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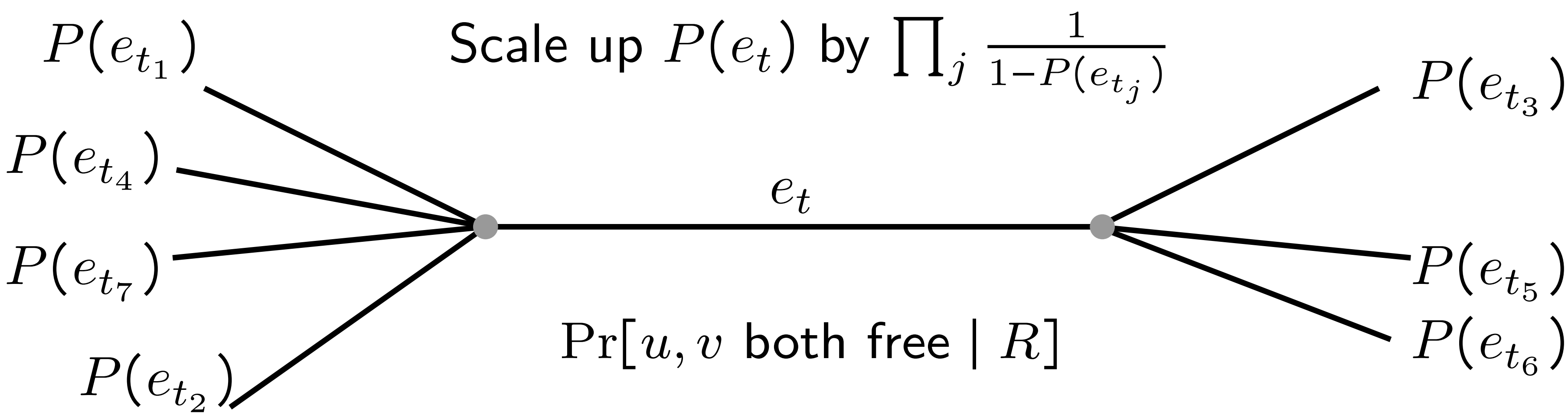

R =randomness outside of incident edges.

Alternative Natural Algorithm

Algorithm 2 (MatchingAlgorithm).

Initialization: *Set* $F_1(v) \leftarrow 1$ *for every vertex* v *and* $M_1 \leftarrow \emptyset$.

At the arrival of edge $e_t = (u, v)$ at time t:

- *Sample* $X_t \sim Uni[0, 1]$.
- *Define*

$$P(e_t) = \begin{cases} \frac{1}{\Delta+q} \cdot \frac{1}{F_t(u) \cdot F_t(v)} & \text{if } u \text{ and } v \text{ are unmatched in } M_t, \\ 0 & \text{otherwise.} \end{cases}$$

  and

$$\hat{P}(e_t) = \begin{cases} P(e_t) & \text{if } \min\{F_t(u), F_t(v)\} \cdot (1 - P(e_t)) \geqslant q/(4\Delta) \\ 0 & \text{otherwise.} \end{cases}$$

- *Set*
  - $F_{t+1}(u) \leftarrow F_t(u) \cdot (1 - \hat{P}(e_t))$;
  - $F_{t+1}(v) \leftarrow F_t(v) \cdot (1 - \hat{P}(e_t))$;
  - $M_{t+1} \leftarrow \begin{cases} M_t \cup \{e_t\} & \text{if } X_t < \hat{P}(e_t), \\ M_t & \text{otherwise.} \end{cases}$

A More Fine-Grained Bayesian Approach

Our Alternative Algorithm: $p_t := \frac{1/(\Delta+q)}{\Pr[u,v \text{ both free in current execution}]}$

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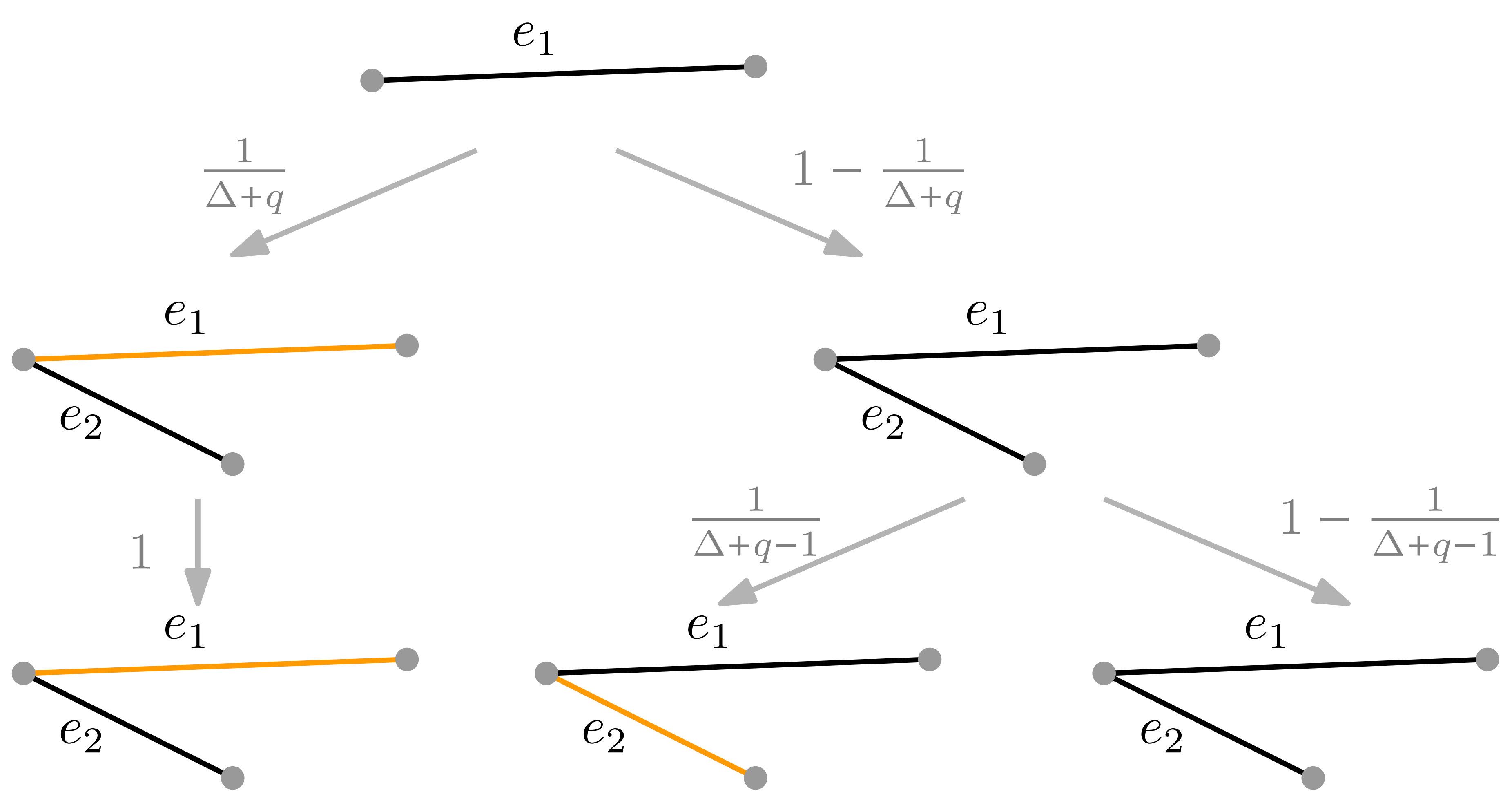

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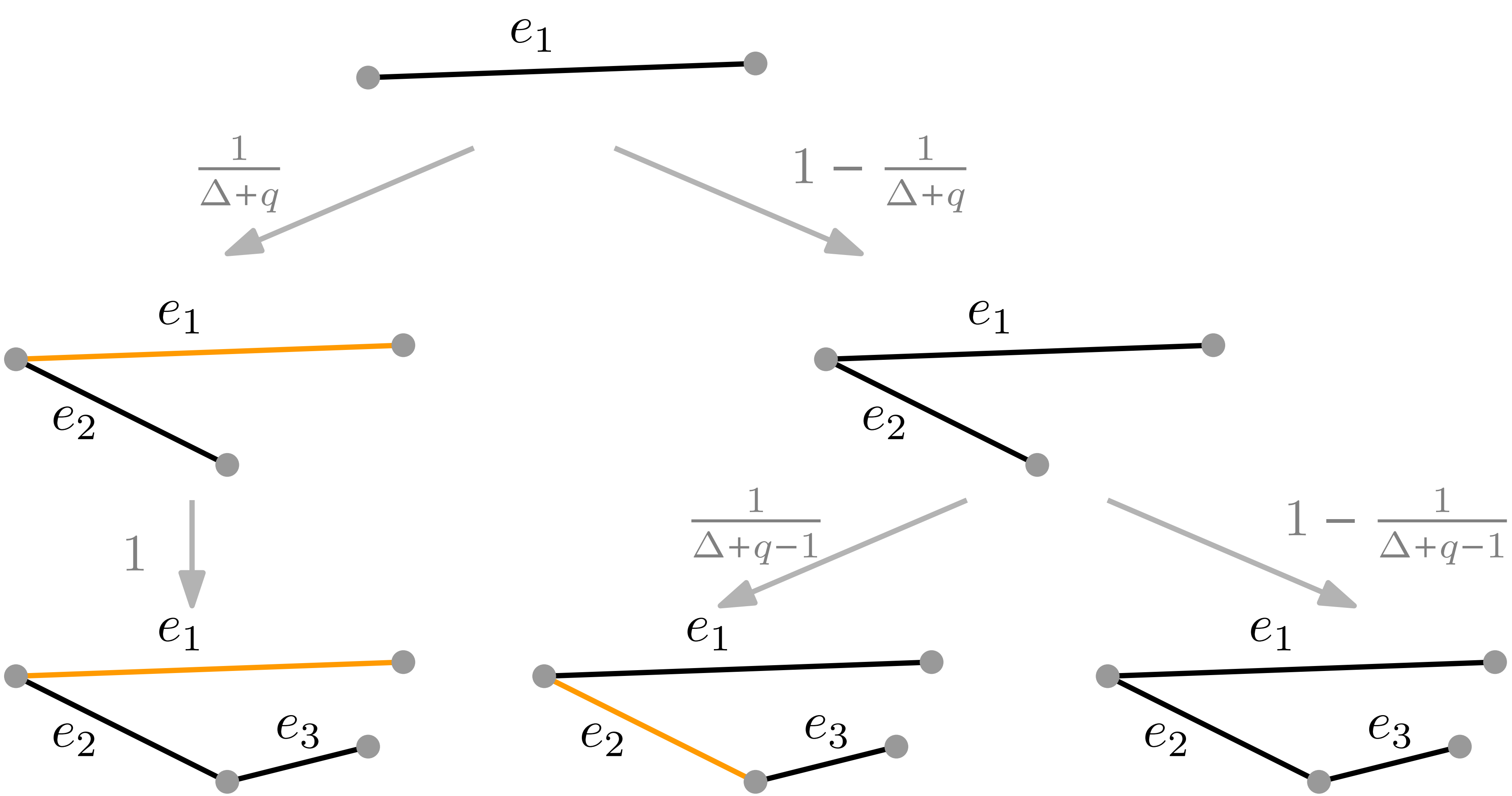

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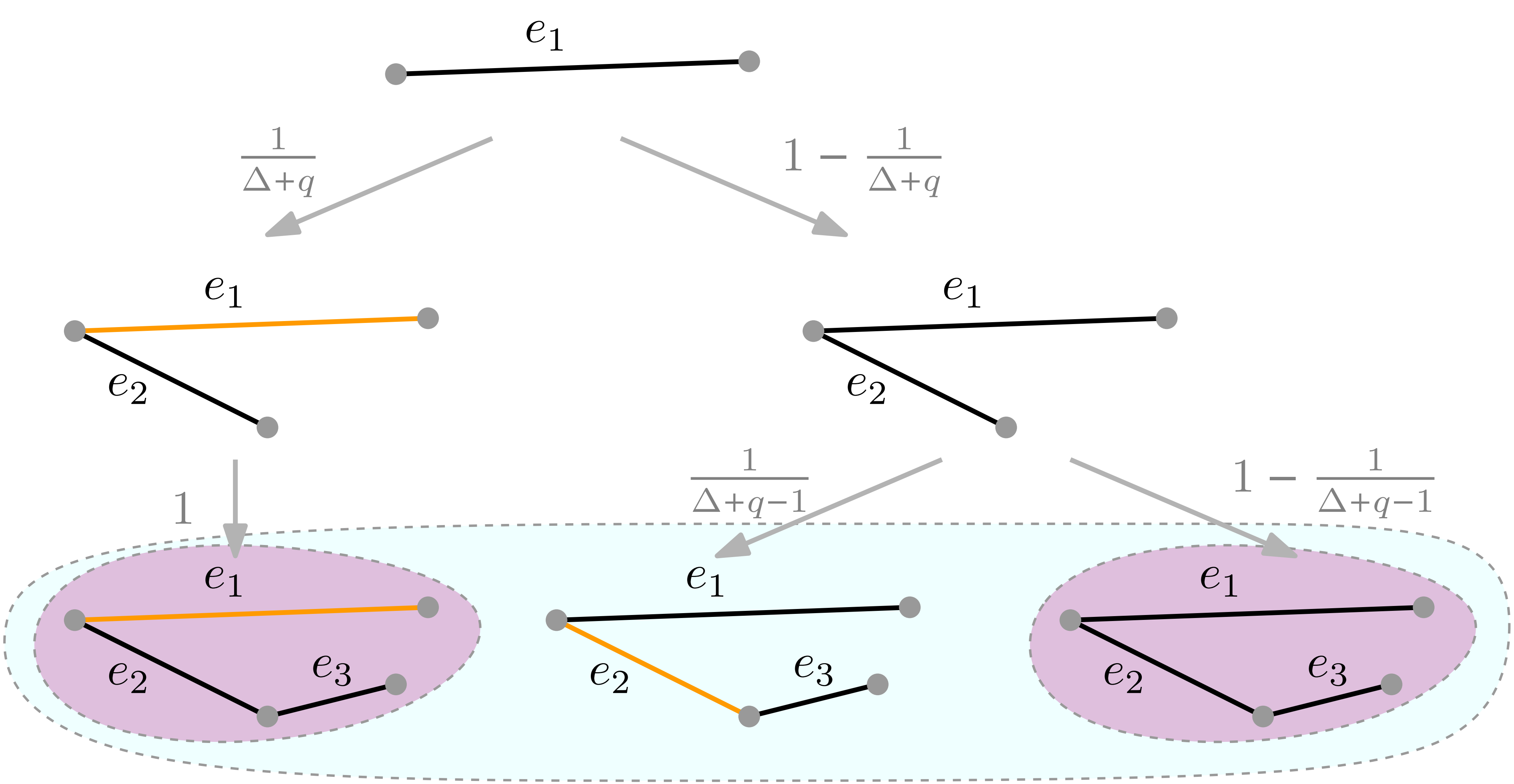

Old Algo: $\Pr[u,v \text{ free}] = (1 - \frac{1}{\Delta+q})$ Uses same scaling factor

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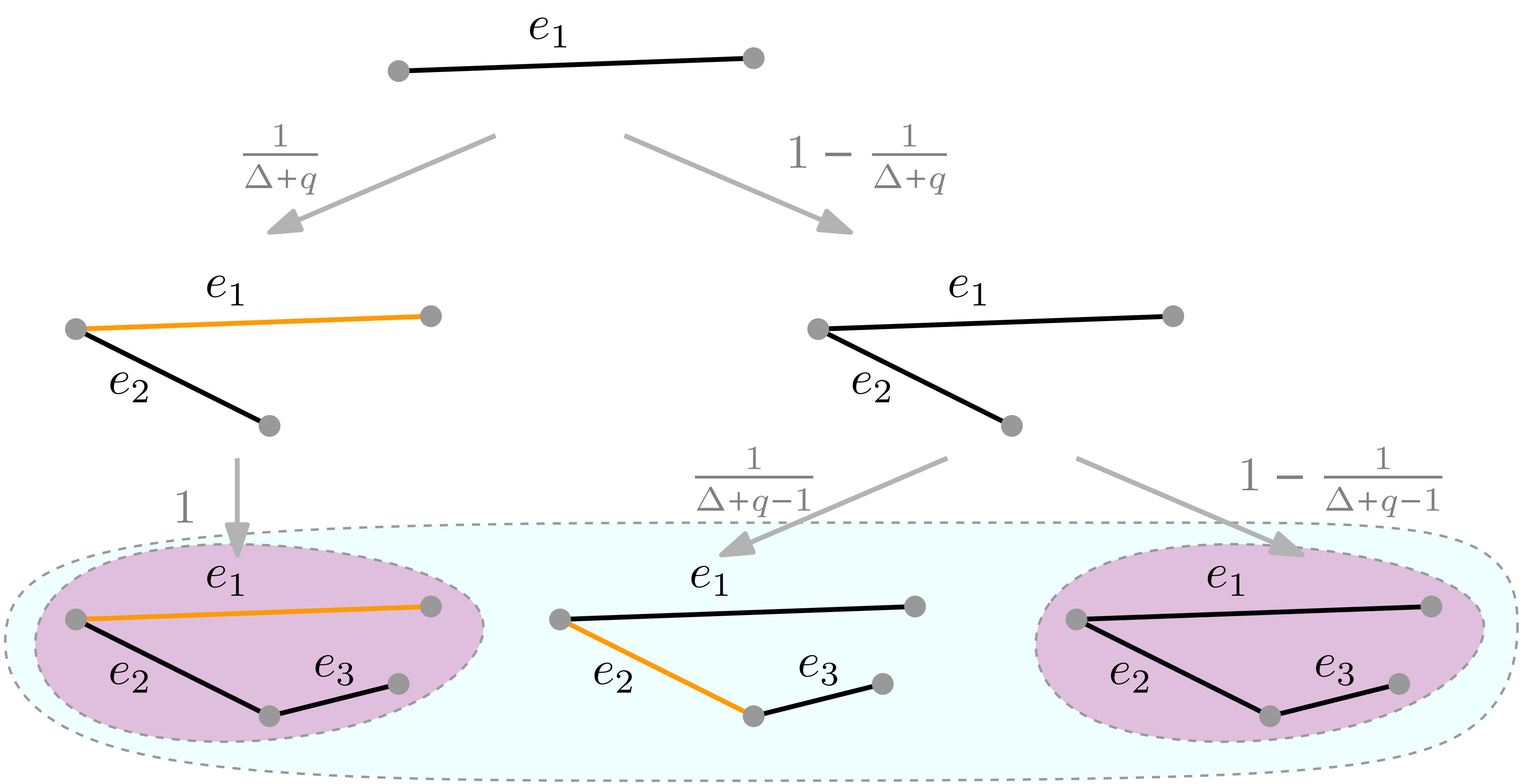

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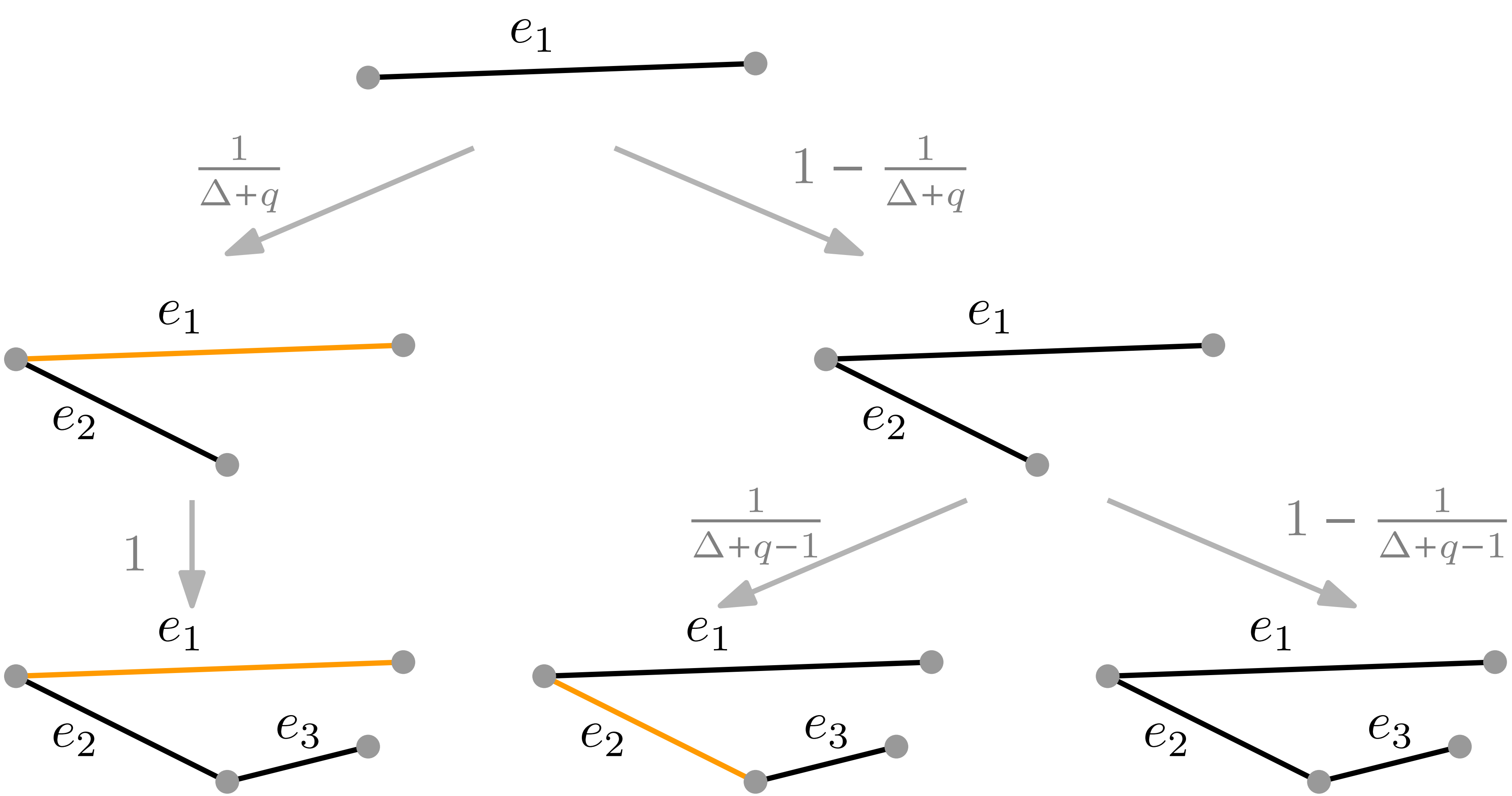

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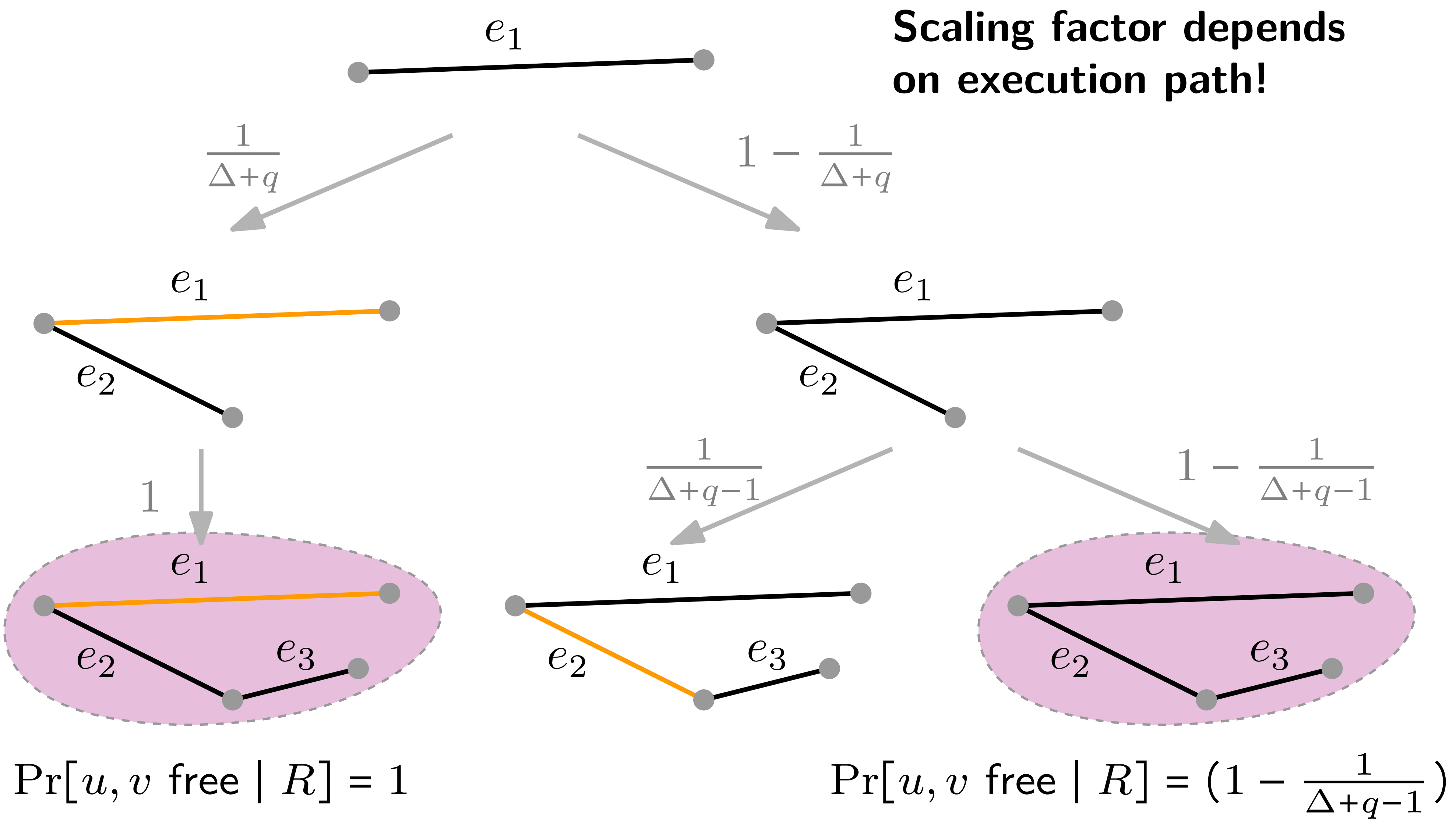
Scaling factor depends on execution path!



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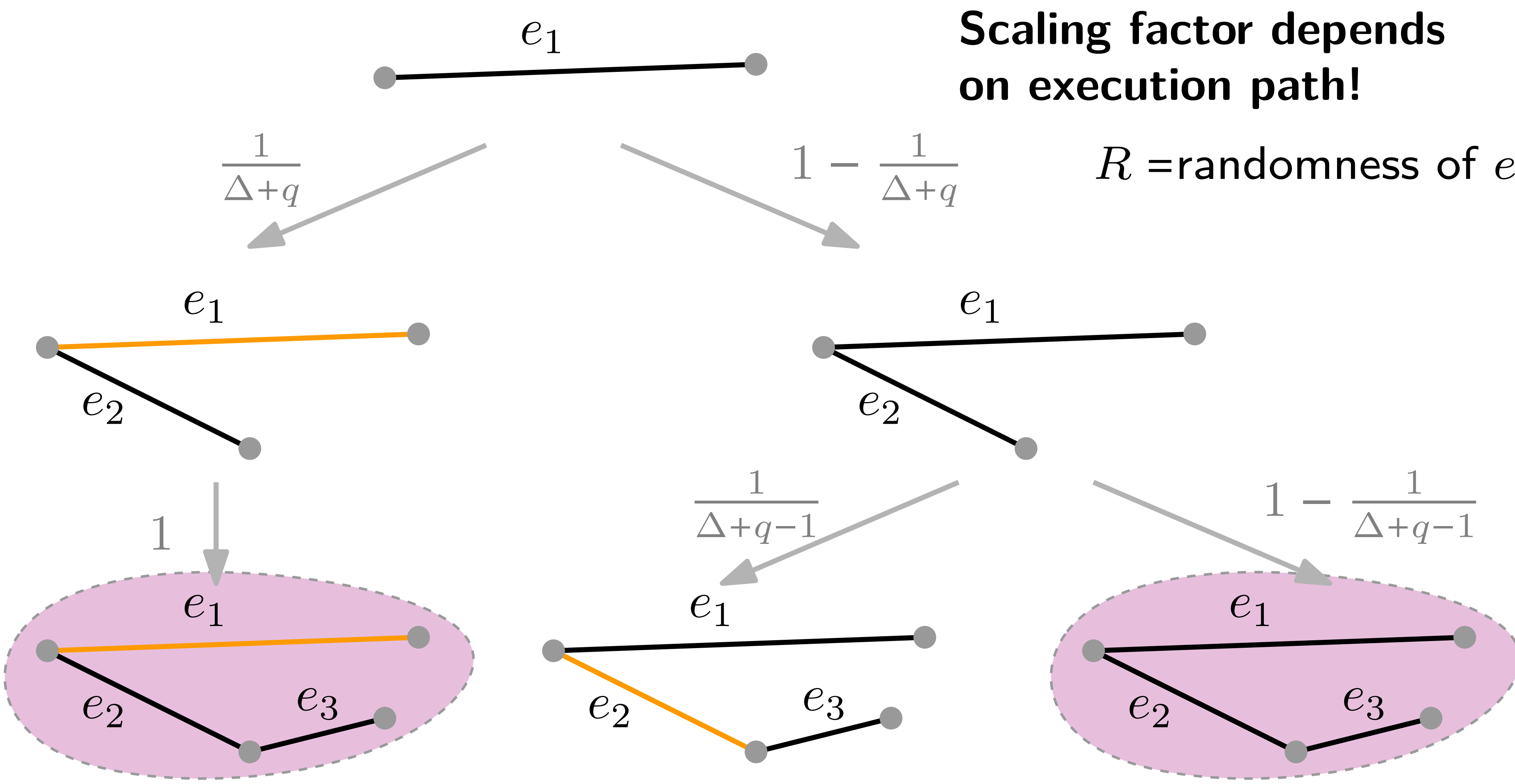
R =randomness of e_1



$\Pr[u,v \text{ free} \mid R] = 1$

$\Pr[u,v \text{ free} \mid R] = (1-\frac{1}{\Delta+q-1})$

Analysis Idea — Random Walk

Core of Analysis: Prove $P(e_t) \leq 1$

Algorithm 1 (NaturalMatchingAlgorithm).

When an edge $e_t = (u, v)$ arrives, match it with probability

$$P(e_t) \leftarrow \begin{cases} \frac{1}{\Delta+q} \cdot \frac{1}{\prod_{j=1}^{k}(1-P(e_{t_j}))} & \text{if } u \text{ and } v \text{ are still unmatched,} \\ 0 & \text{otherwise,} \end{cases}$$

where $e_{t_1}, \ldots, e_{t_k}$ are those previously-arrived edges incident to the endpoints of e_t.

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Core of Analysis: Prove $P(e_t) \leq \frac{10}{\sqrt{\Delta}}$

Previous work: Control Correlation
Our work: Embrace Correlations

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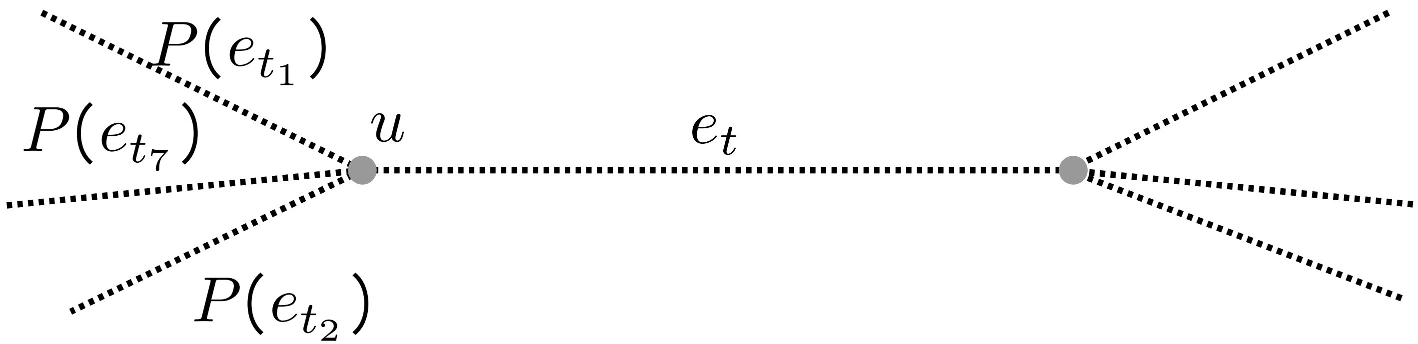

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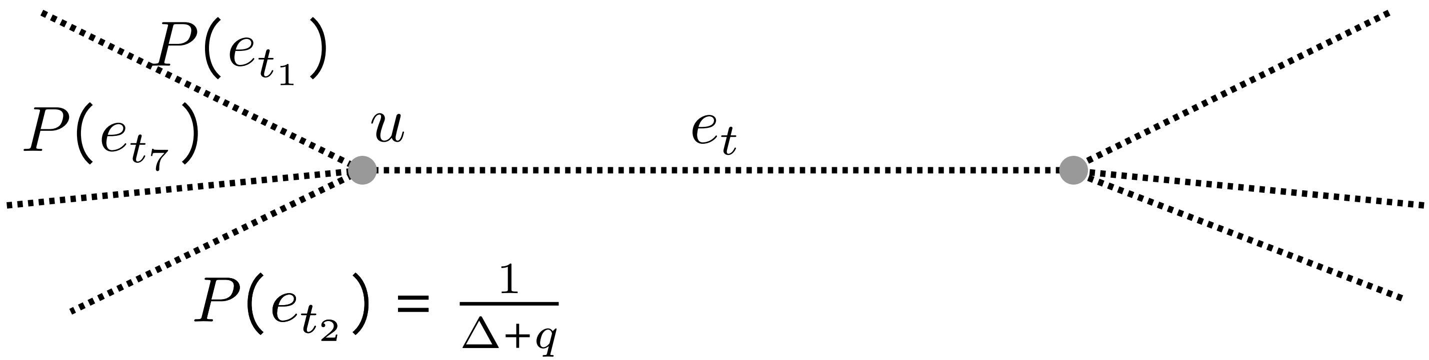

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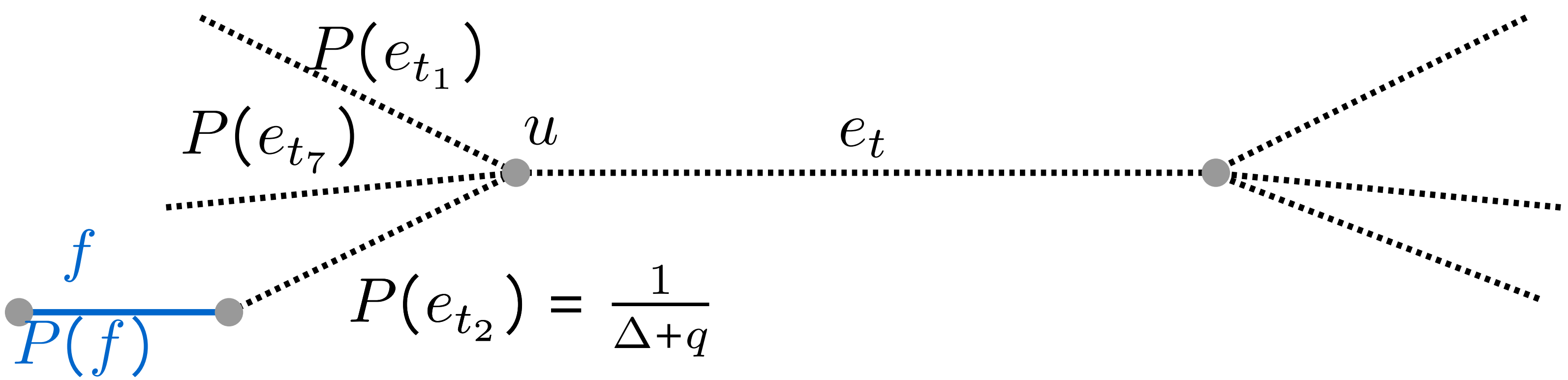

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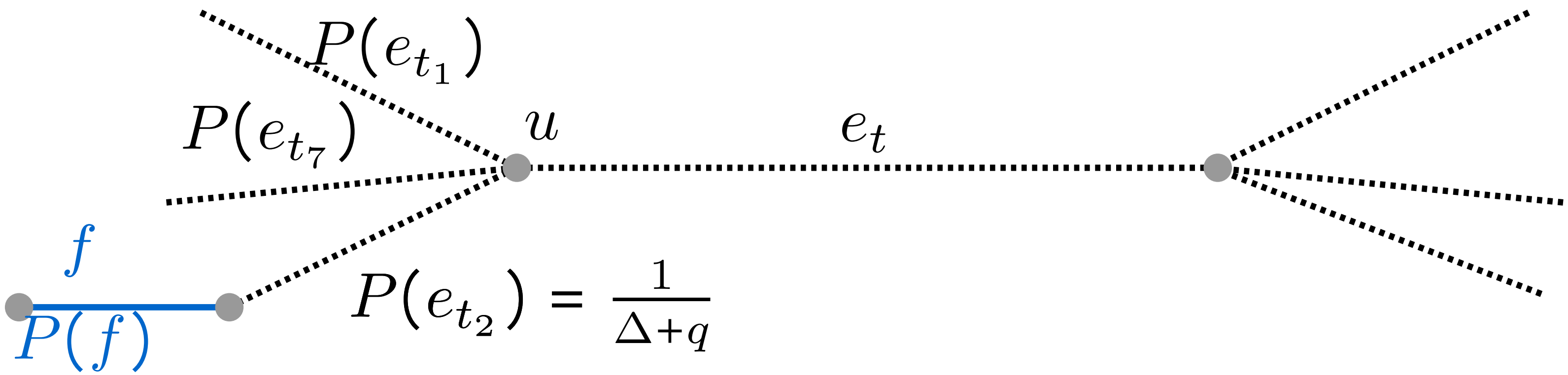

If f matched $\implies P^{new}(e_{t_2}) \leftarrow 0$

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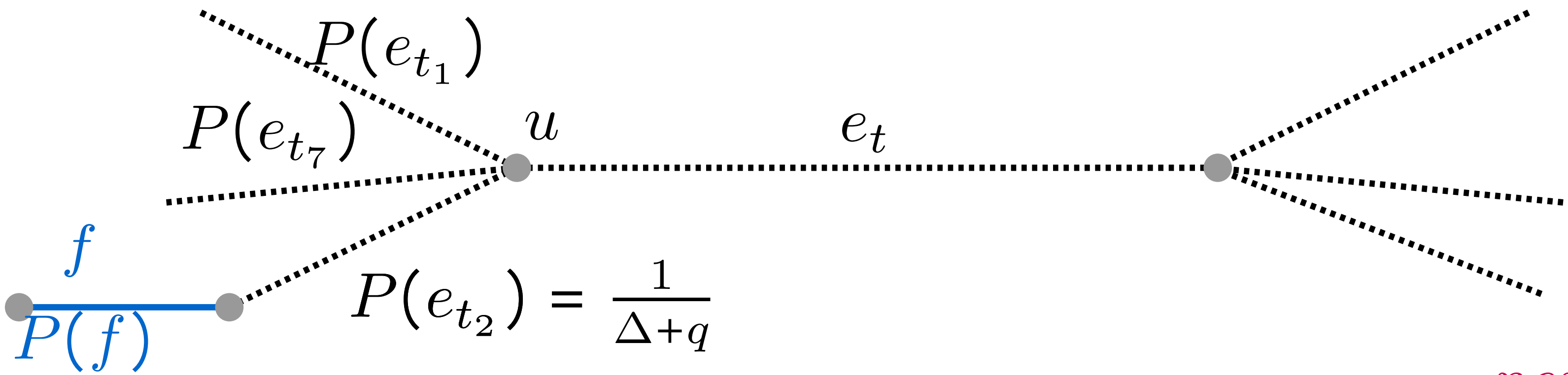

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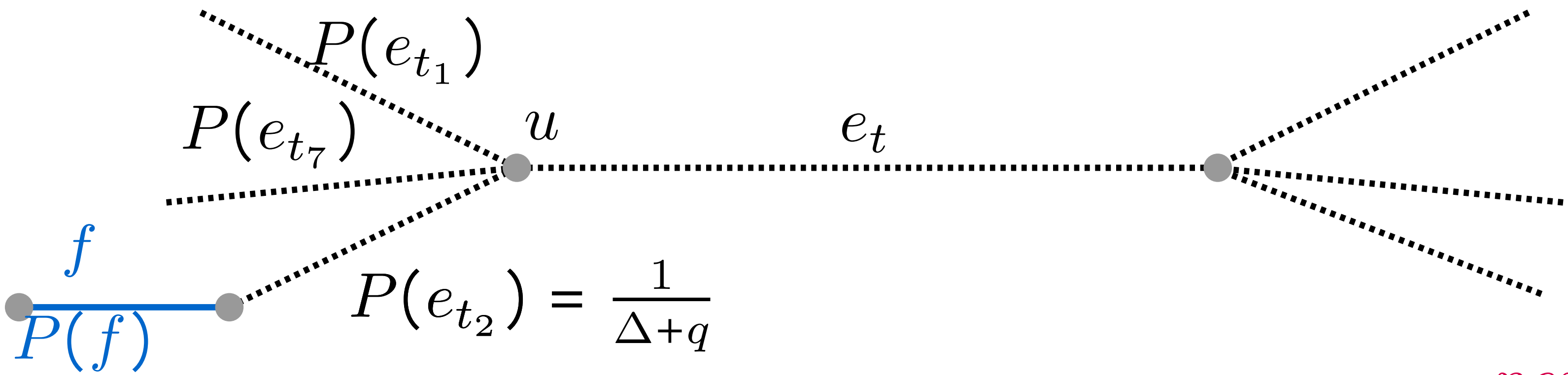

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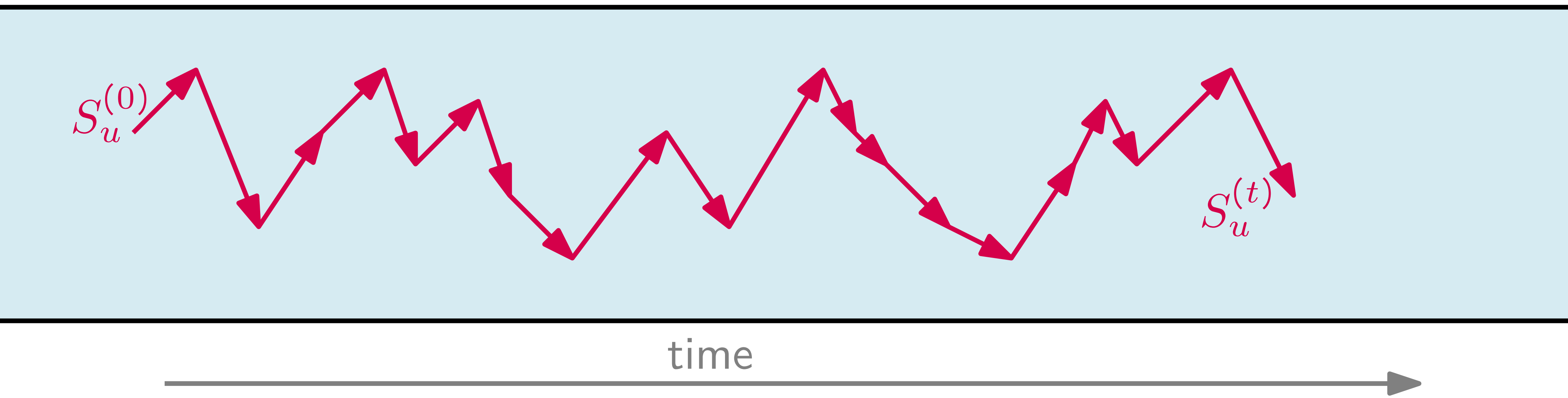

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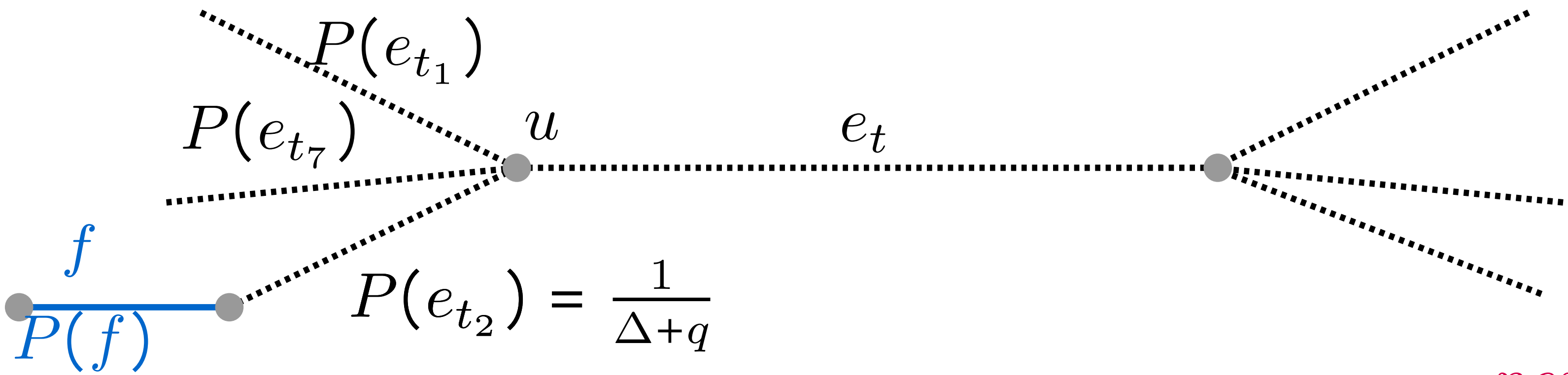

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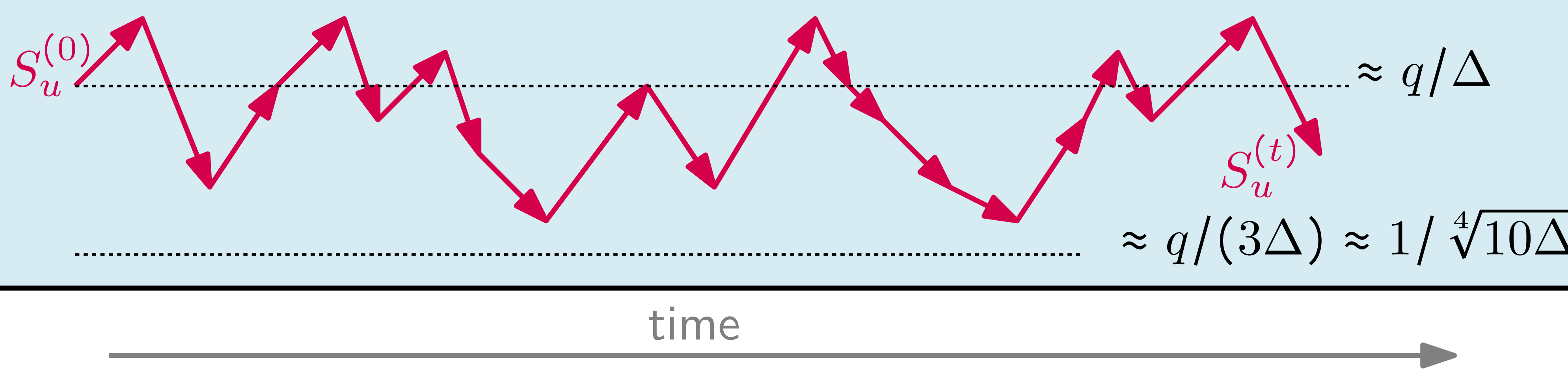
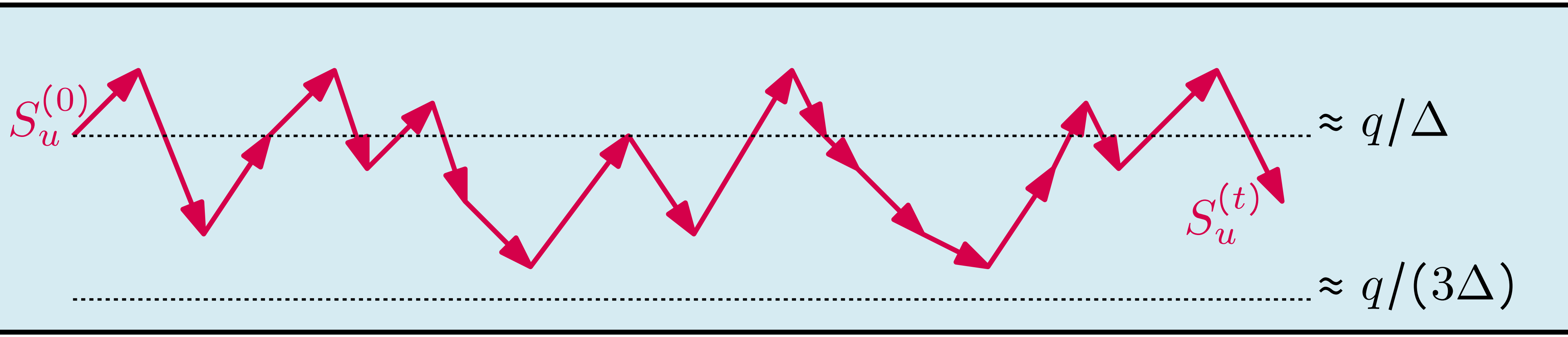


Martingale Process



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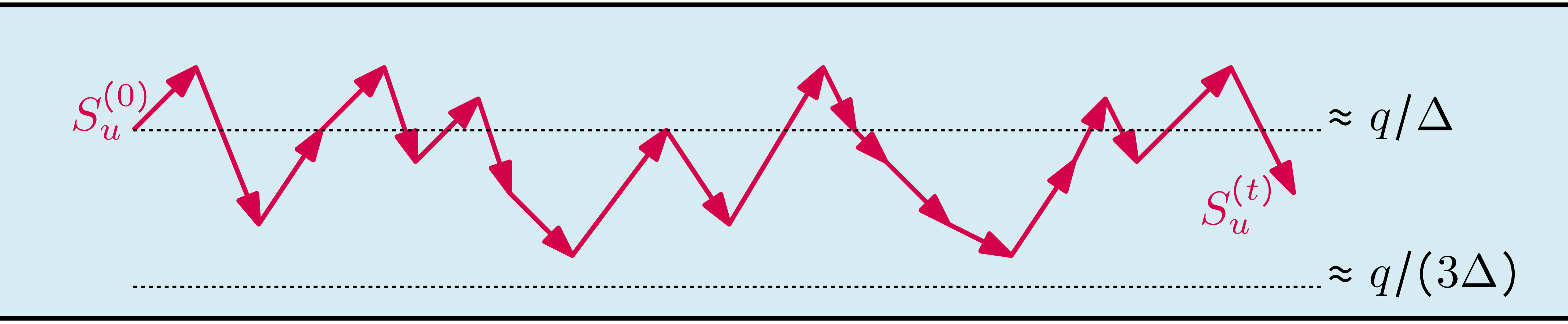

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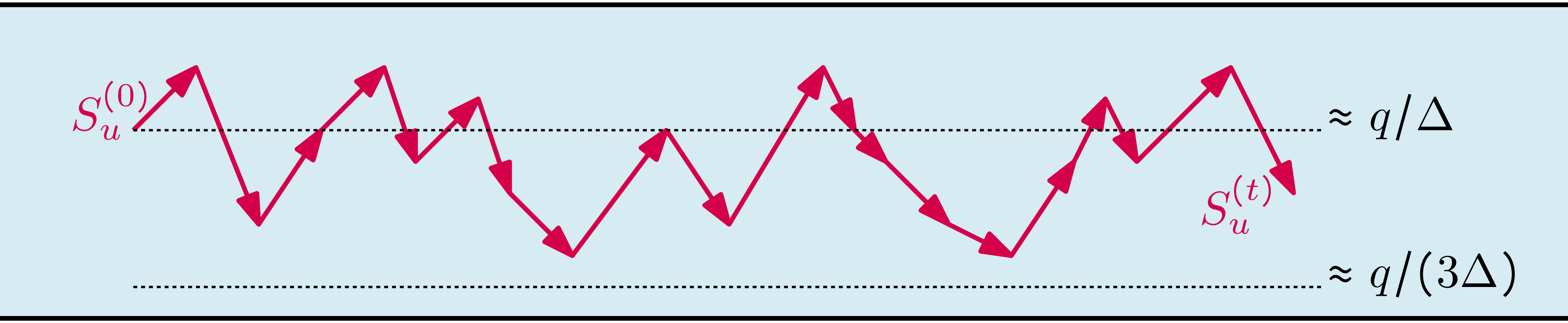

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- Martingale $\mathbb{E}[Z_{t+1} - Z_t \mid Z_1, Z_2, \ldots, Z_t] = 0$
- Step size $|Z_{t+1} - Z_t| \leq A$
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$$\implies \Pr\left[|Z_t - Z_0| \geq \varepsilon\right] \leq 2\exp\left(-\frac{\varepsilon^2}{2(\sigma^2 + A\varepsilon/3)}\right)$$

Fair Matching Result

Main Technical Result:
There is an online algorithm which outputs a random matching M so that

$$\Pr[e \in M] \geq \frac{1}{\Delta + q} \quad \forall e \in E, \quad \text{where} \quad q = O(\Delta^{3/4}\sqrt{\log \Delta})$$

Summary and Open Problems

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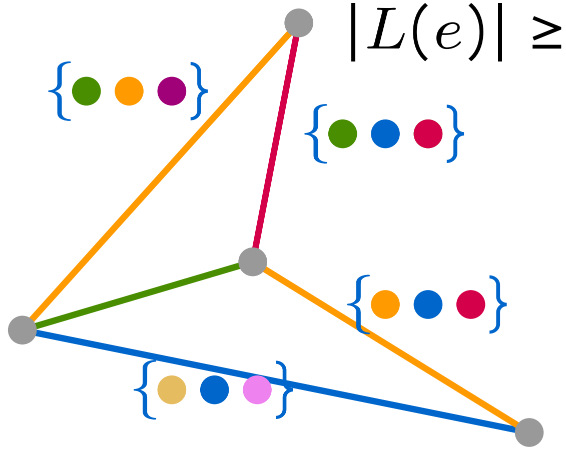

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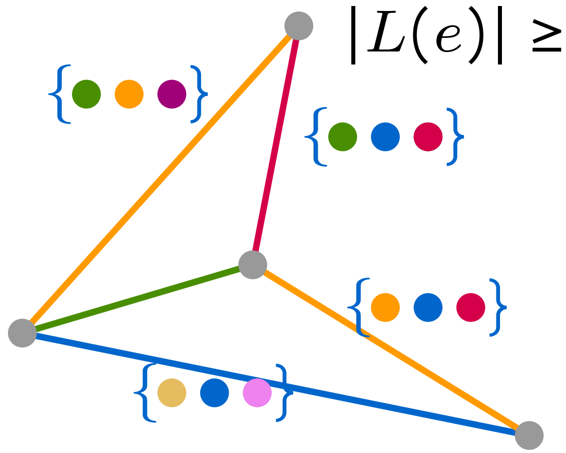
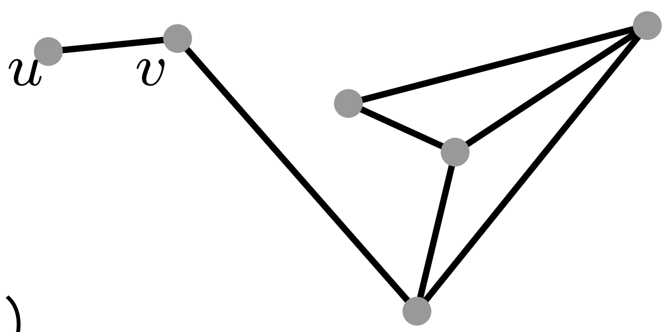


Local edge coloring



color(u, v)
$\leq (1 + o(1)) \max(\deg(u), \deg(v))$
$+O(\log n)$

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Given (online) fractional matching $x \in \mathbb{R}^E$ satisfying $x_e \leq \varepsilon^5$,
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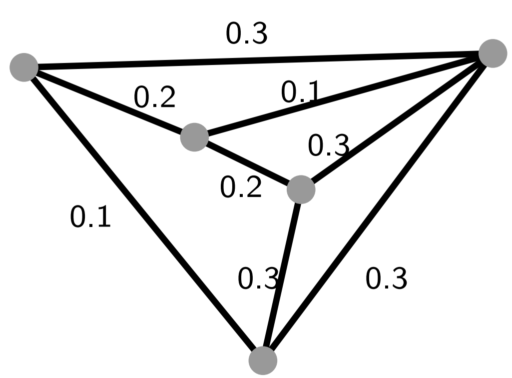

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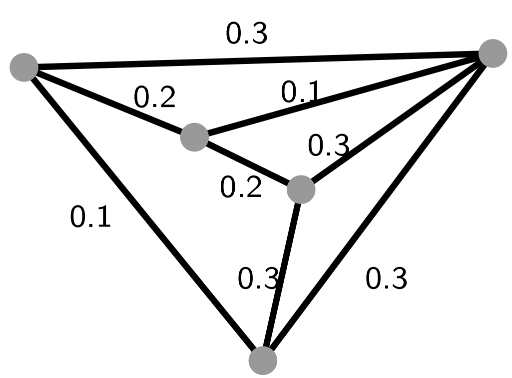

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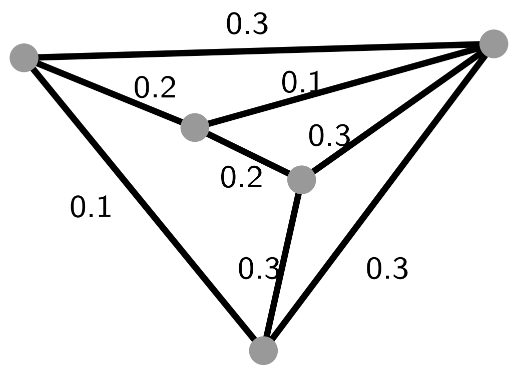

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$x_e := \frac{1}{\Delta}$ recovers fair matching theorem

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  - (Or equiv. randomized vs adaptive aversary)
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