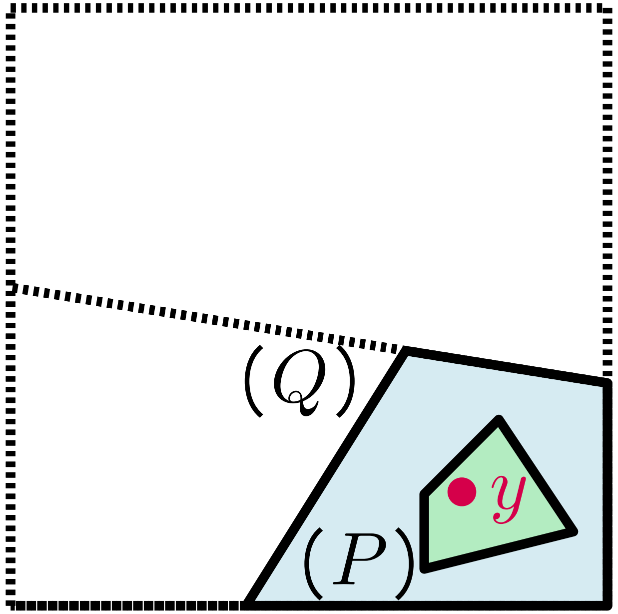
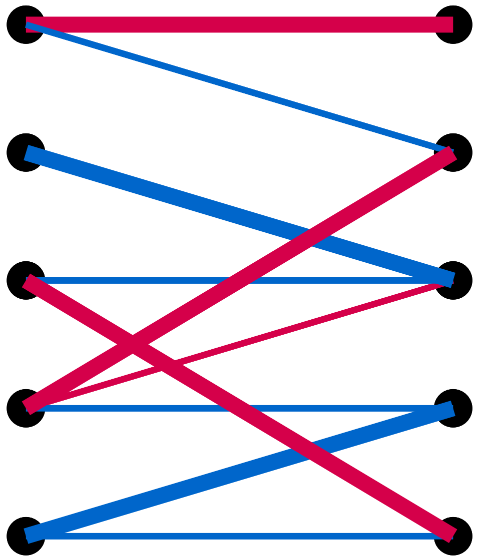
Nearly Optimal Communication and Query Complexity of Bipartite Matching

Joakim Blikstad

KTH Royal Institute of Technology





To appear in FOCS'22

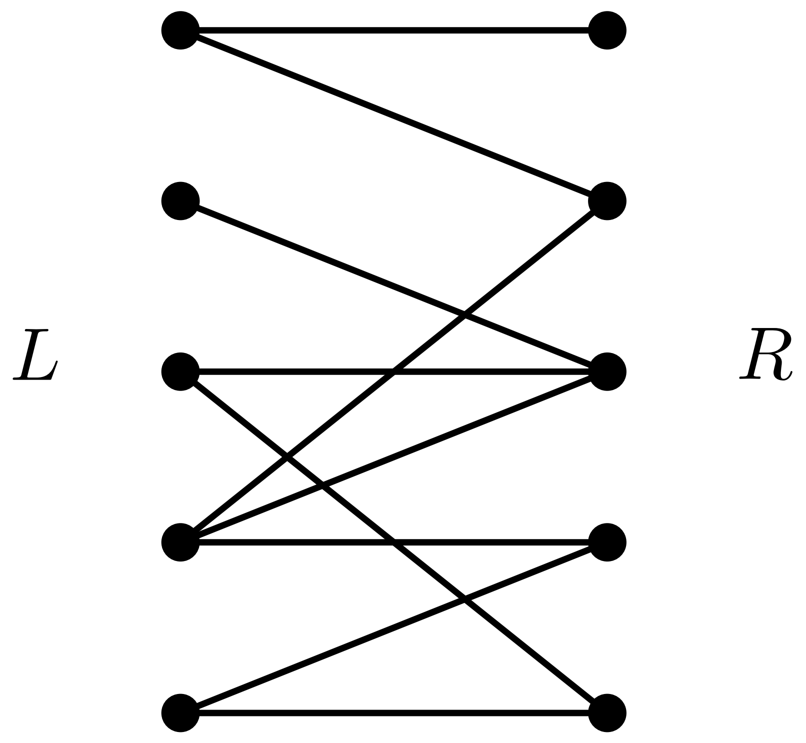
Joint work with: Jan van den Brand, Yuval Efron, Danupon Nanongkai, and Sagnik Mukhopadhyay.

TCS+ talk, fall 2022

Biparite Matching

Given: Graph $G = (L \cup R, E)$ with $|L| = |R| = n$, $|E| = m$

Goal: Find a maximum matching $M \subseteq E$ of G



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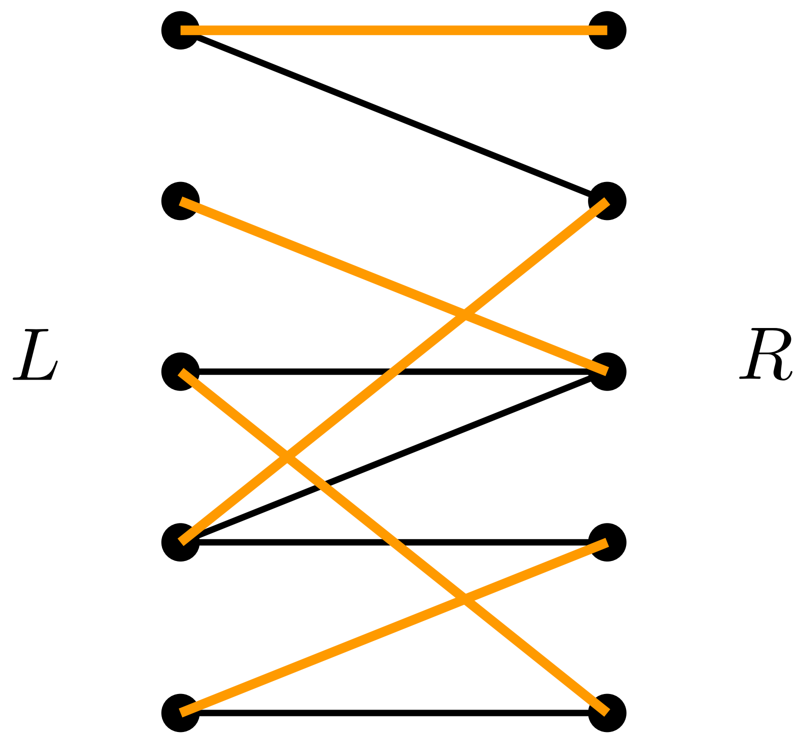


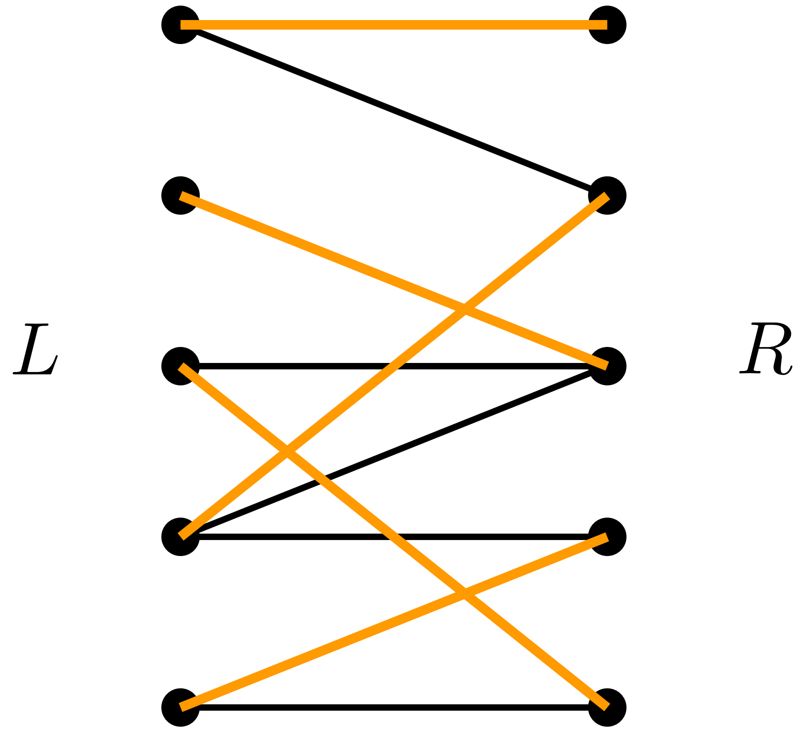
Biparite Matching

Given: Graph $G = (L \cup R, E)$ with $|L| = |R| = n$, $|E| = m$

Goal: Find a maximum matching $M \subseteq E$ of G

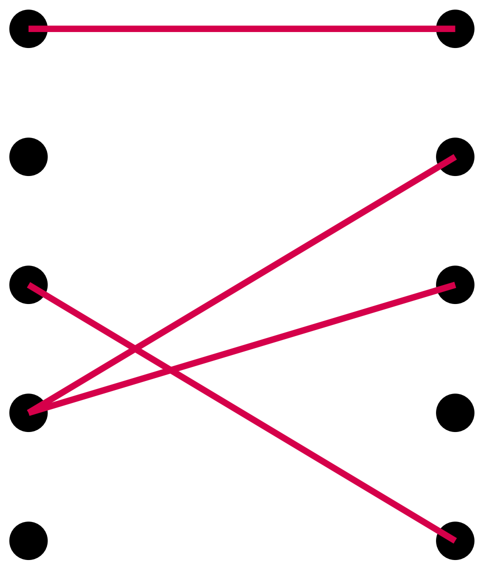
Solve (sequentially) in:

- $\tilde{O}(m + n\sqrt{n})$ [vdBLNPSSSW'20]
- $O(m^{1+o(1)})$ [CKLPGS'22]



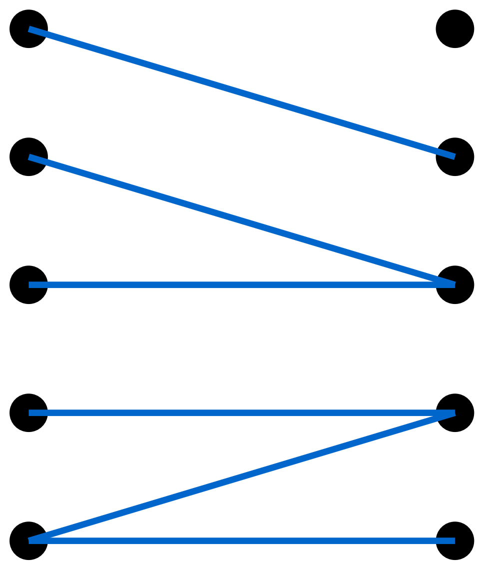
Two-Party Communication Model

Alice



E_A

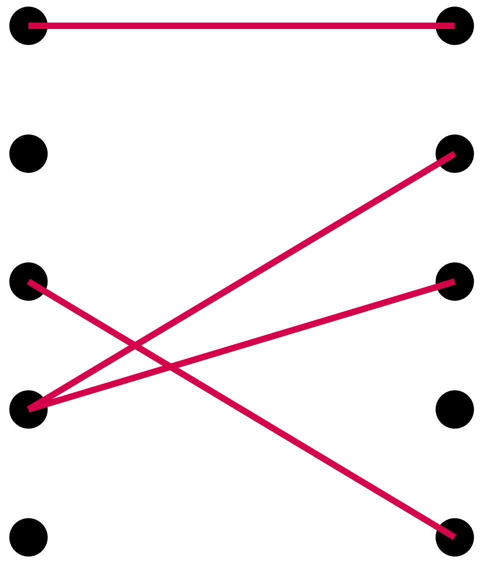
Bob



E_B

Two-Party Communication Model

Alice **Bob**



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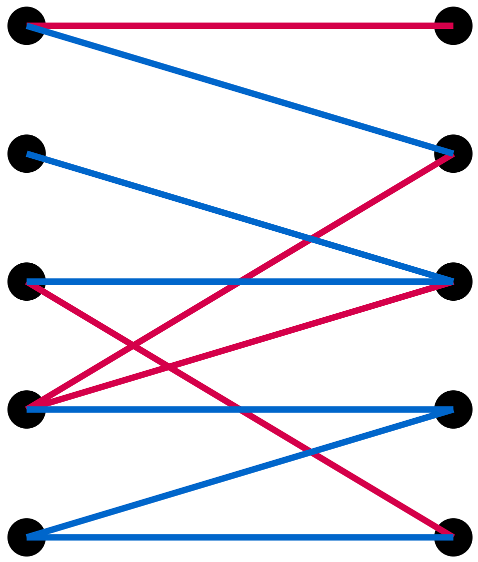
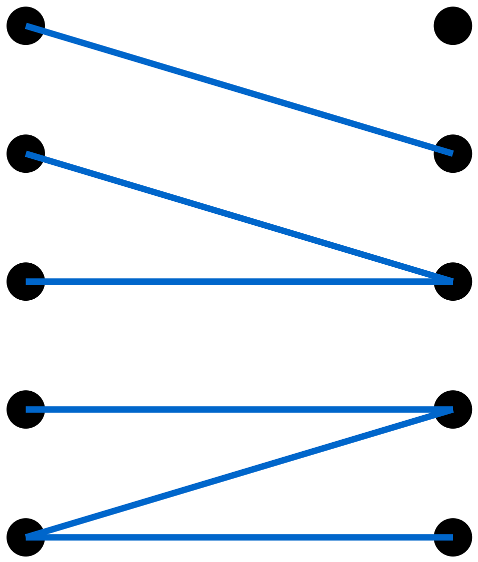


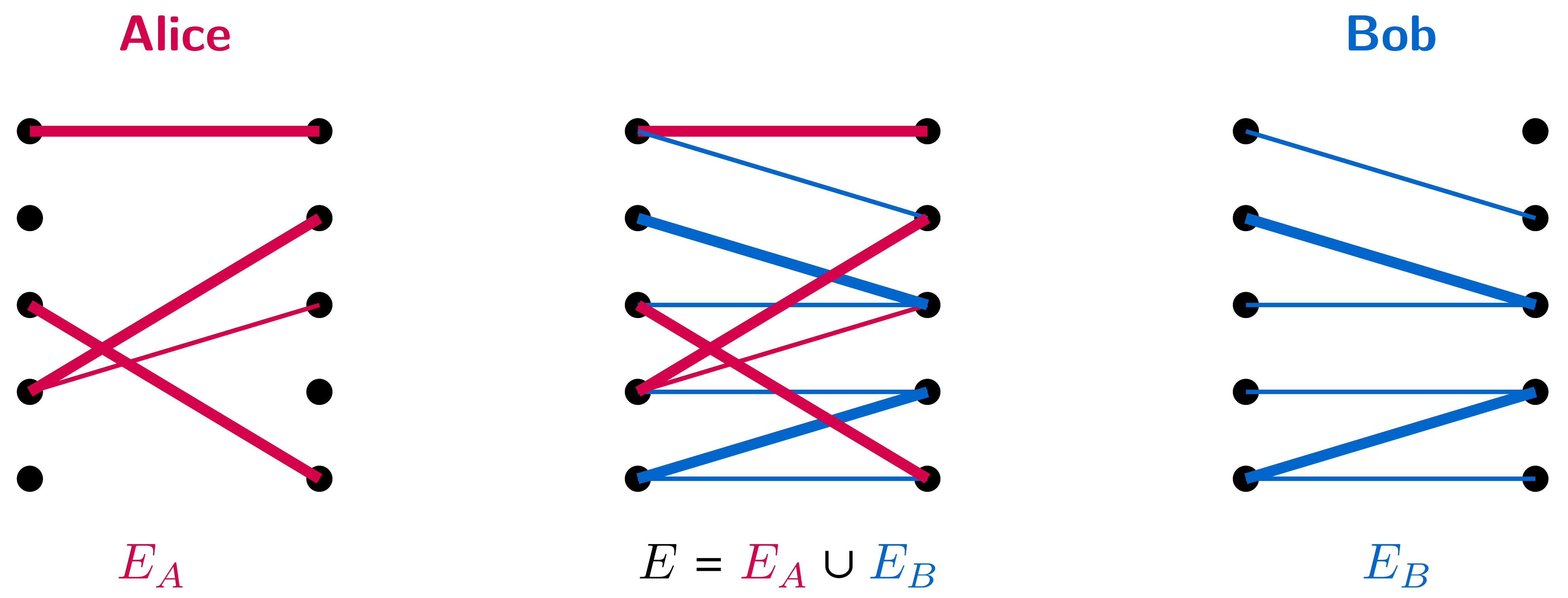
$E = E_A \cup E_B$



E_B

Goal: Solve matching on the union of their graphs

Two-Party Communication Model



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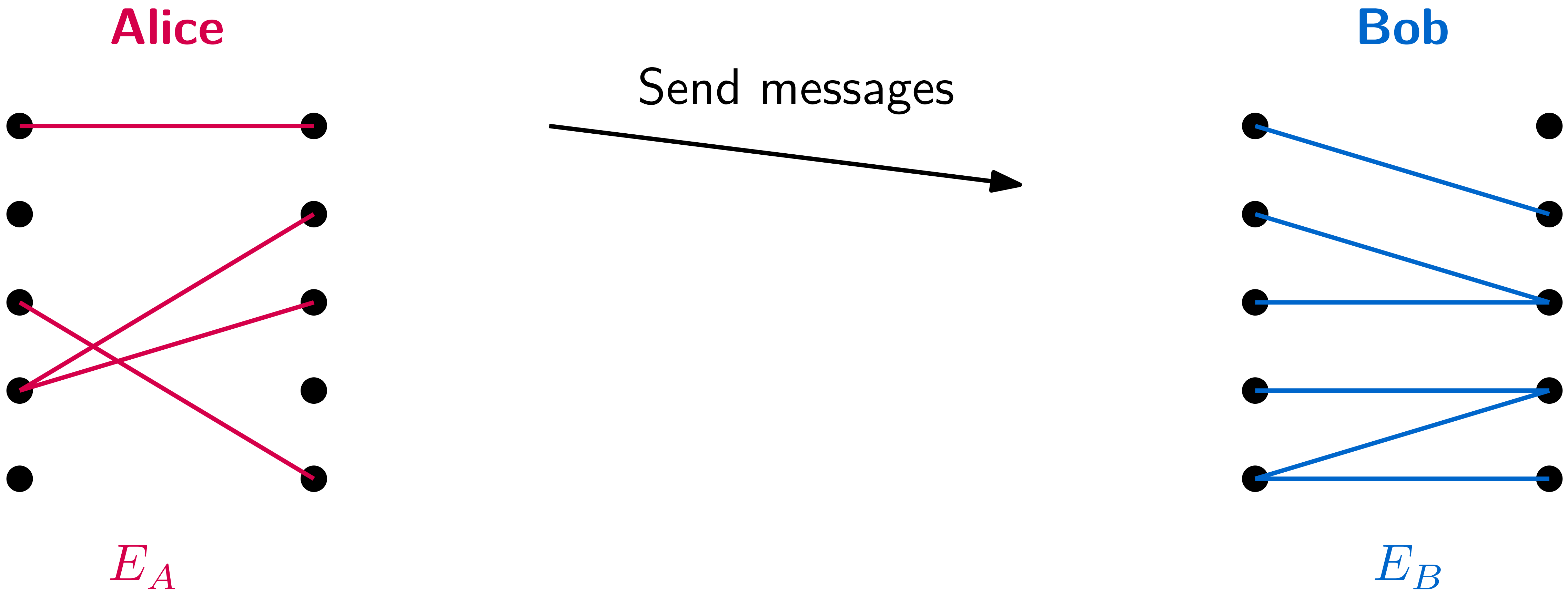

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With as few bits of communication!

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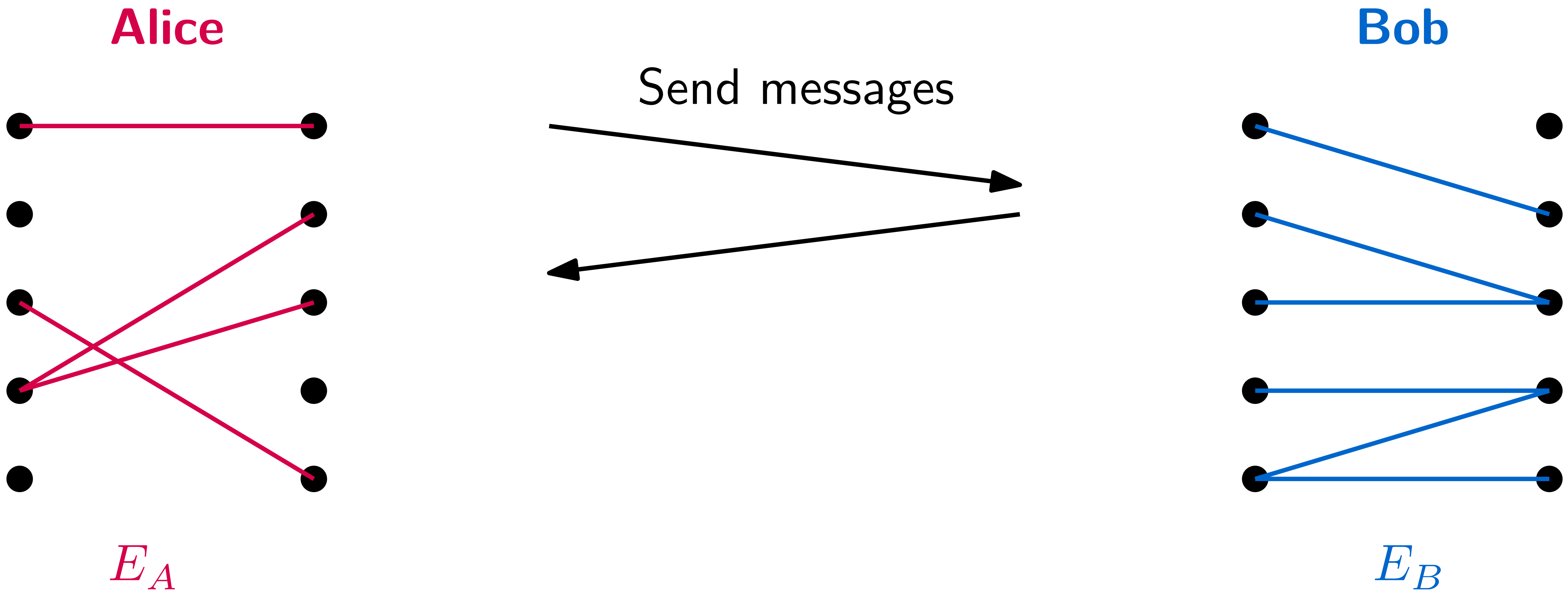

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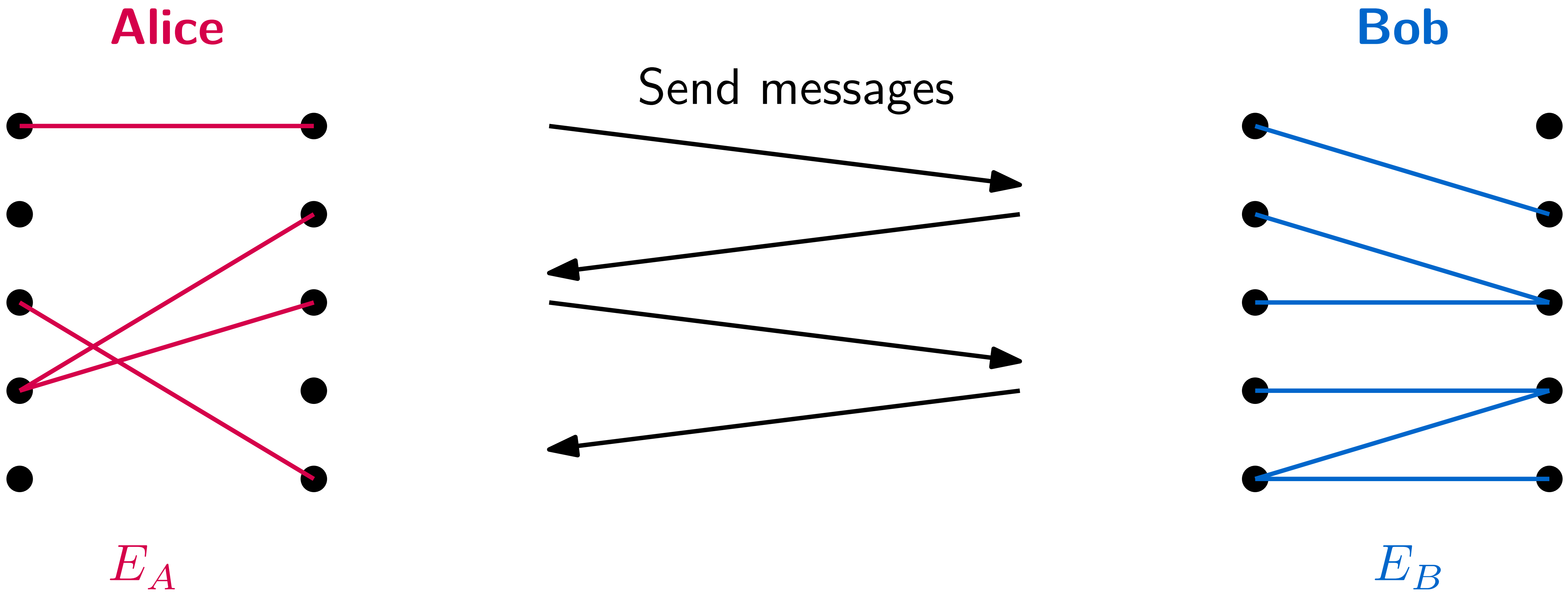

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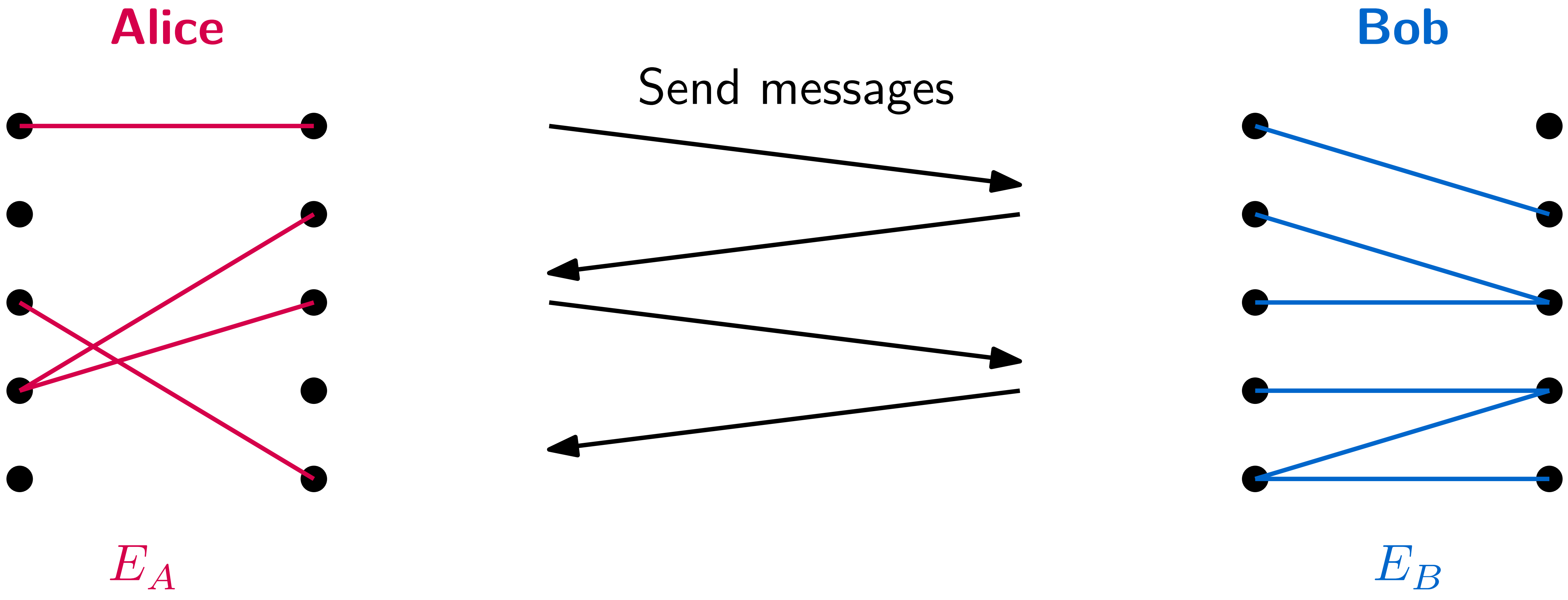

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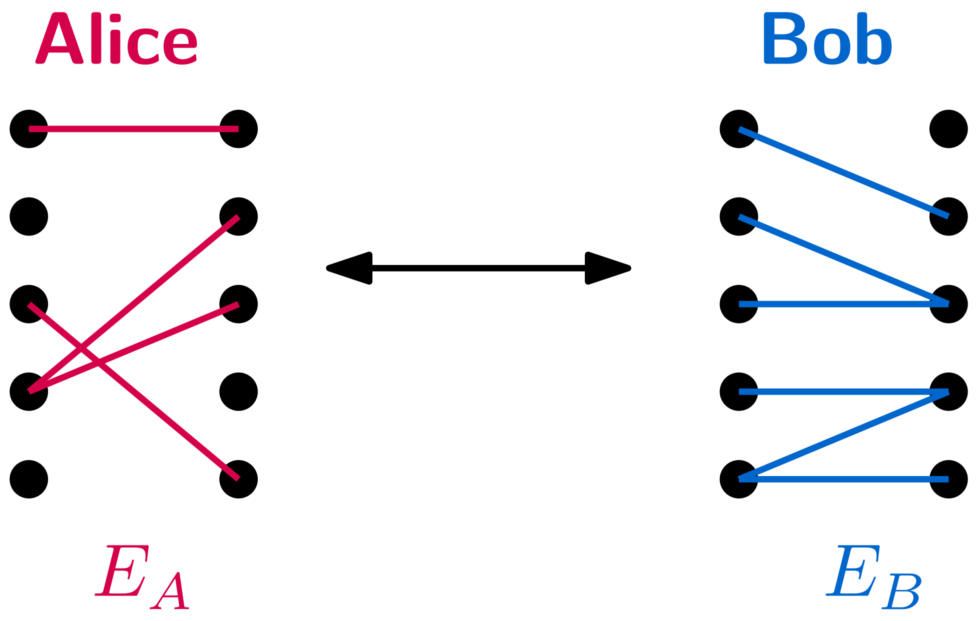
Note: Do not care about internal running time

First tries — Upper Bounds



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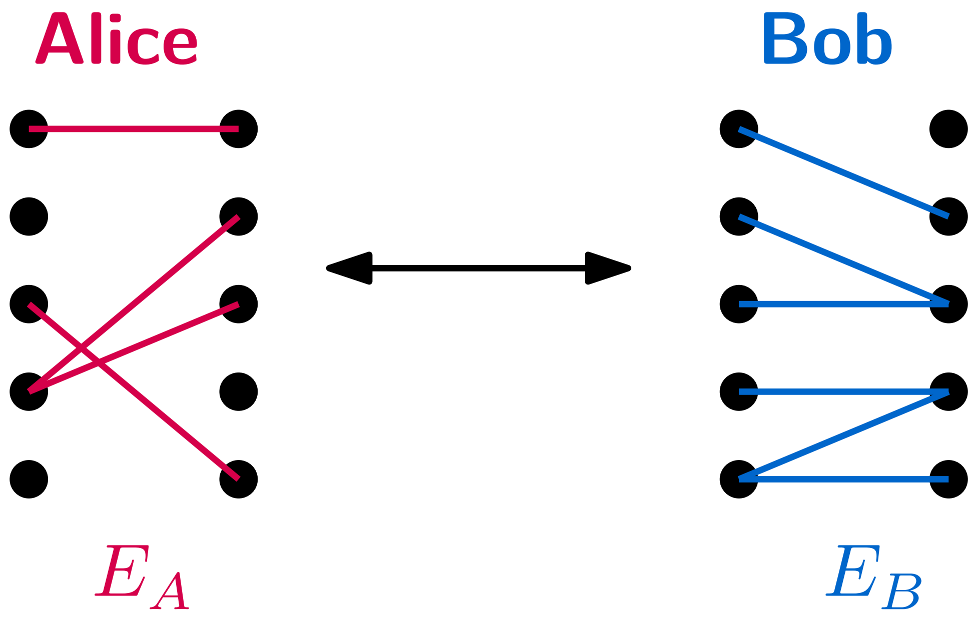
Sending an edge: $O(\log n)$ bits



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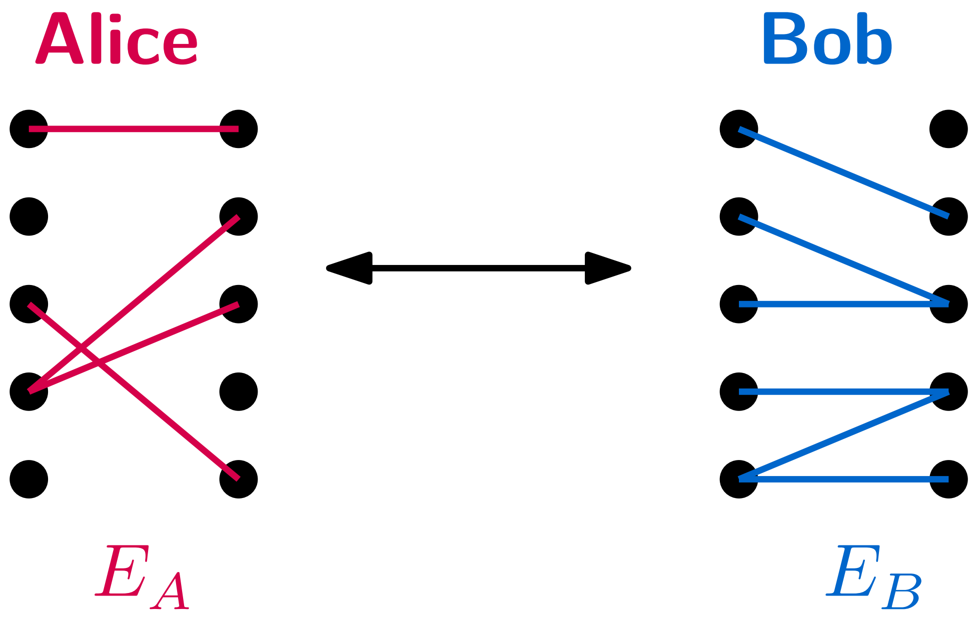
Trivial Protocol:

Alice sends all her edges to Bob:

- $O(m \log n)$
- $O(n^2)$



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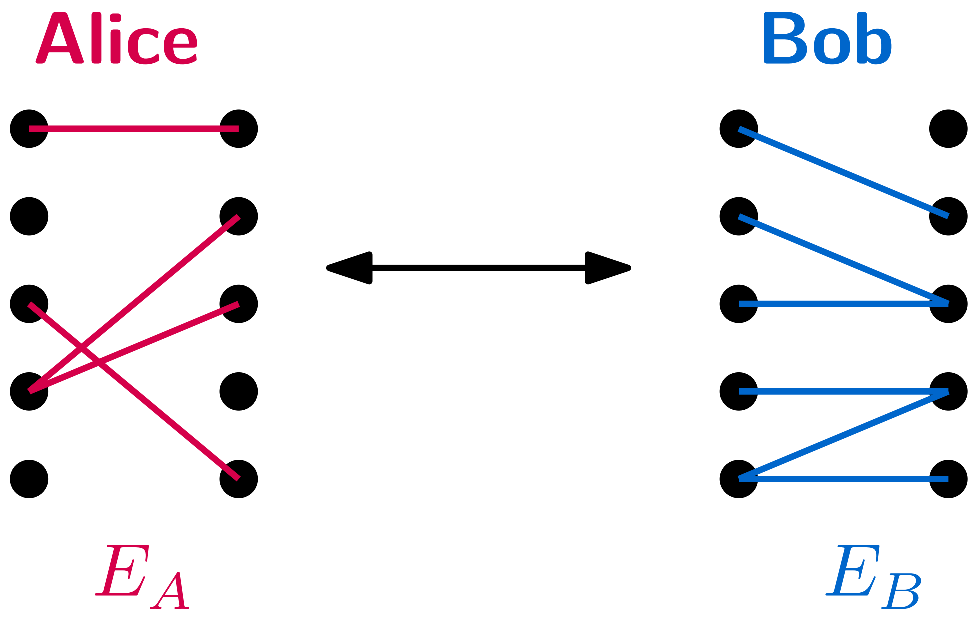

Hopcroft-Karp: (Blocking-Flow)

- Sequential: $O(m\sqrt{n})$ running time

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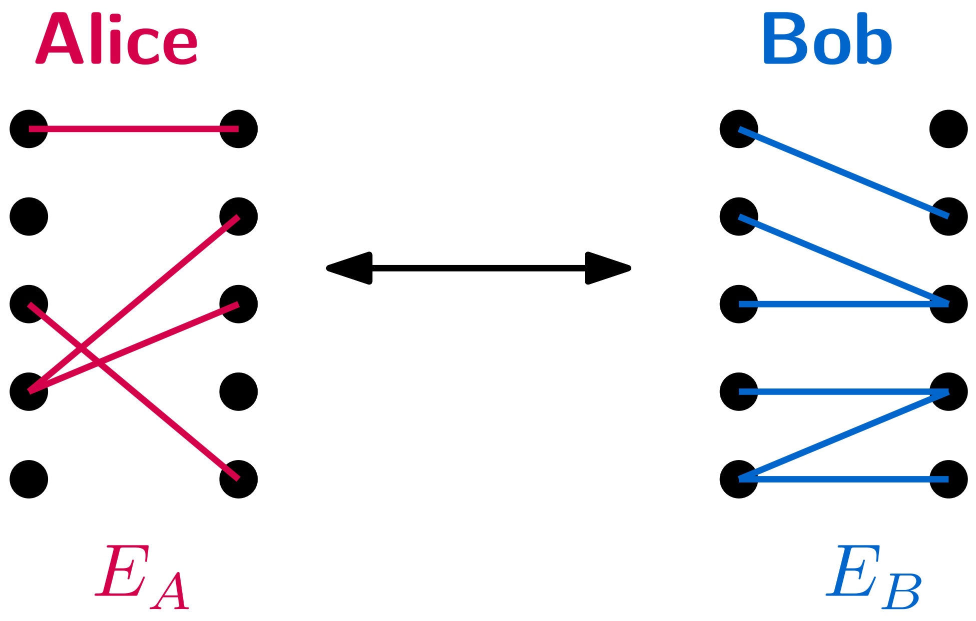

Hopcroft-Karp: (Blocking-Flow)

- Sequential: $O(m\sqrt{n})$ running time
- Communication: $O(n\sqrt{n} \log n)$ bits

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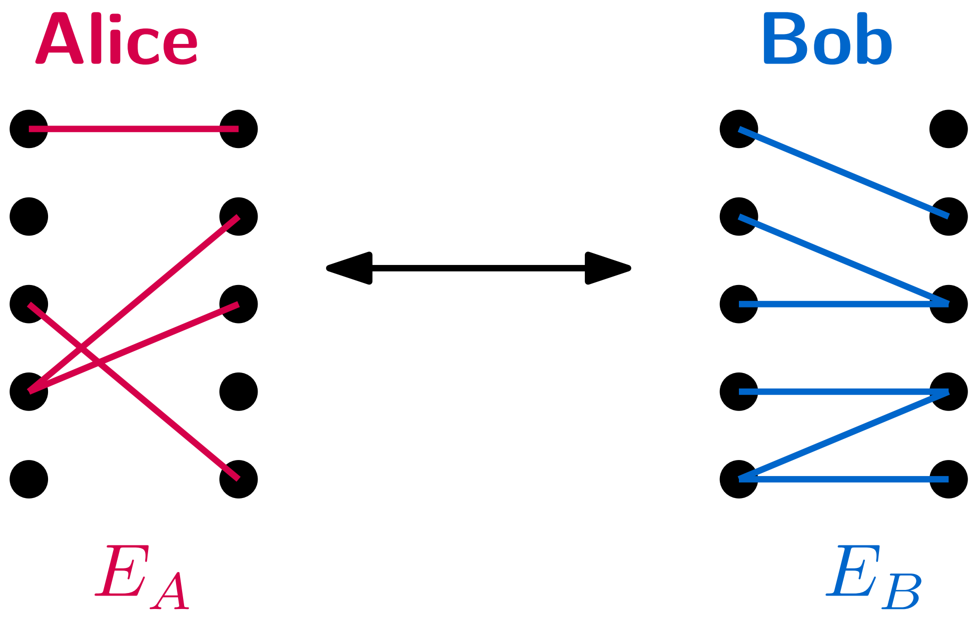

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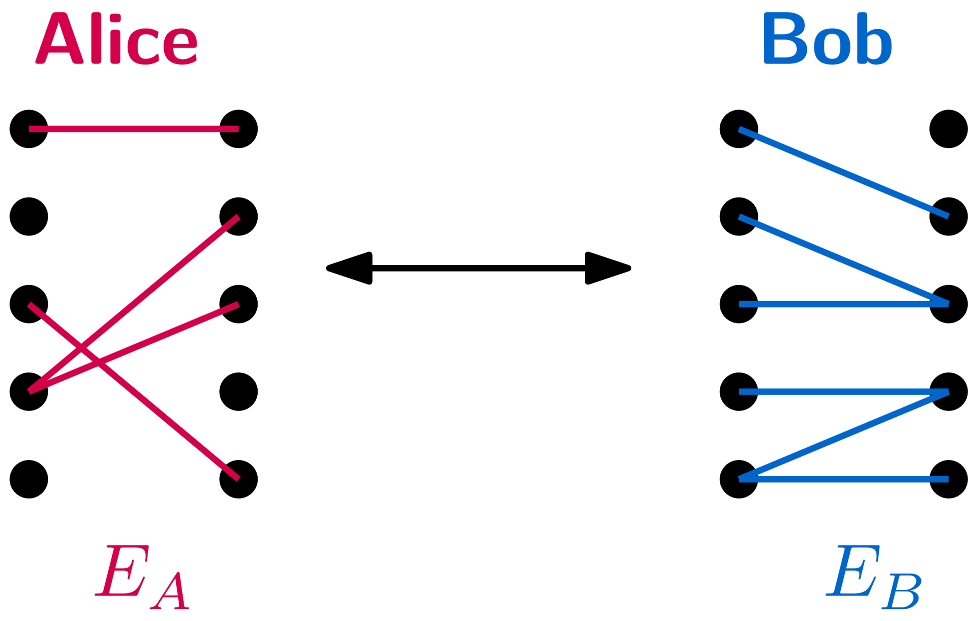

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- Sequential: $O(m\sqrt{n})$ running time
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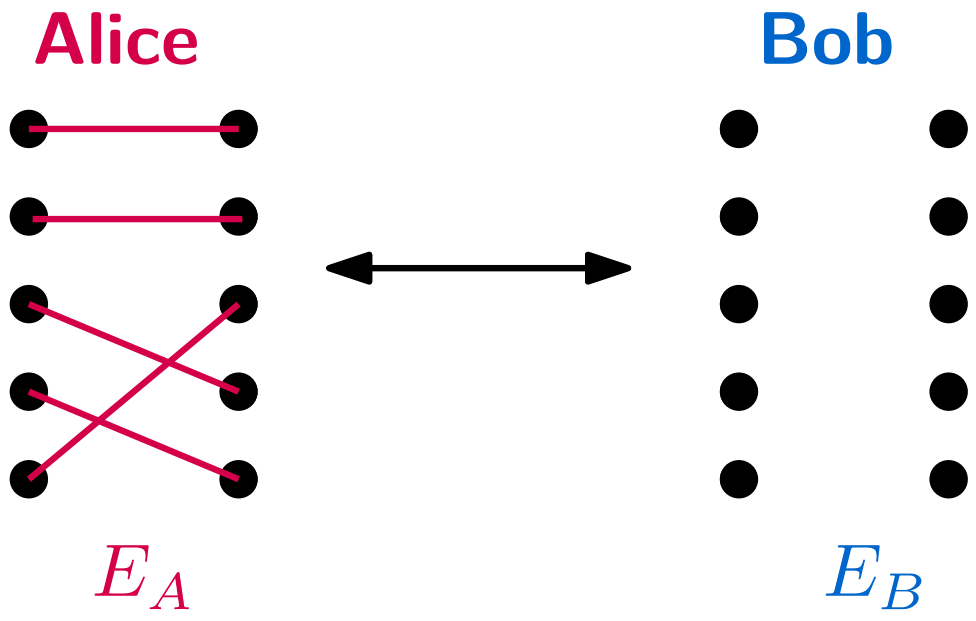
Idea: BFS / DFS need only $O(n \log n)$ bits of communication

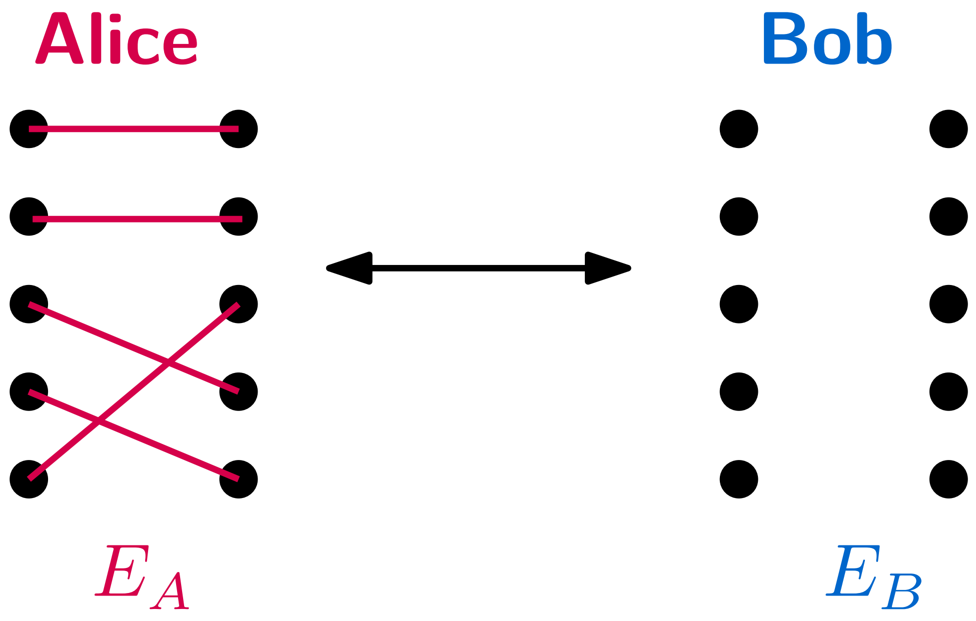
Converting $O(m^{1+o(1)})$ sequential $\longrightarrow$ $O(n^{1+o(1)})$ communication seems difficult

Lower Bounds



Lower Bounds

If Bob needs to output the matching:
$\Omega(n \log n)$ bits lower bound



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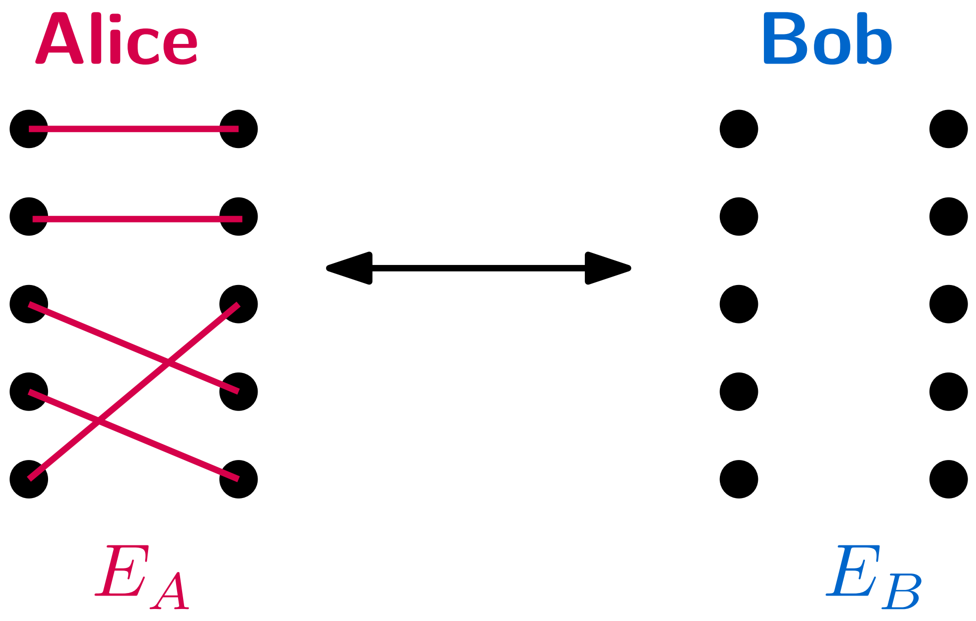

Theorem: [HMT'88]
$\Omega(n \log n)$ bits are needed to output the **size** of the maximum matching

$\uparrow$ only deterministic

$\Omega(n)$ randomized

$\Omega(n \log n)$ $O(n\sqrt{n} \log n)$

Major Question†**:** What is the Communication Complexity of Bipartite Matching?

† [Hajnal, Maass, Turan STOC'88];[Ivanyos, Klauck, Lee, Santha, de Wolf FSTTCS'12]; [Dobzinski, Nisan, Oren STOC'14]; [Nisan SODA'21]; [Beniamini, Nisan STOC'21]; [Zhang ICALP'04]

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Main Result:
One can solve bipartite matching in $O(n \log^2 n)$ bits of communication.

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Main Result:
One can solve bipartite matching in $O(n \log^2 n)$ bits of communication.

Highlights:

- Follow from **simple** applications of known techniques (cutting planes method)
- Very slow runtime, but efficient communication
- Only "finds" $O(n \log n)$ edges

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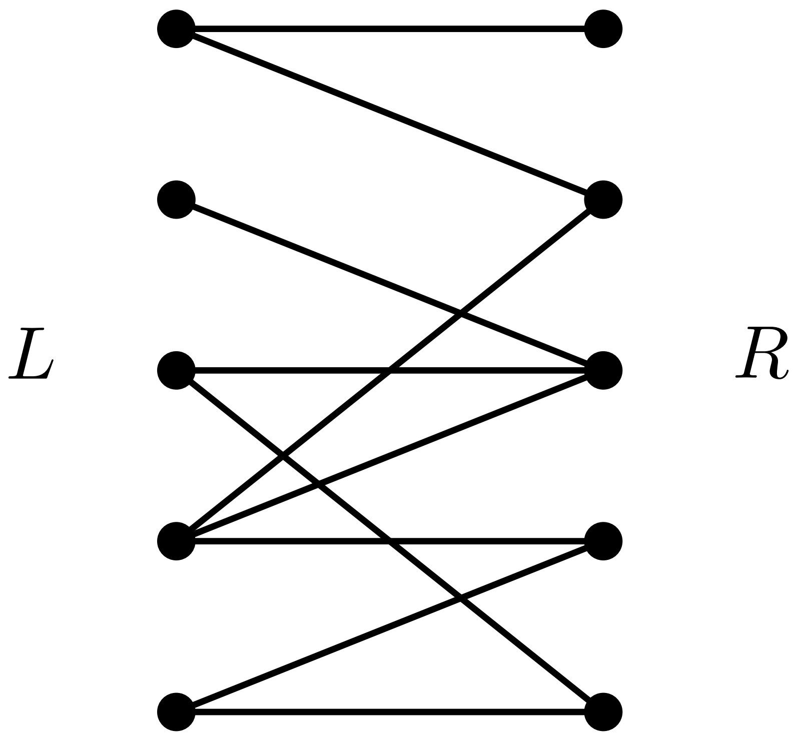
Query Models

- Hidden biparite graph $G = (L \cup R, E)$



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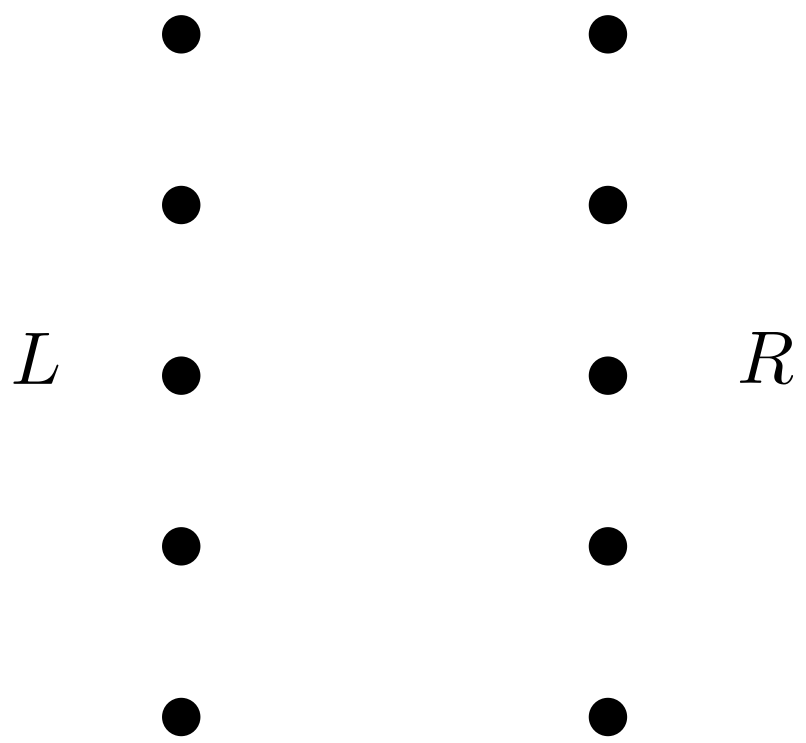

Query Models

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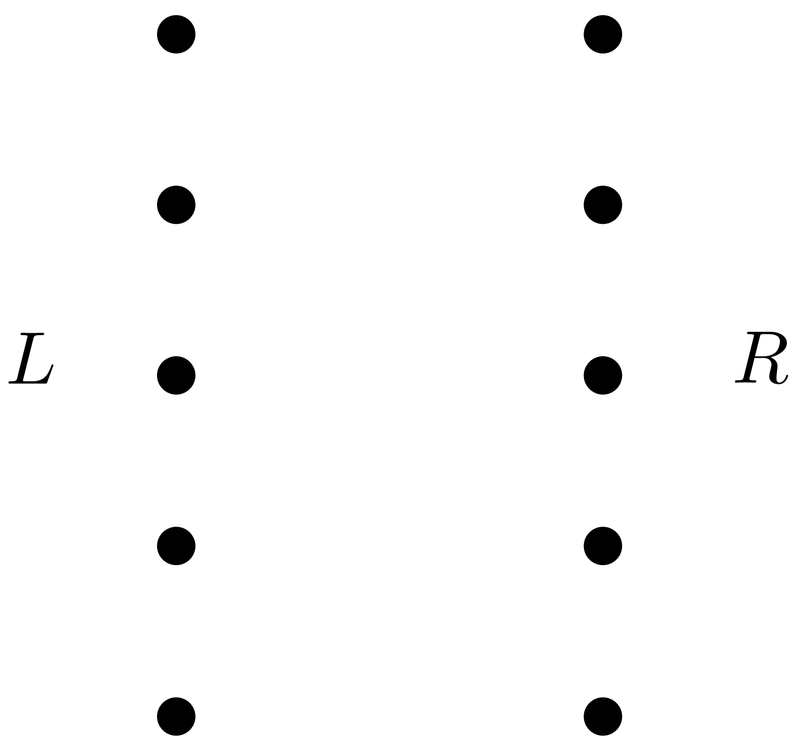

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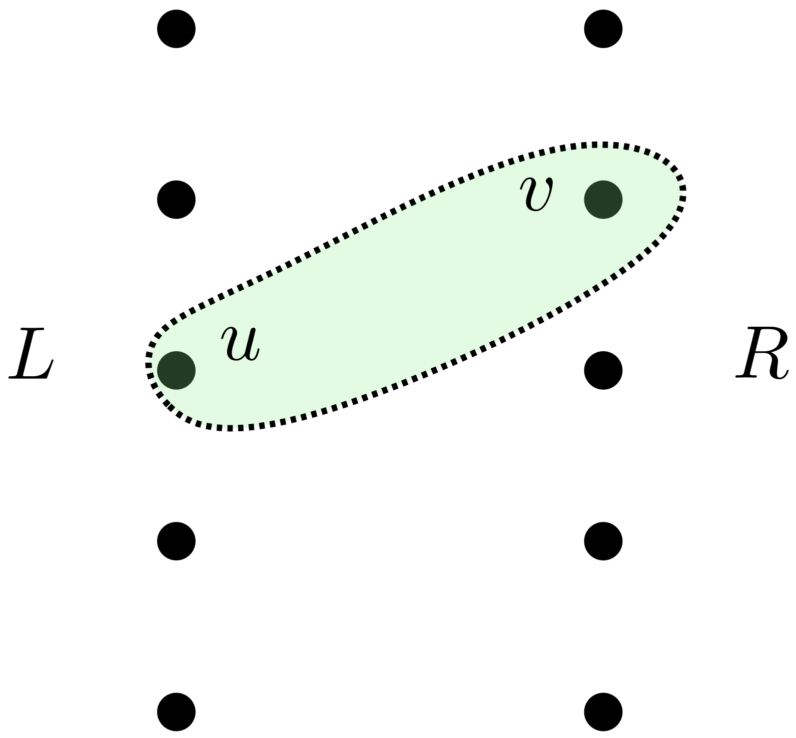

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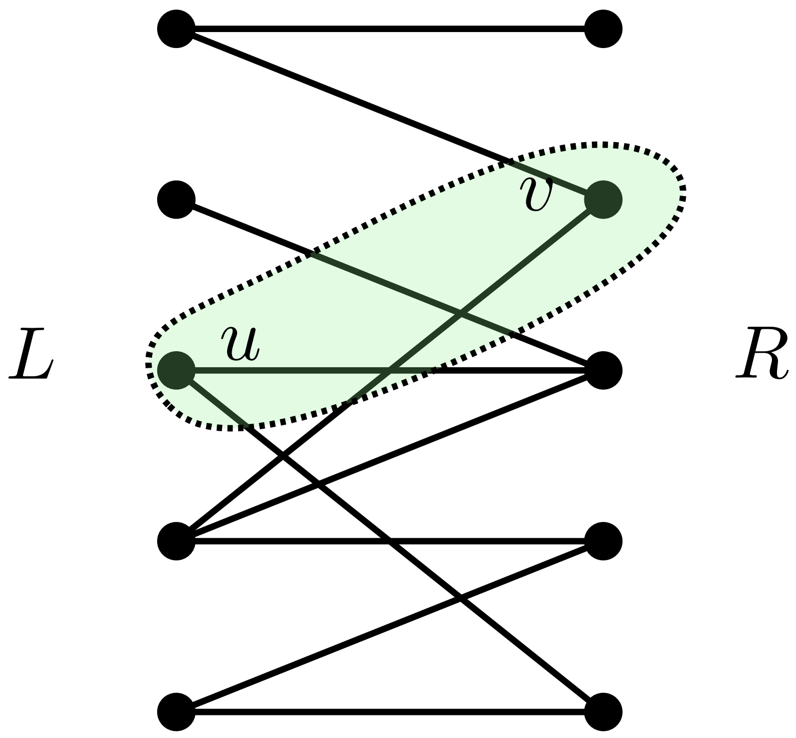

Query Models

$(S \subseteq L \times R)$

- Hidden biparite graph $G = (L \cup R, E)$
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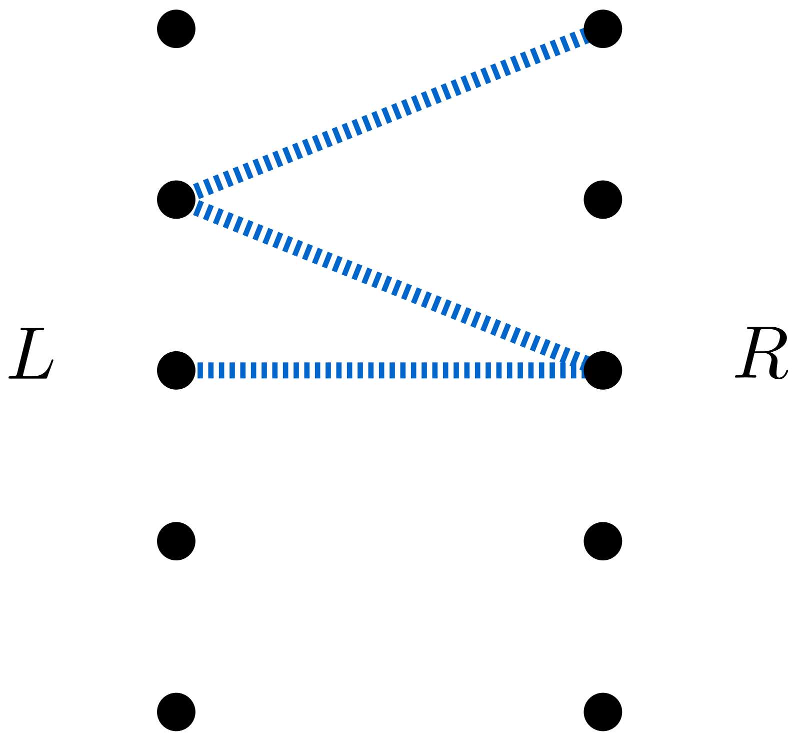
  - Edge-Query: "Is $(u, v) \in E$?"
  - OR-Query: "Is $|S \cap E| \geq 1$?" "YES"
  - XOR-Query: "Is $|S \cap E|$ odd?" "NO"
  - AND-Query: "Is $|S \cap E| = |\mathrm{S}|$" "NO"



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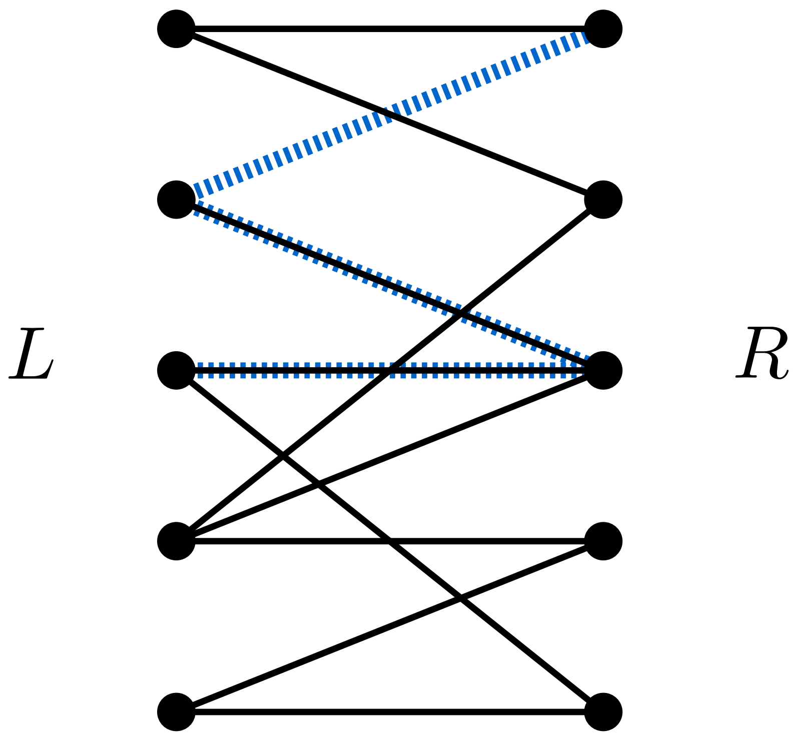

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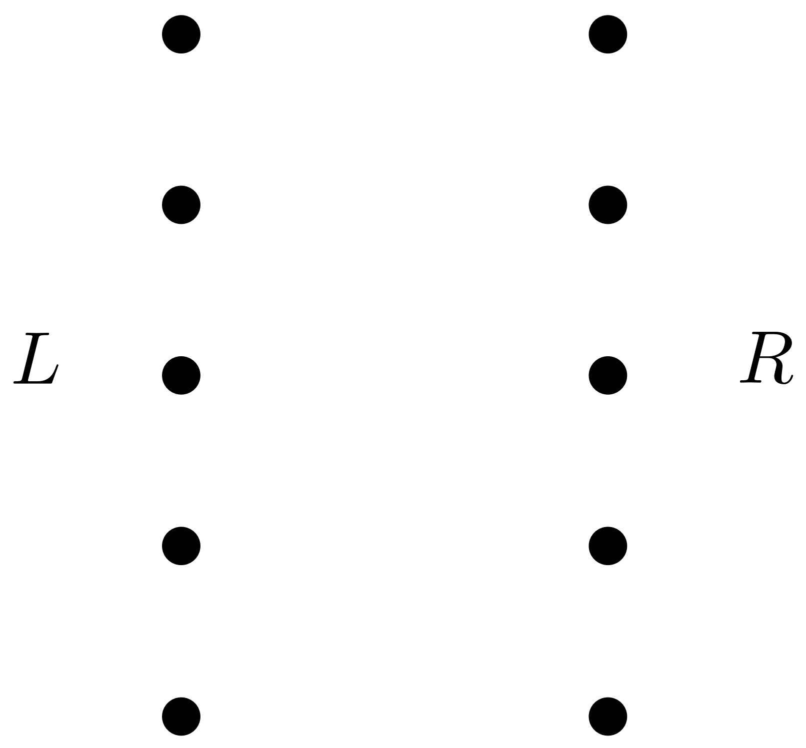
  - Quantum-Edge-Query



Query Models

- Hidden biparite graph $G = (L \cup R, E)$
- Query access:

	Deterministic:	Randomized:
Edge-Query: "Is $(u,v) \in E$?"	$\Theta(n^2)$	$\Theta(n^2)$
OR-Query: "Is $\lvert S \cap E\rvert \geq 1$?"	$\tilde{\Omega}(n), \tilde{O}(n\sqrt{n})$	$\Omega(n), \tilde{O}(n\sqrt{n})$
XOR-Query: "Is $\lvert S \cap E\rvert$ odd?"	$\Theta(n^2)$	$\Omega(n), \tilde{O}(n\sqrt{n})$
AND-Query: "Is $\lvert S \cap E\rvert = \lvert S\rvert$"	$\Theta(n^2)$	$\Omega(n), O(n^2)$
Quantum-Edge-Query	—	$\tilde{\Omega}(n\sqrt{n}), \tilde{O}(n^{7/4})$

[Yao'88], [Zha'04], [DHHM'06], [IKLSdW'12], [LL'15], [BN'15], [Nis'15], [DNO'19], [Ben'22]

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Quantum-Edge-Query	—	$\tilde{\Theta}(n\sqrt{n})$

Green: new tight upper-bound!

Red: new tight lower-bound!

[Yao'88], [Zha'04], [DHHM'06], [IKLSdW'12], [LL'15], [BN'15], [Nis'15], [DNO'19], [Ben'22]

The Algorithms

Our Algorithms

Key Idea: Apply Cutting Planes Method to the Dual Vertex Cover LP.

Think "Ellipsoid Method"

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Think "Ellipsoid Method"

- [Vempala, Wang, Woodruff SODA'20]:
  Solving general LPs in Communication Model with Cutting Planes:
  $\tilde{O}(\mathrm{dimension}^3 \cdot \#\text{bits per constraint})$ communication

Our Algorithms

Key Idea: Apply Cutting Planes Method to the Dual Vertex Cover LP.

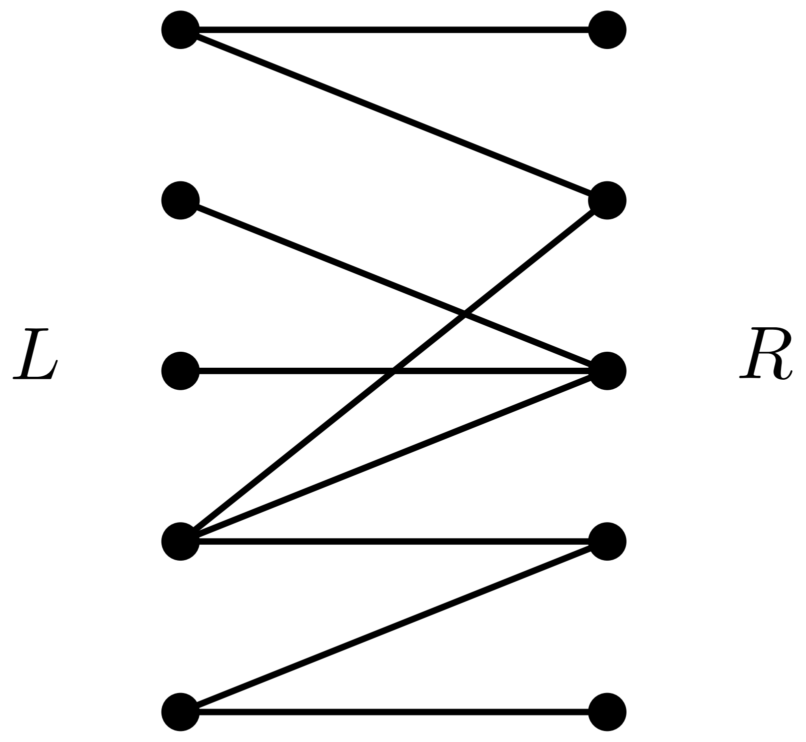
Think "Ellipsoid Method"

- [Vempala, Wang, Woodruff SODA'20]:
  Solving general LPs in Communication Model with Cutting Planes:
  $\tilde{O}(\mathrm{dimension}^3 \cdot \#\text{bits per constraint})$ communication
- Crucial properties of Dual Vertex Cover LP:
  - Low dimension (n instead of m)
  - Constraints are "short" (low support = cheap to send)
  - Volume is small
  - … but not too small

Dual: Minimum Vertex Cover

Given: Graph $G = (L \cup R, E)$ with $|L| = |R| = n$, $|E| = m$

Goal: Find smallest set C of vertices covering all edges



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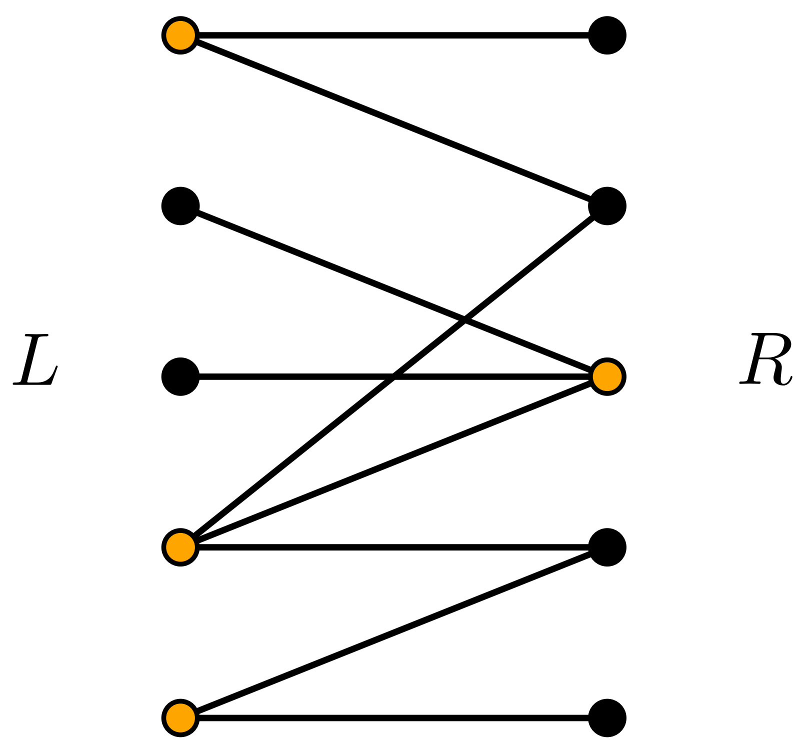

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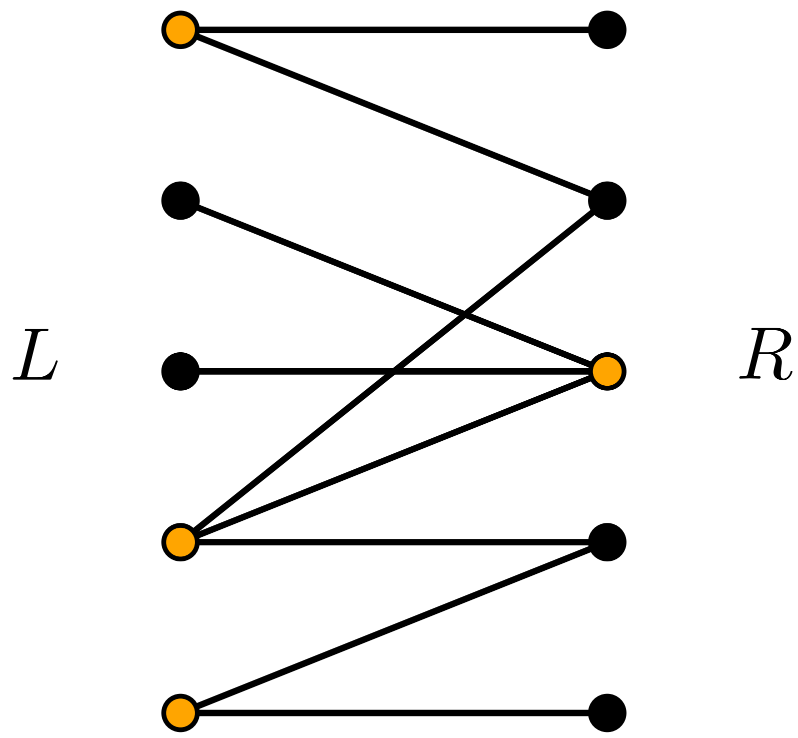
Kőnig's Theorem:
|max-matching| = |min-vertex cover|
(in bipartite graphs!)



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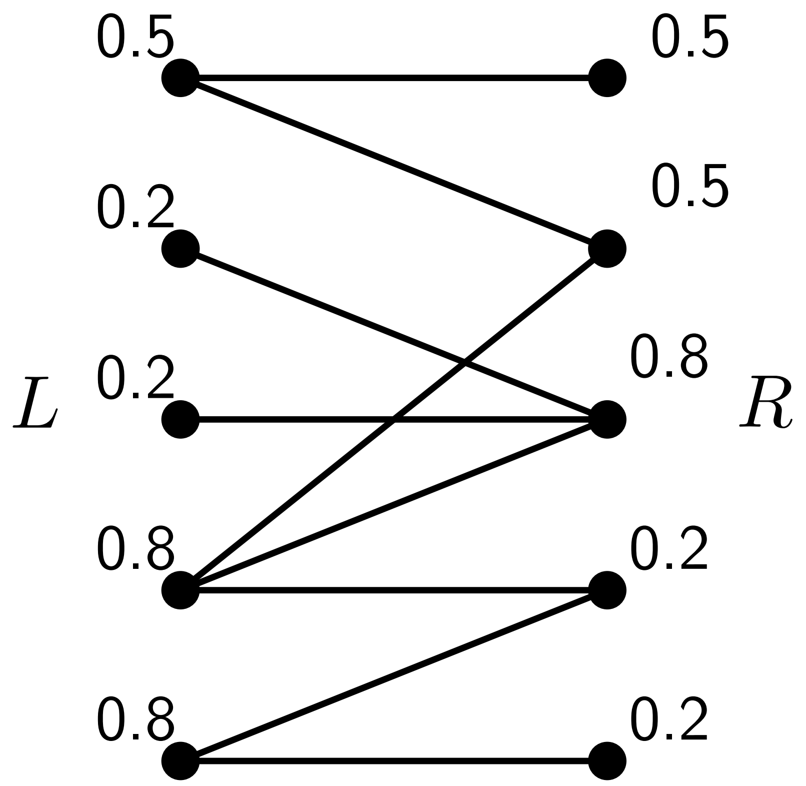

Kőnig's Theorem:
|max-matching| = |min-vertex cover|
(in bipartite graphs!)

Def: Fractional vertex cover x:
$x_u + x_v \geq 1$ for all edges (u, v)

Dual Linear Program: Minimum Vertex Cover

$$
\begin{array}{lll}
\min & \sum_{v \in V} x_v & \\
\text{s.t.} & x_v + x_u \geq 1 & \forall (u,v) \in E \\
 & 0 \leq x \leq 1 &
\end{array}
\qquad (P)
$$

Dual Linear Program: Minimum Vertex Cover

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\begin{array}{lll}
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\text{s.t.} & x_v + x_u \geq 1 \quad \forall (u,v) \in E_A & \\
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$$

Dual Linear Program: Minimum Vertex Cover

$$\begin{aligned} &\textstyle\sum_{v \in V} x_v \leq n - 1 \\ &x_v + x_u \geq 1 \quad \forall (u,v) \in E_A \\ &x_v + x_u \geq 1 \quad \forall (u,v) \in E_B \\ &0 \leq x \leq 1 \end{aligned} \qquad (P)$$

- (P) feasible $\iff$ No perfect matching exists

Dual Linear Program: Minimum Vertex Cover

$$\begin{aligned}
&\textstyle\sum_{v\in V} x_v \le n - \frac{1}{2} \\
&x_v + x_u \ge 1 \quad \forall (u,v) \in E_A \\
&x_v + x_u \ge 1 \quad \forall (u,v) \in E_B \\
&0 \le x \le 1
\end{aligned} \qquad (P)$$

- (P) feasible $\iff$ No perfect matching exists
- (P) feasible $\implies \mathrm{Vol}(P) \ge \left(\frac{1}{20n}\right)^{5n}$

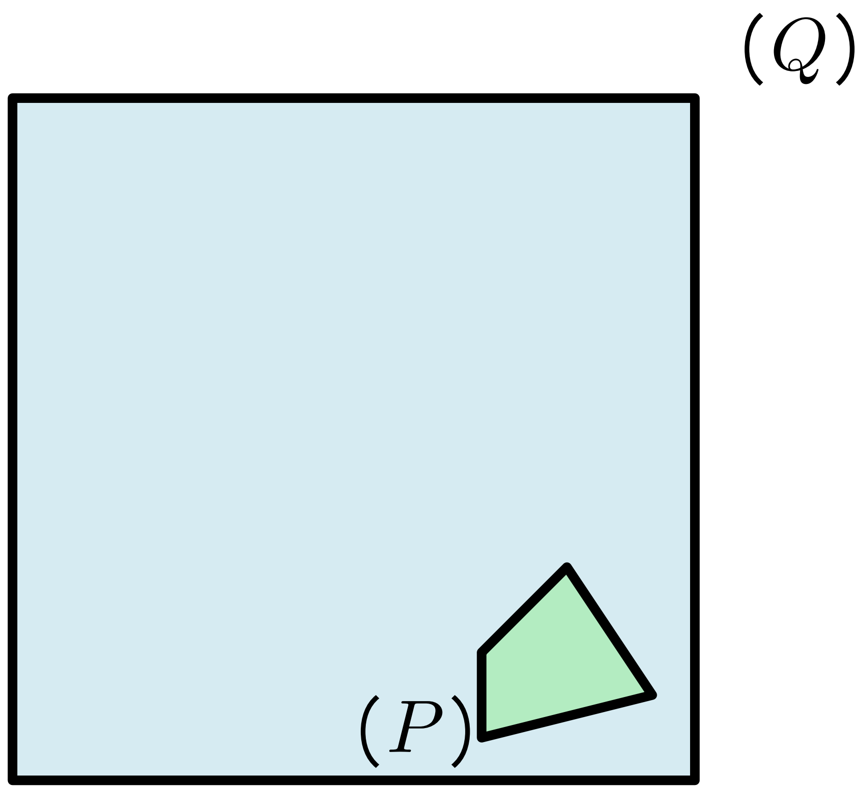
Cutting Planes Method

Separation Oracle:

Given $y \in \mathbb{R}^n$, return either:

- "y is in (P)"
- Violated hyperplane: "$c^\top x \leq d$"
  - valid for all $x \in (P)$
  - not valid for y

Center-of-gravity Cutting Planes [Levin'65] [Newman'65]



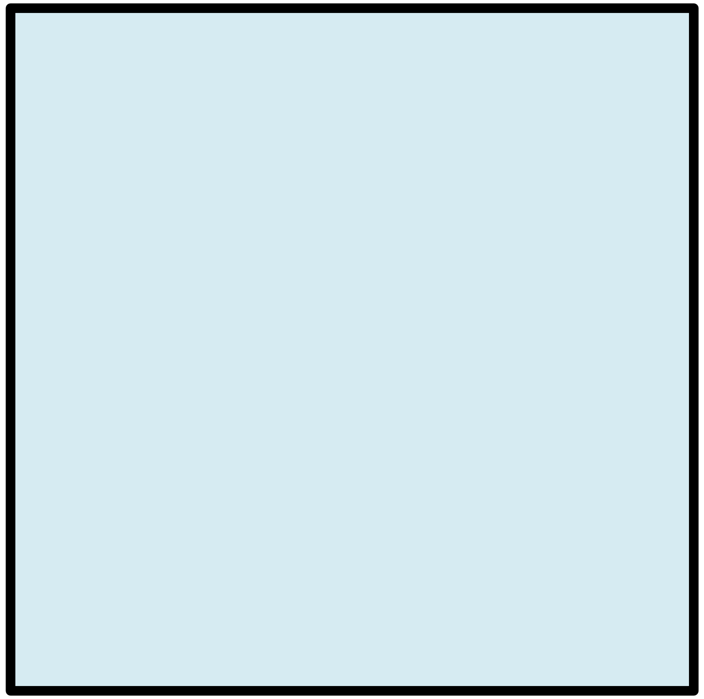
Goal: Find point in unknown polytope (P)

Given: (Q) containing (P)

Separation Oracle

Center-of-gravity Cutting Planes [Levin'65] [Newman'65]

(Q)



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Separation Oracle

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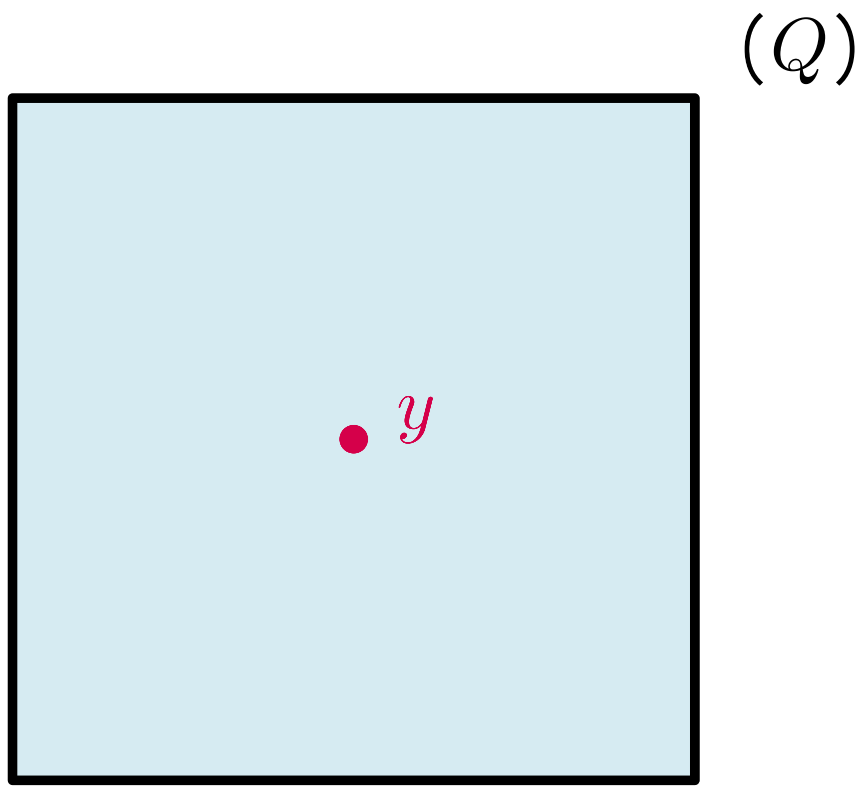

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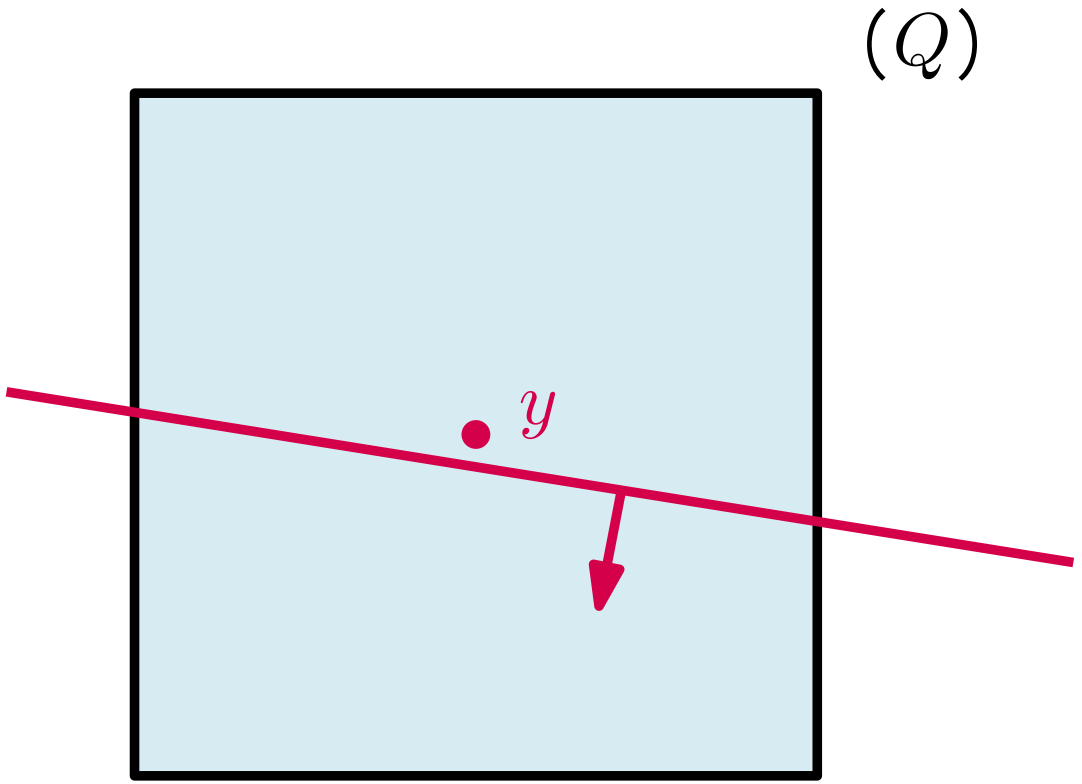

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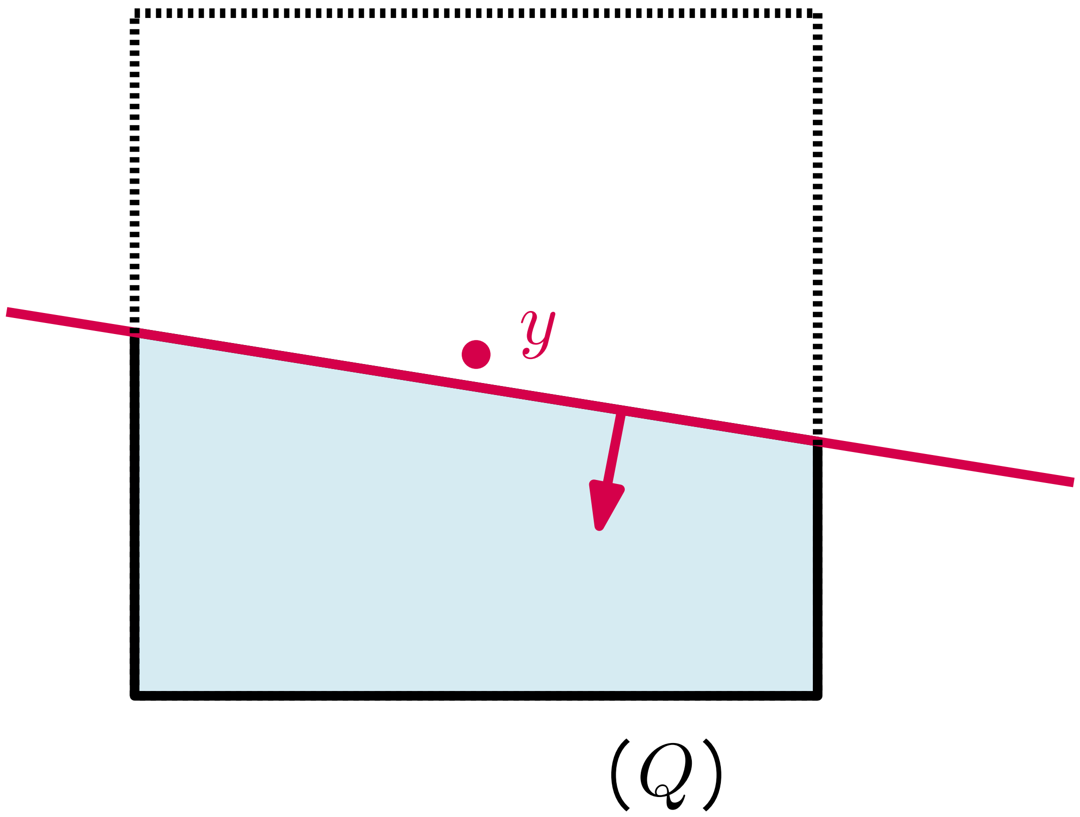

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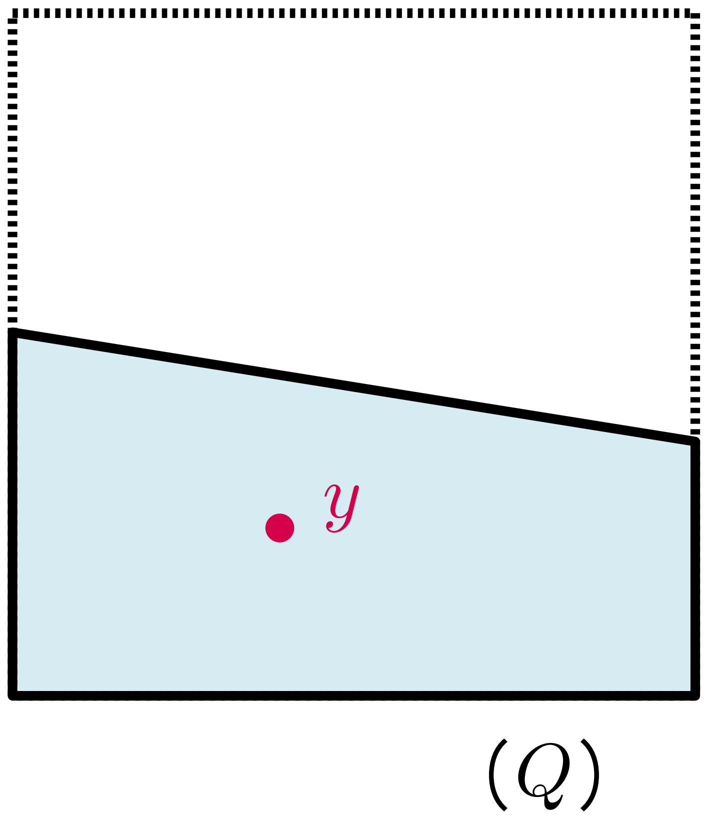

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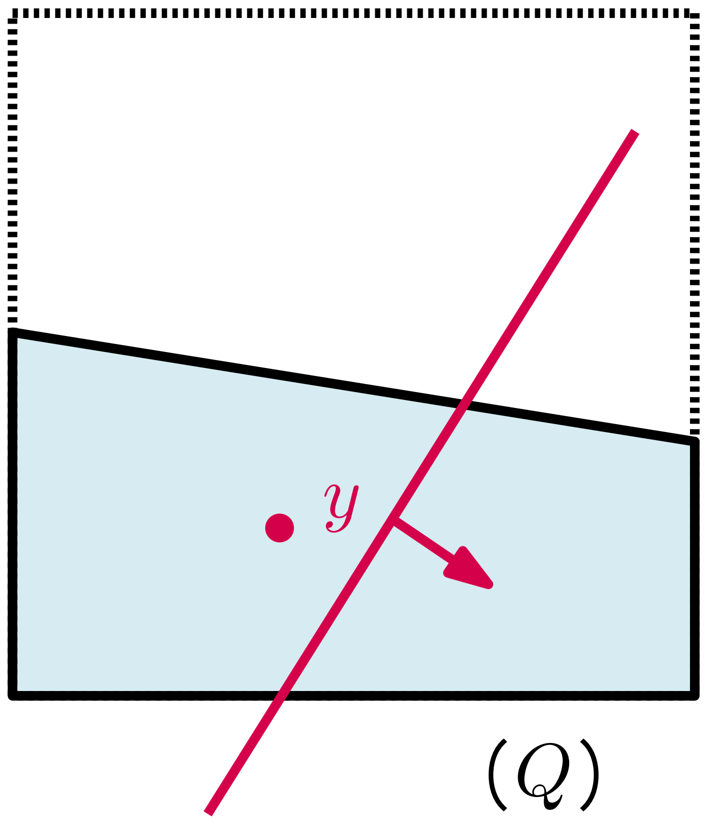

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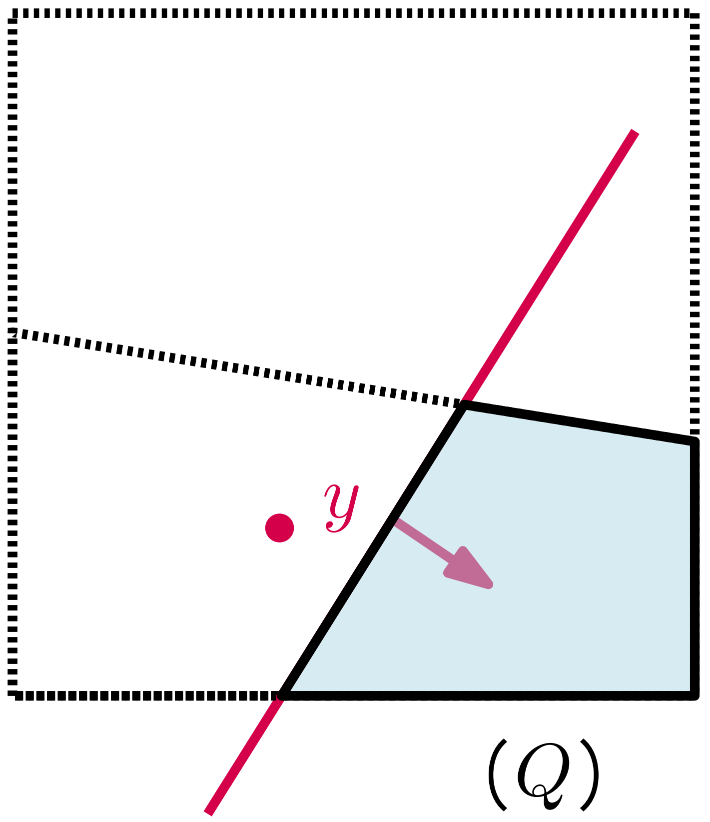

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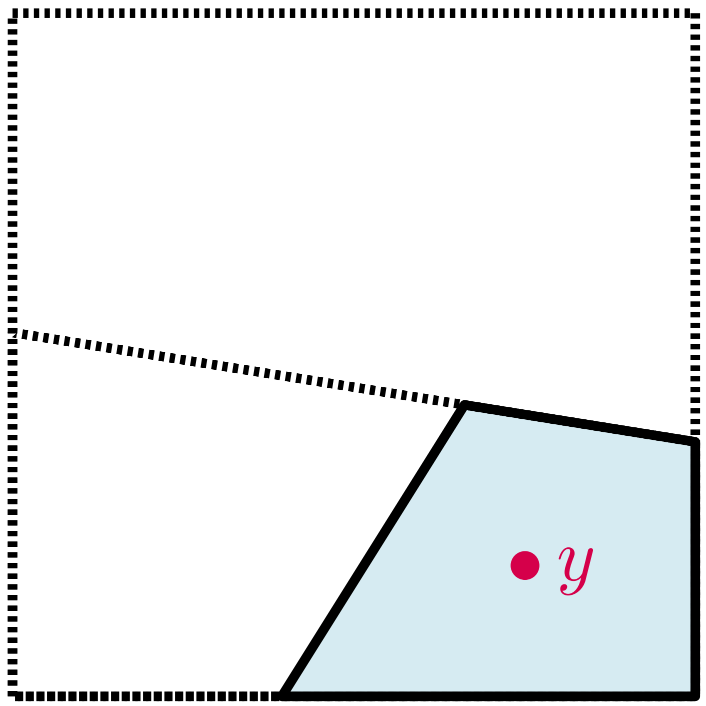

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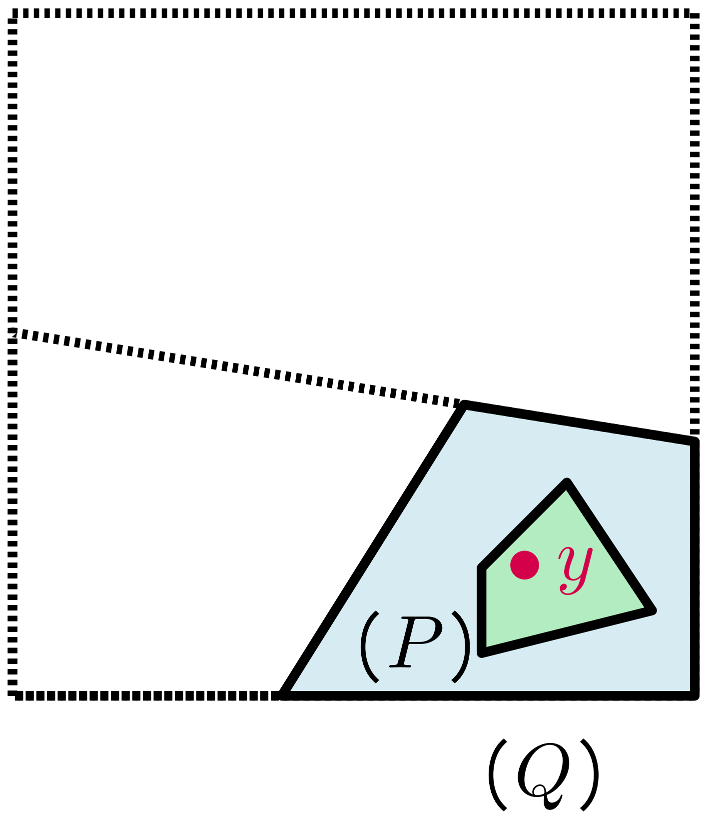

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Goal: Find point in unknown polytope (P)

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$\#P$-hard to compute
We don't care! $\longrightarrow$

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Cutting Planes for Biparite Matching

$$\sum_{v \in V} x_v \leq n - \frac{1}{2} \qquad (Q)$$

$$0 \leq x \leq 1$$

$$\sum_{v \in V} x_v \leq n - \frac{1}{2} \qquad (P)$$

$$x_v + x_u \geq 1 \quad \forall (u,v) \in E_A$$

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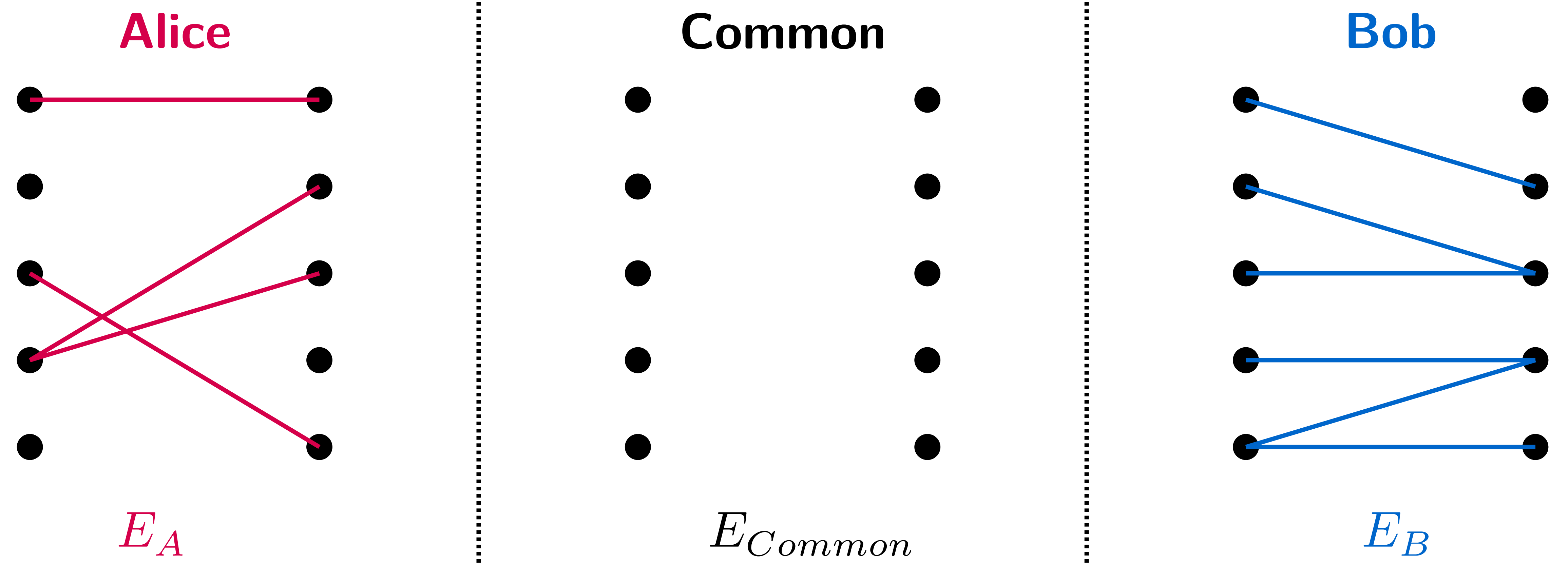

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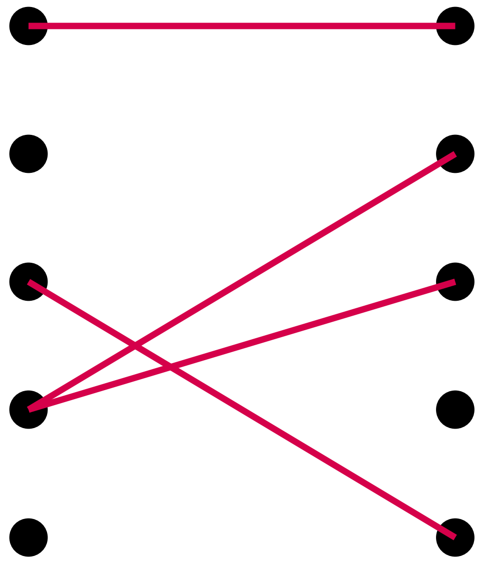
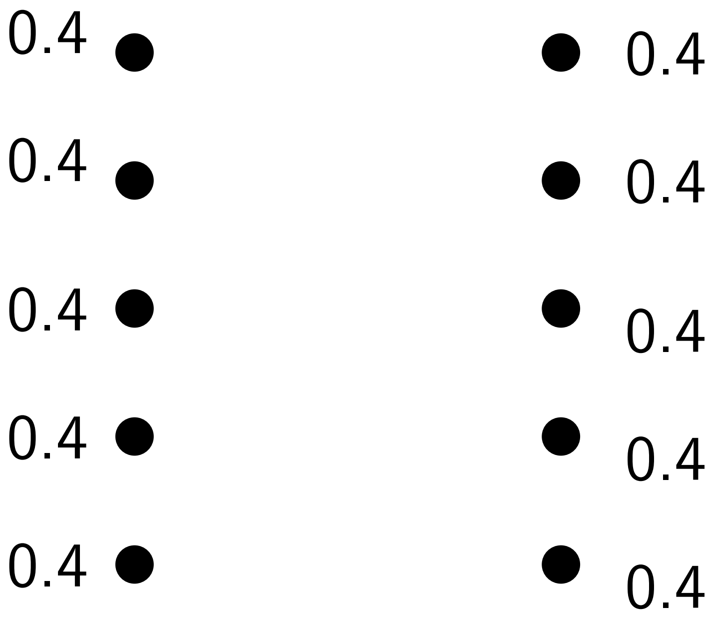


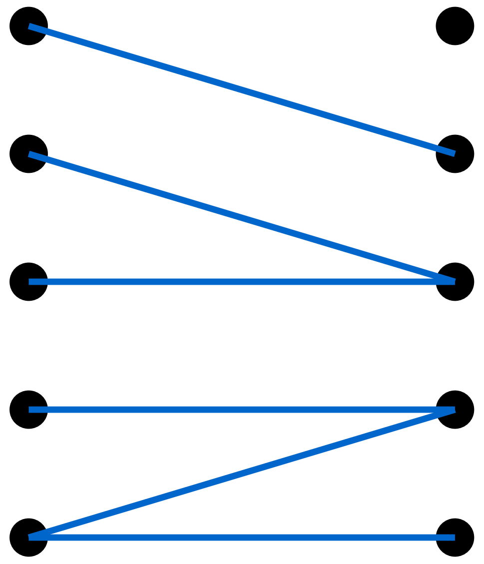
E_A

Common



E_{Common}

Bob



E_B

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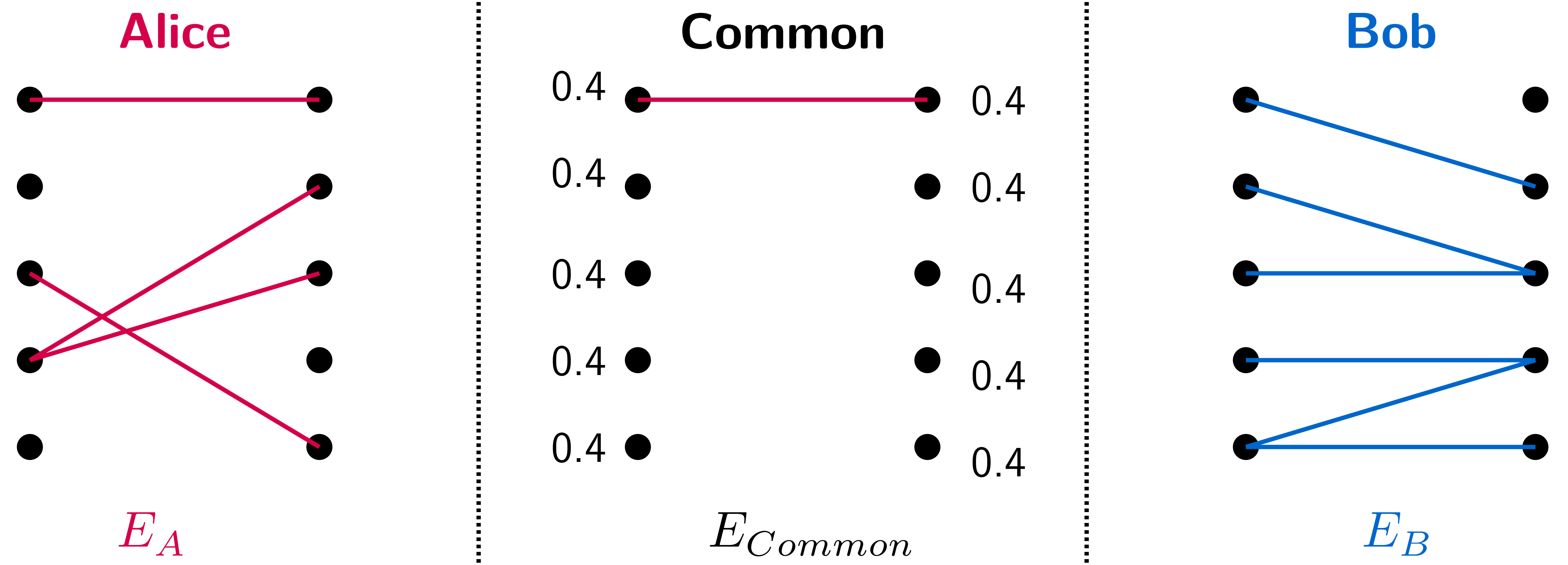

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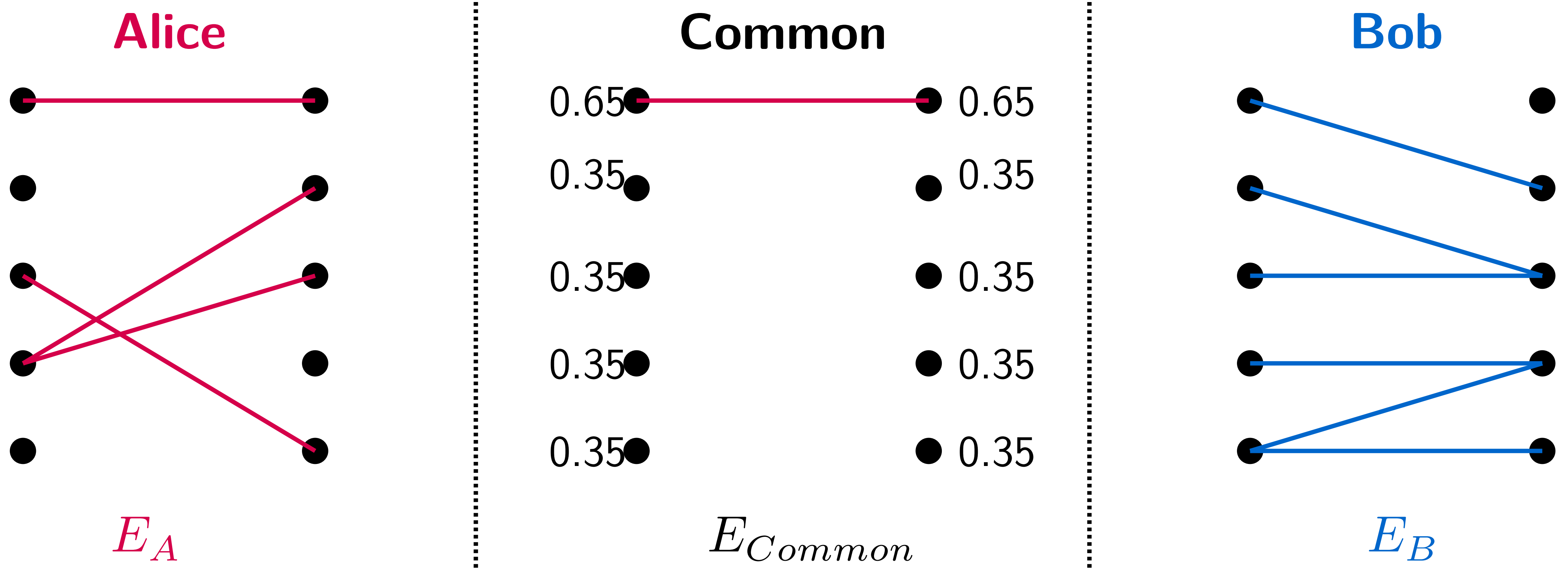

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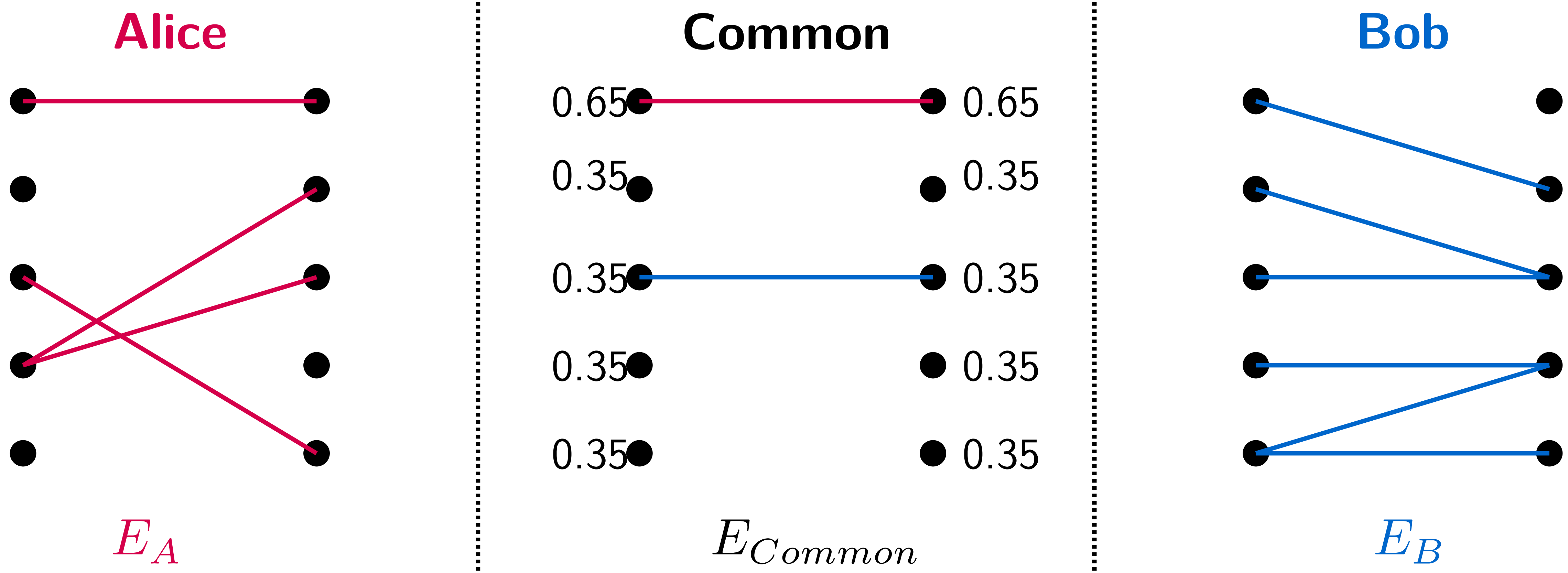

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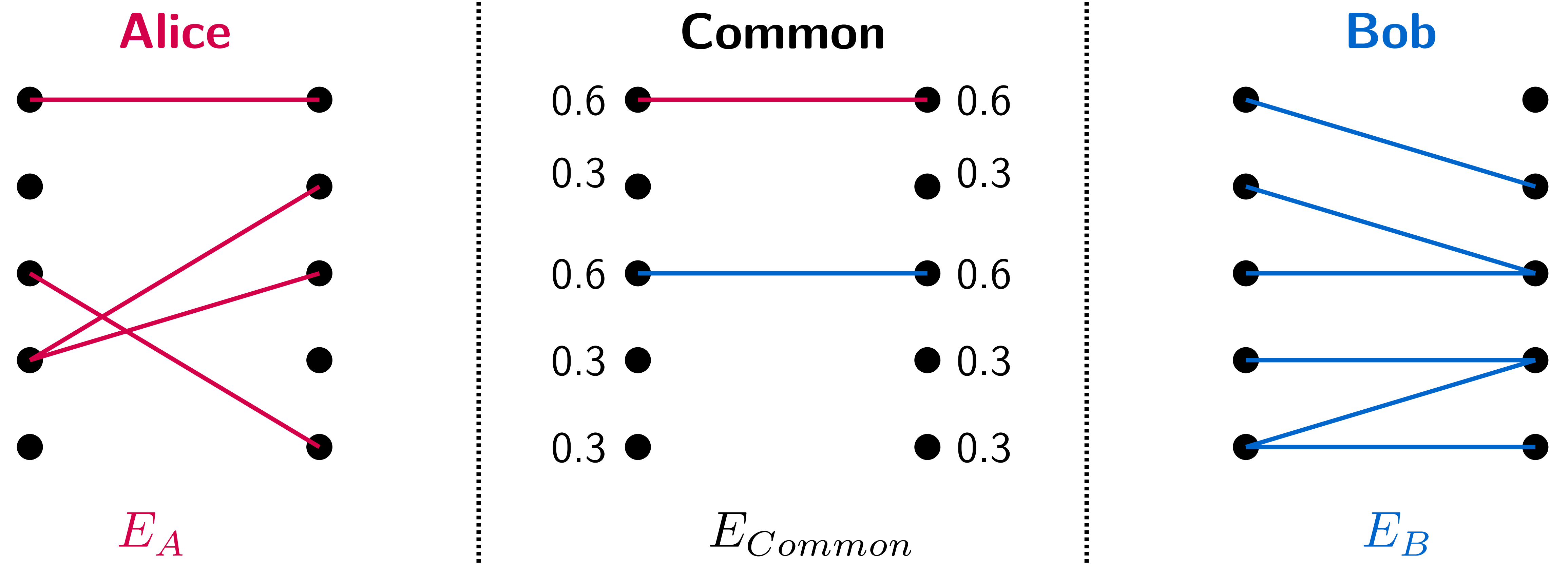

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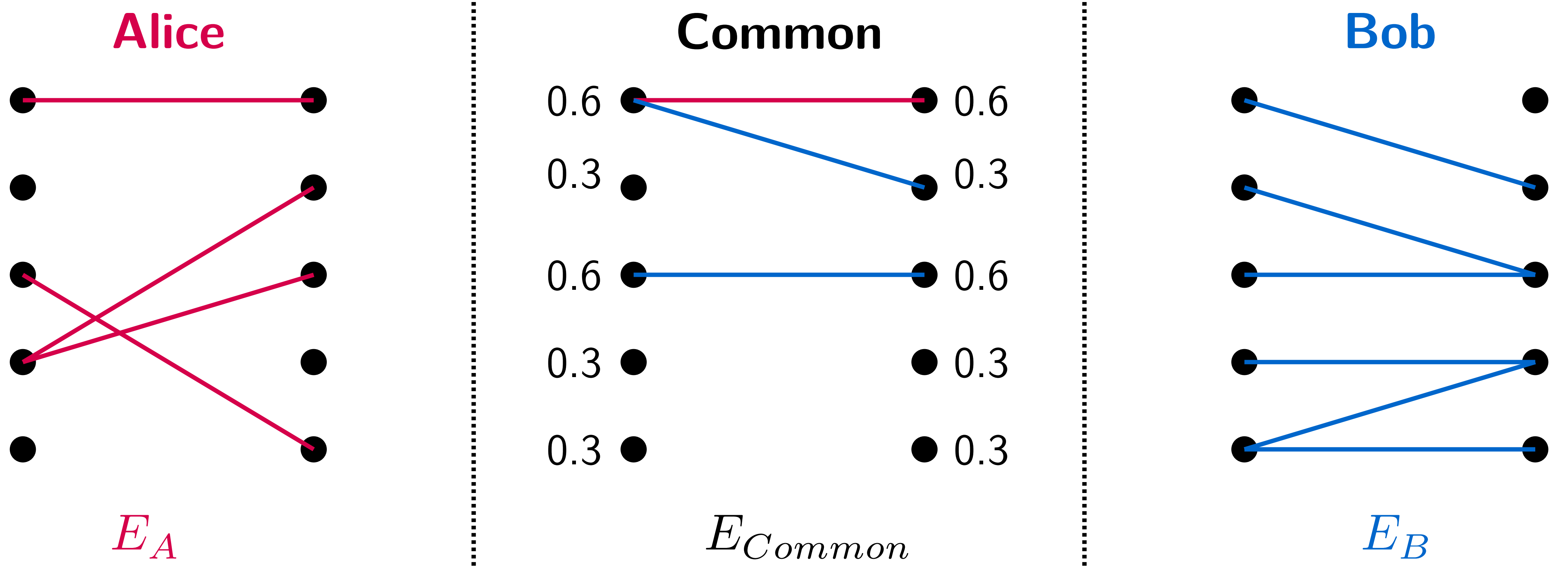

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- $E_{common} = \varnothing, \quad Q = \left\{x \in [0,1]^V : \sum x_v \leq n - \frac{1}{2}\right\}$
- While $vol(Q) > 0$:
  - Let c = center-of-gravity(Q) "fractional vertex cover"
  - If either Alice or Bob have an edge (u, v) violating c: add it to E_{common} and add "$x_v + x_u \geq 1$" to (Q)
  - If not, **return** c as a fractional vertex cover
- E_{common} must now contain a perfect matching.

OR-Query Algorithm

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  - Let c = center-of-gravity(Q) "fractional vertex cover"
  - Binary search with OR-queries to find violated edge in $S = \{(u,v) \in L \times R : c_u + c_v < 1\}$
  - If not, **return** c as a fractional vertex cover
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Main Result:
One can solve bipartite matching in $O(n \log^2 n)$ bits of communication.

Extensions

Weights and Demands!

$$
\begin{array}{lll}
\min & \sum_{v \in V} x_v & \\
\text{s.t.} & x_v + x_u \geq 1 & \forall (u,v) \in E_A \\
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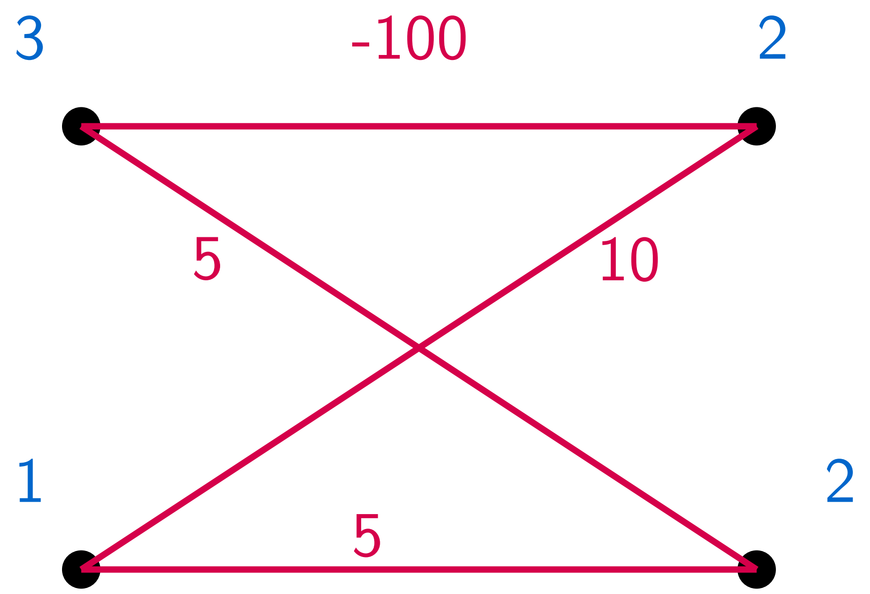
Weights and Demands!

$$\begin{array}{llll} \min & \sum_{v \in V} b_v x_v & & \\ \text{s.t.} & x_v + x_u \geq c_{uv} & \forall (u,v) \in E_A \\ & x_v + x_u \geq c_{uv} & \forall (u,v) \in E_B \\ & 0 \leq x \leq W & \end{array}$$

- $W := \max\{|c_{uv}|, |b_v|, 1\}$
- Maximum-cost b-matching



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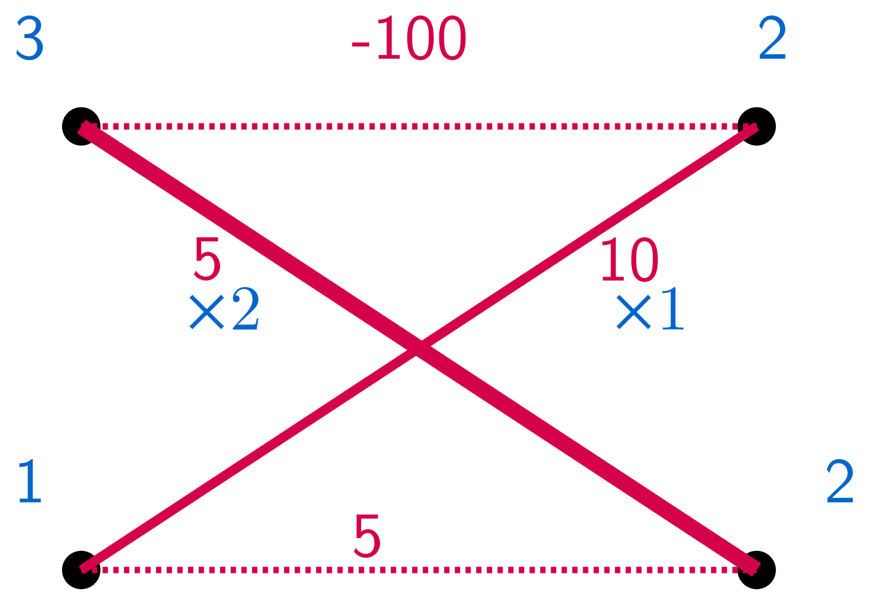

Other (Equivalent & Weaker) Problems

Theorem:
If weights/costs/capacities/demands are poly(n), then we can solve the following using $O(n \log^2 n)$ communication:

- Maximum-cost bipartite perfect b-matching
- Maximum-cost bipartite b-matching
- Vertex-capacitated minimum-cost (s, t)-flow
- Transshipment
- Negative-weight single source shortest path
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Note: All these have $O(n)$ edges in their answer!

Query Lower-Bounds

- AND-query $S = \{(u, v) \in L \times R\}$:
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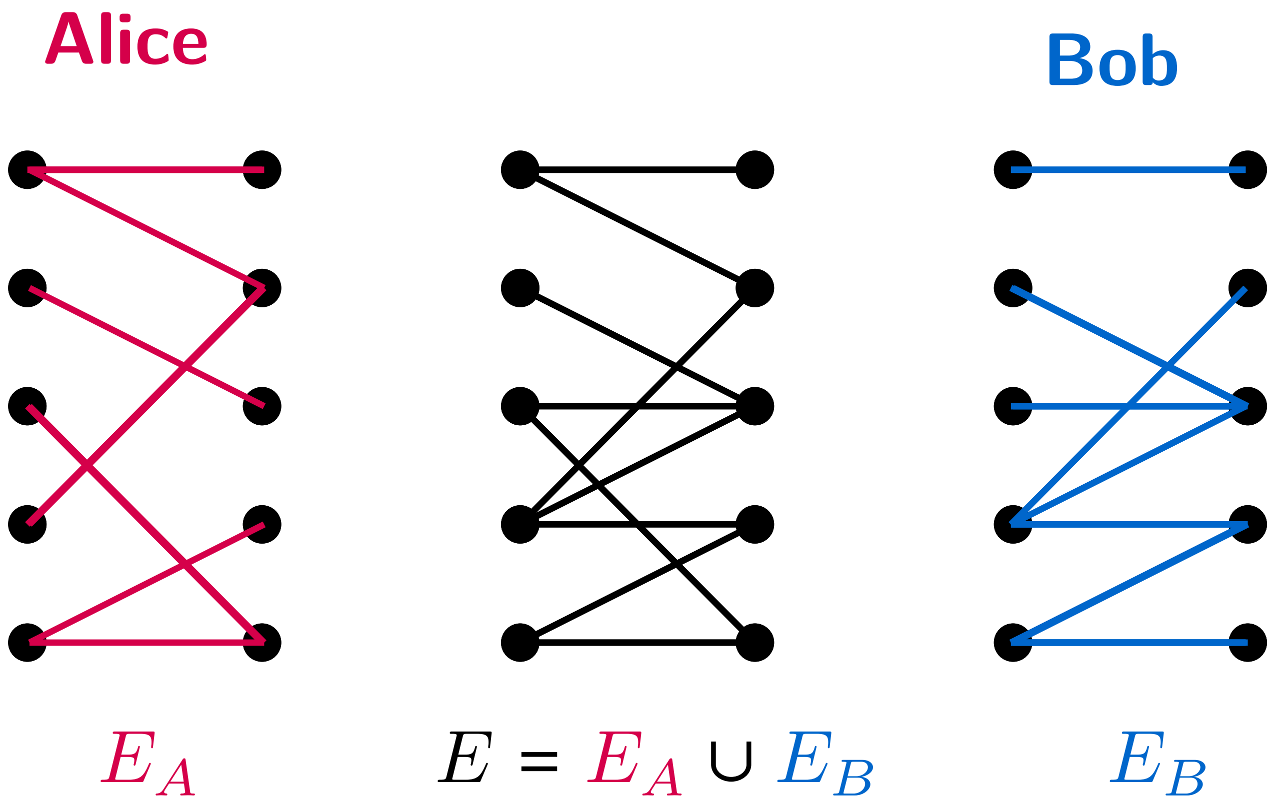

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- AND-query $S = \{(u,v) \in L \times R\}$:
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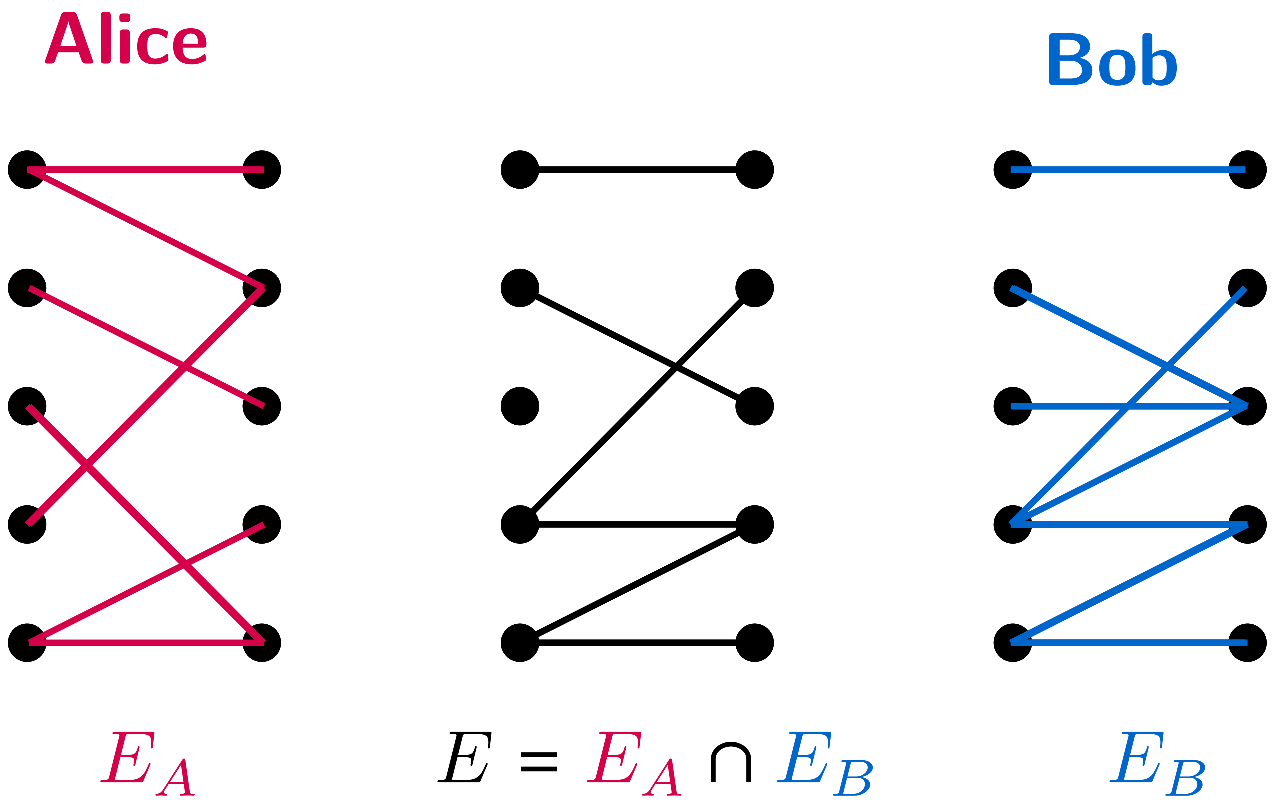
- AND-query algorithm $\implies$ communication protocol on intersection graph



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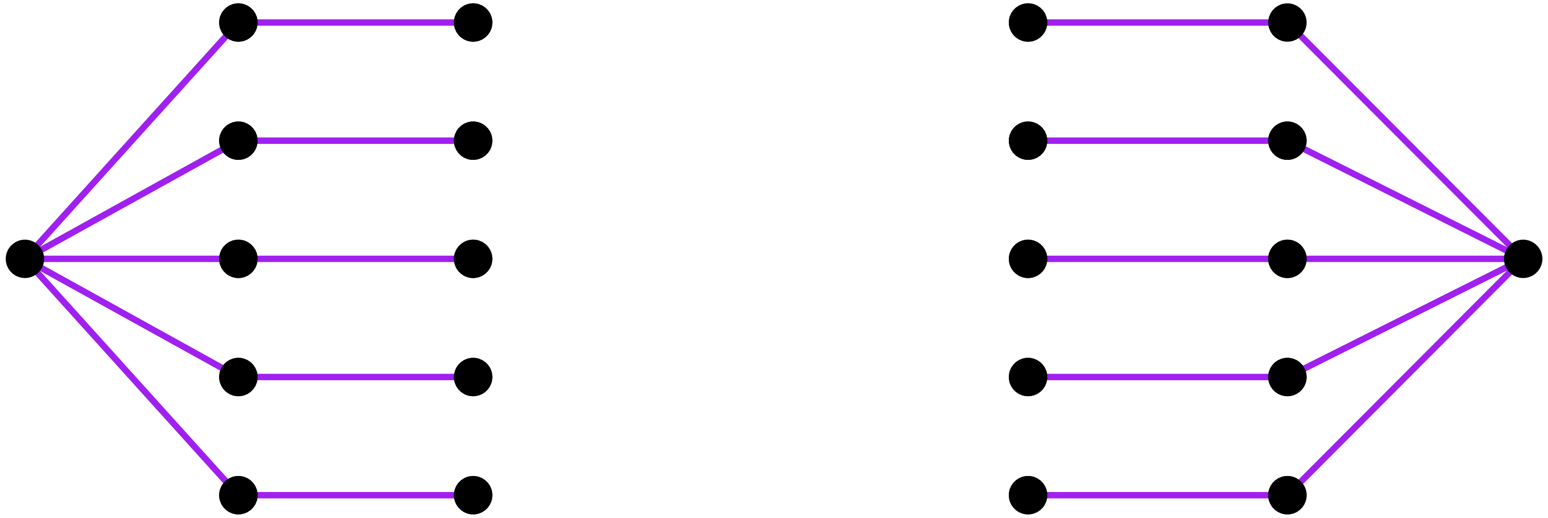

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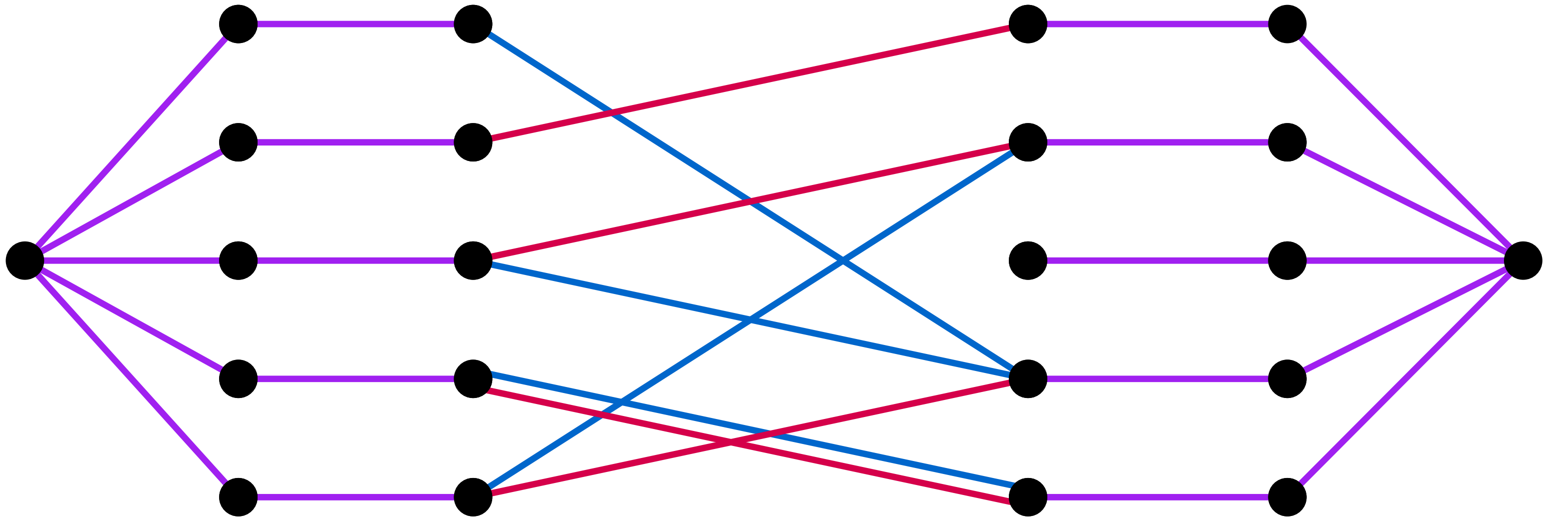


- Perfect matching $\iff$ edges intersect
- Set-Intersection on $\approx n^2$ bits. Needs $\Omega(n^2)$ communication!

Summary — Results

Models	**Previous papers**		**This paper**
	Lower bounds	Upper bounds	
Two-party communication	$\Omega(n)$ Rand, $\Omega(n \log n)$ Det, Footnote 1 and 2	$\tilde{O}(n^{1.5})$ [DNO19, IKL+12]	$O(n \log^2 n)$, Det Theorem 1.1
Quantum edge query	$\Omega(n^{1.5})$ [Zha04, Ben22b]	$O(n^{1.75})$ [LL15]	$\tilde{O}(n^{1.5})$ Theorem 1.3
OR-query	$\Omega(n)$ Rand, $\Omega(n \log n)$ Det, [BN21]	$\tilde{O}(n^{1.5})$ Det, [Nis21]	$O(n \log^2 n)$, Det Theorem 1.3
XOR-query	$\Omega(n)$ Rand $\Omega(n^2)$ Det [BN21]	$\tilde{O}(n^{1.5})$ Rand Lemma 2.14 and [Nis21]	$O(n \log^2 n)$, Rand Theorem 1.3
AND-query	$\Omega(n)$ Rand, $\Omega(n^2)$ Det [BN21]	$O(n^2)$ Trivial	$\Omega(n^2)$, Rand Theorem 1.3

Open Problems :)

Open Problem — Round vs Communication Tradeoff

- Restricting the #rounds:
  - Streaming
  - Distributed
  - MPC
  - ...

	Rounds	**Communication**
trivial:	1	$\Theta(n^2)$
	?	?
cutting-planes:	$O(n \log n)$	$O(n \log^2 n)$

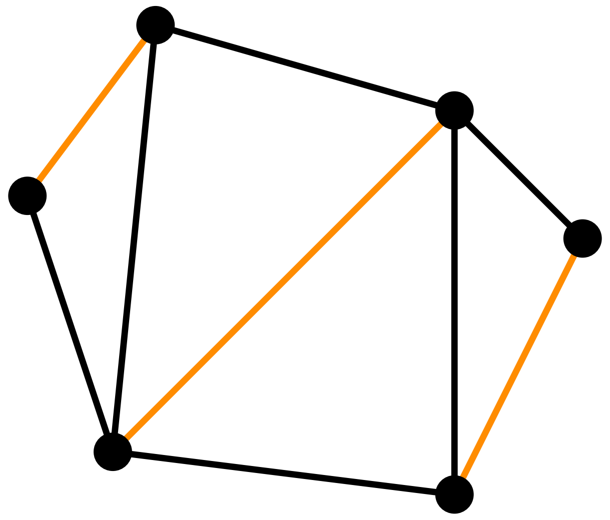
Open Problem — Approximation

- Finding an α-approximation instead? (size version)

Approximation	**Communication**
1	$O(n \log^2 n)$
?	?
2	$O(\log n)$

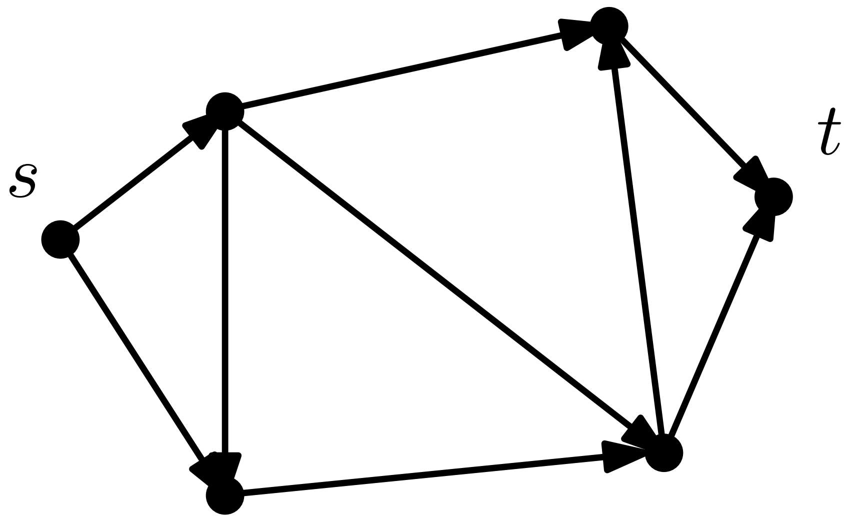
Open Problems — General Matching

- Communication and Query complexity of General Matching?
  - Interplay between *general* and *bipartite* matching unclear. . .
  - Optimal fractional matching by same approach.
  - Answer also has only $O(n)$ edges.
  - Unwieldy Linear Program. . .



Open Problems — Max Flow

- Communication and Query complexity of s,t-(min-cost)-max-flow?
  - Both the dual & primal have $\approx n^2$ variables
  - Answer may include all $\approx n^2$ edges
  - Nondeterministic (certificate) complexity are still low: $\tilde{O}(n)$



Open Problems

- Rounds vs Communication tradeoff
- Approximate bipartite matching
- Communication complexity of other problems?
  - General Matching
  - Max flow
  - Matroid intersection
  - . . .
- Other query models, e.g. *demand queries* (one-sided OR)
- Multiparty communication
- . . .

Thanks!