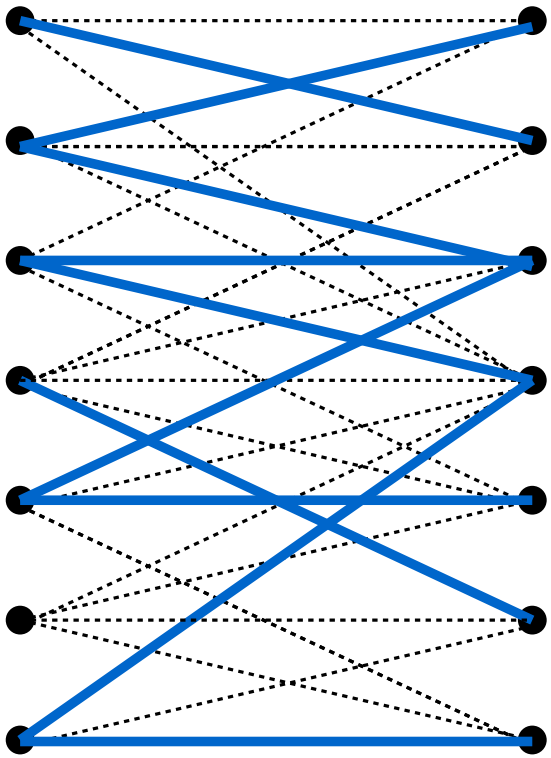
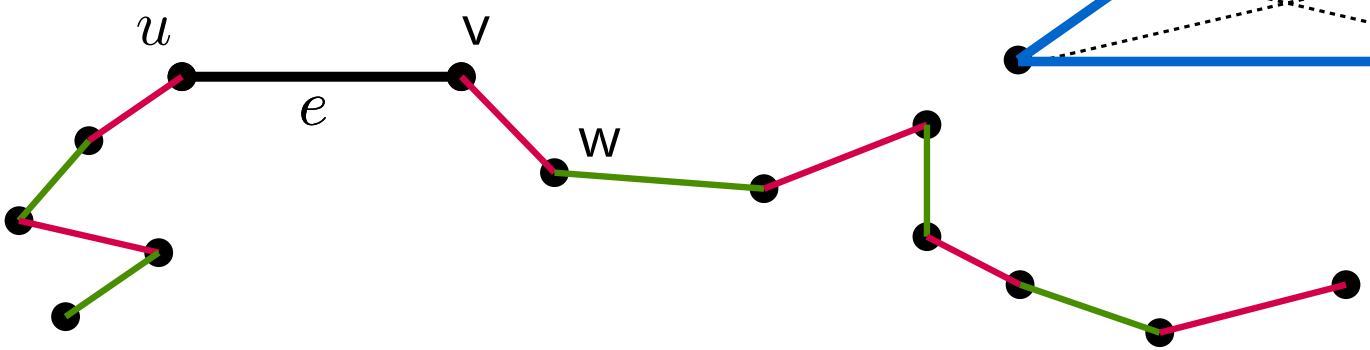
Incremental $(1-\epsilon)$-approximate Dynamic Matching in $O(\mathrm{poly}(1/\epsilon))$ Update Time

Joakim Blikstad[†] & Peter Kiss[*]

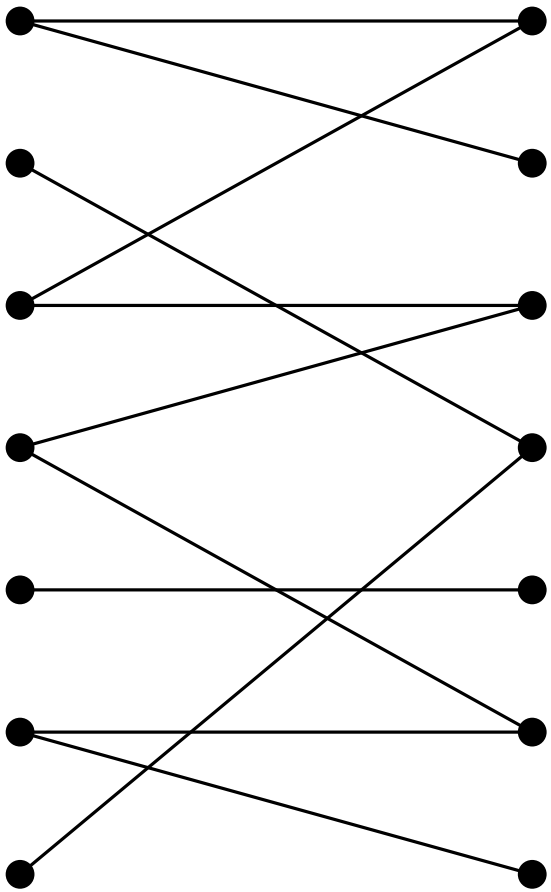
ESA 2023

[†]KTH Royal Institute of Technology
[*]University of Warwick
[†*]Max-Planck-Institut für Informatik





Bipartite Matching



Bipartite Graph $G = (V, E)$

- $n = |V|$, $m = |E|$

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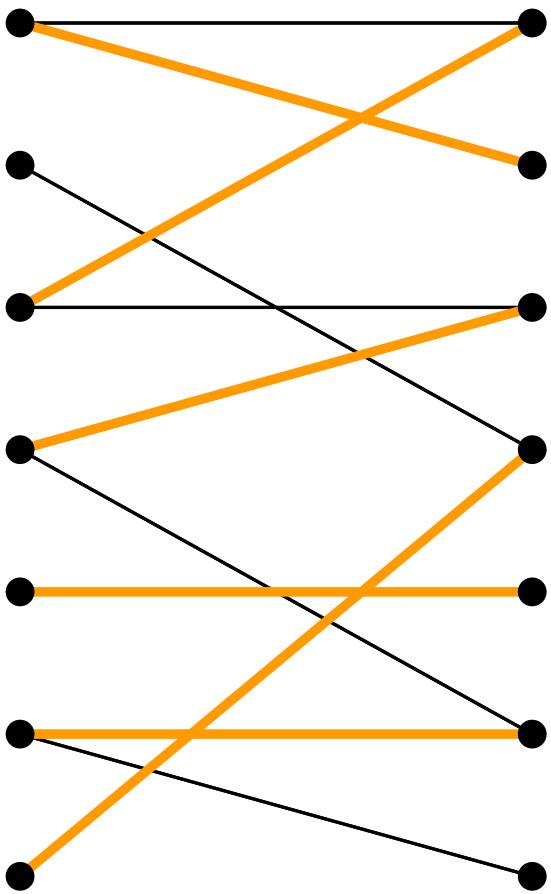


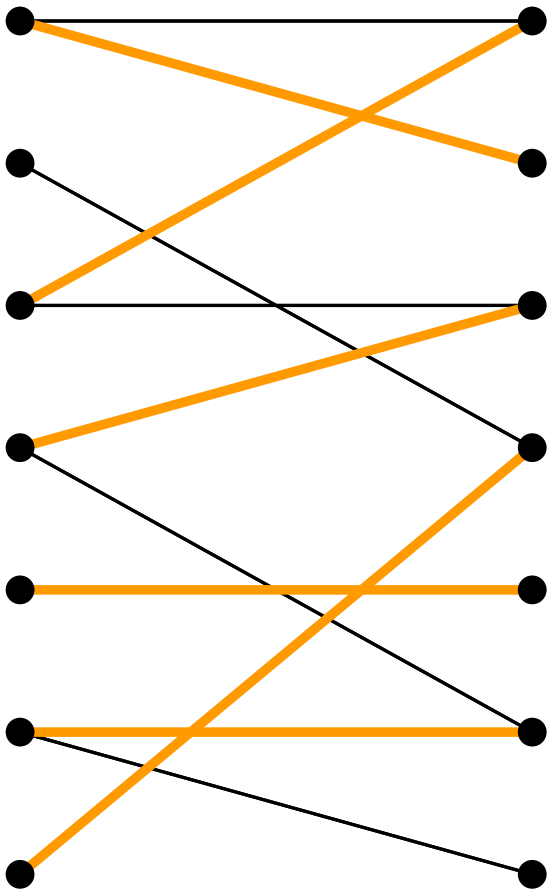
Bipartite Graph $G = (V, E)$

- $n = |V|$, $m = |E|$

Matching $M \subseteq E$

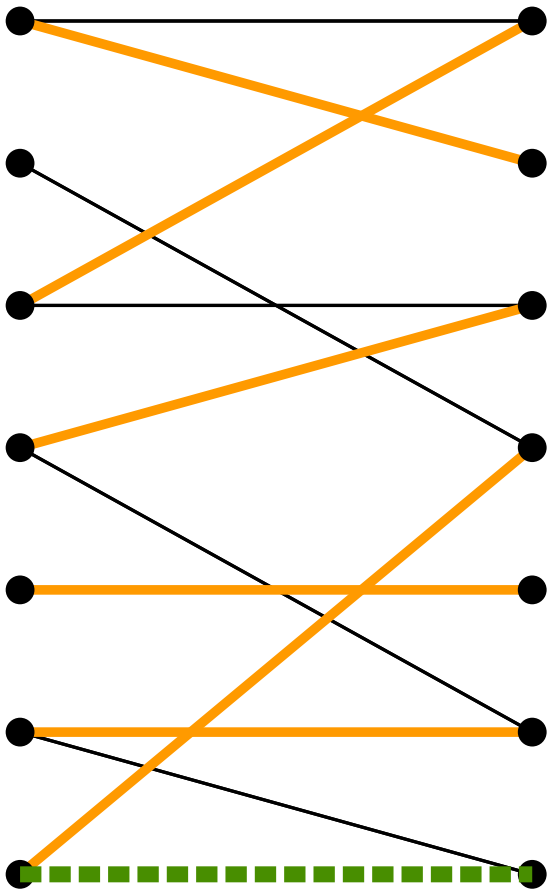
- share no vertices
- $\mu(G) = |\text{maximum matching}|$

Dynamic Model



Graph G changes over time

Dynamic Model



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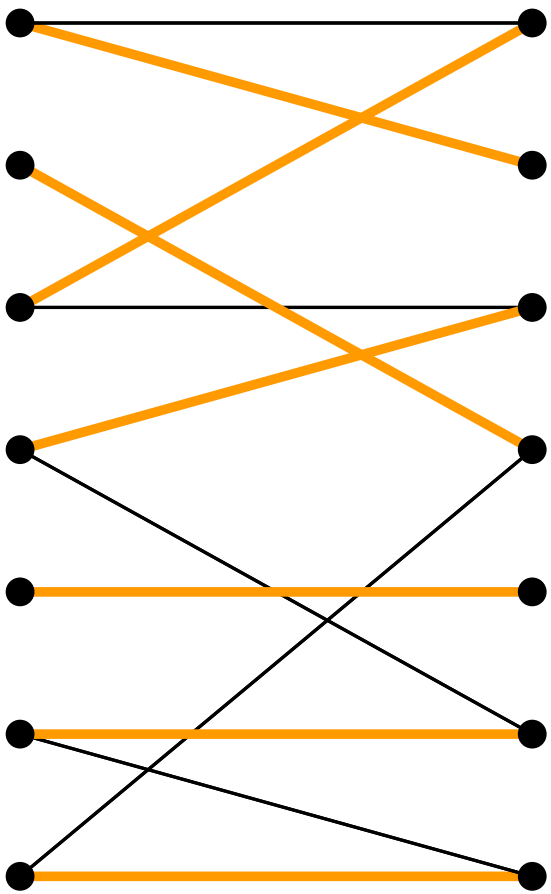

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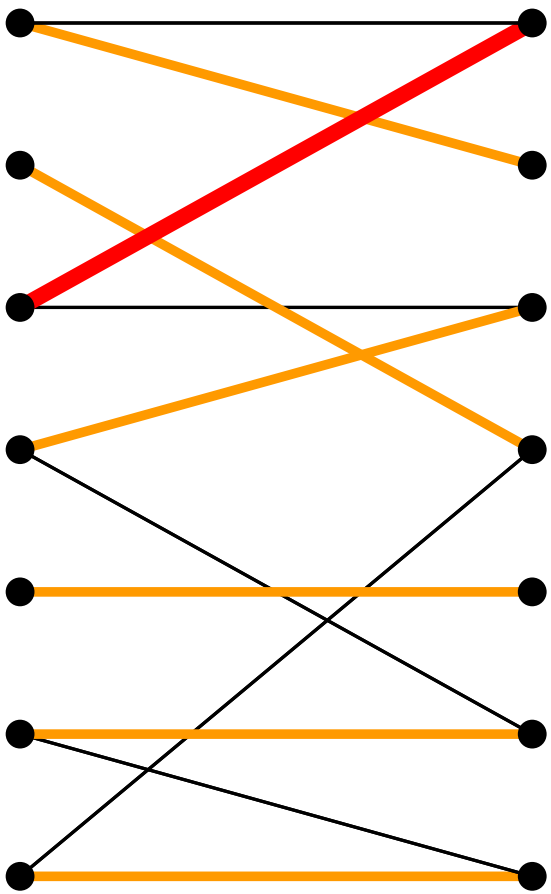

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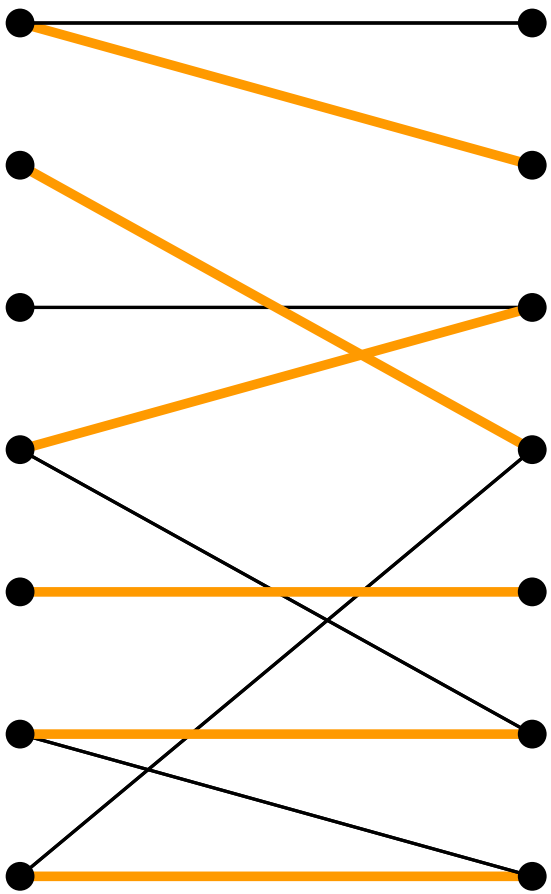

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$O(\text{polylog}(n))$ update time for $\frac{1}{2}$-approx or 0.585-approx [Baswana-Gupta-Sen FOCS'11] [Behnezhad-Khanna SODA'23] [Bhattacharya-Kiss-Saranurak-Wajc SODA'23]

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YES — Our Result

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Bonus: It's quite simple :)

Our Techniques

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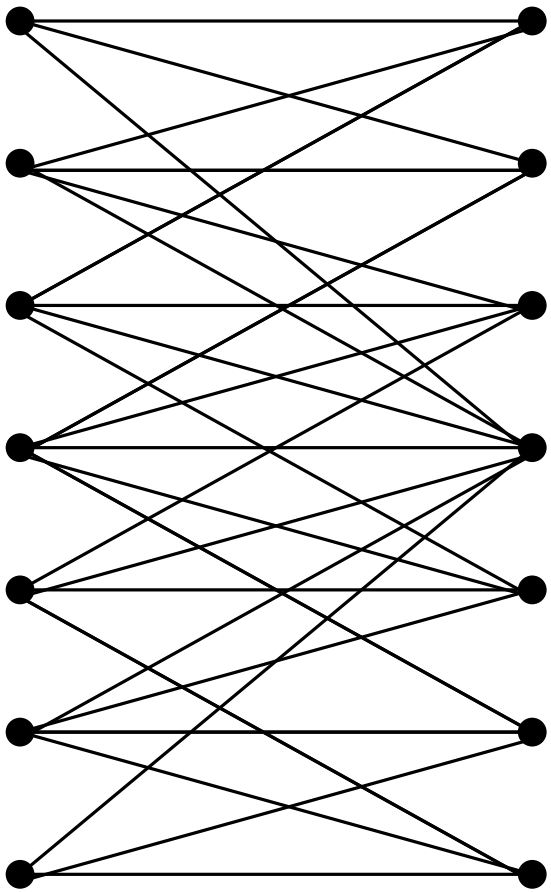

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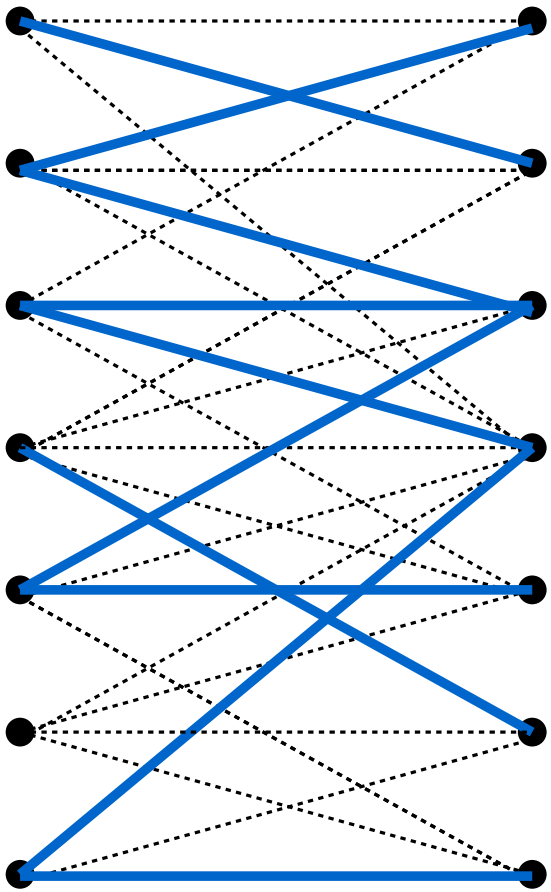

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Keep track of a subgraph H:

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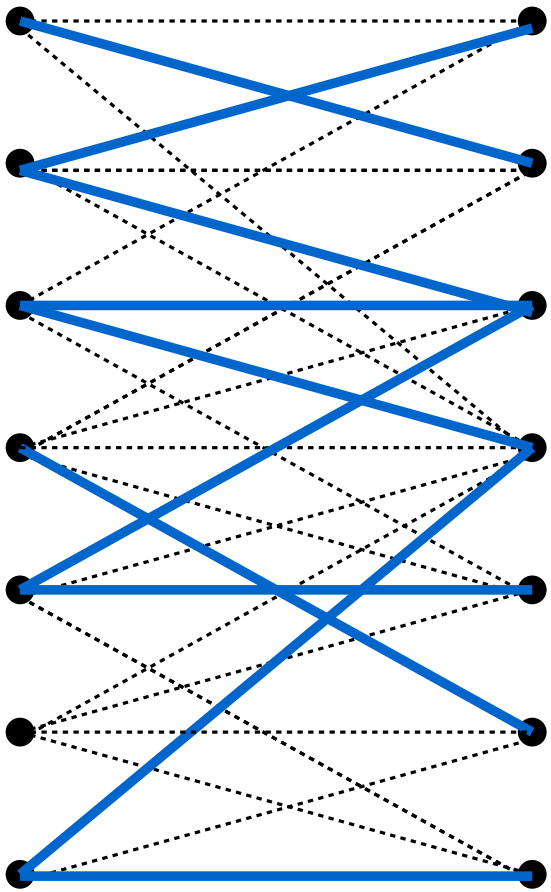

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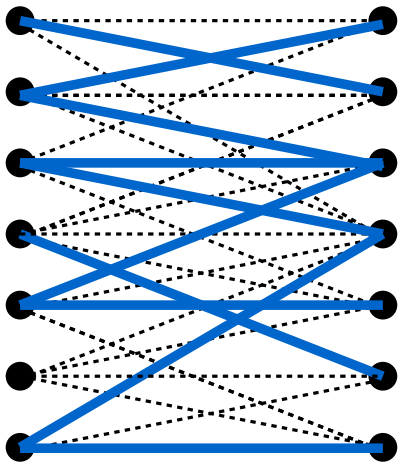

G, H, and good matching

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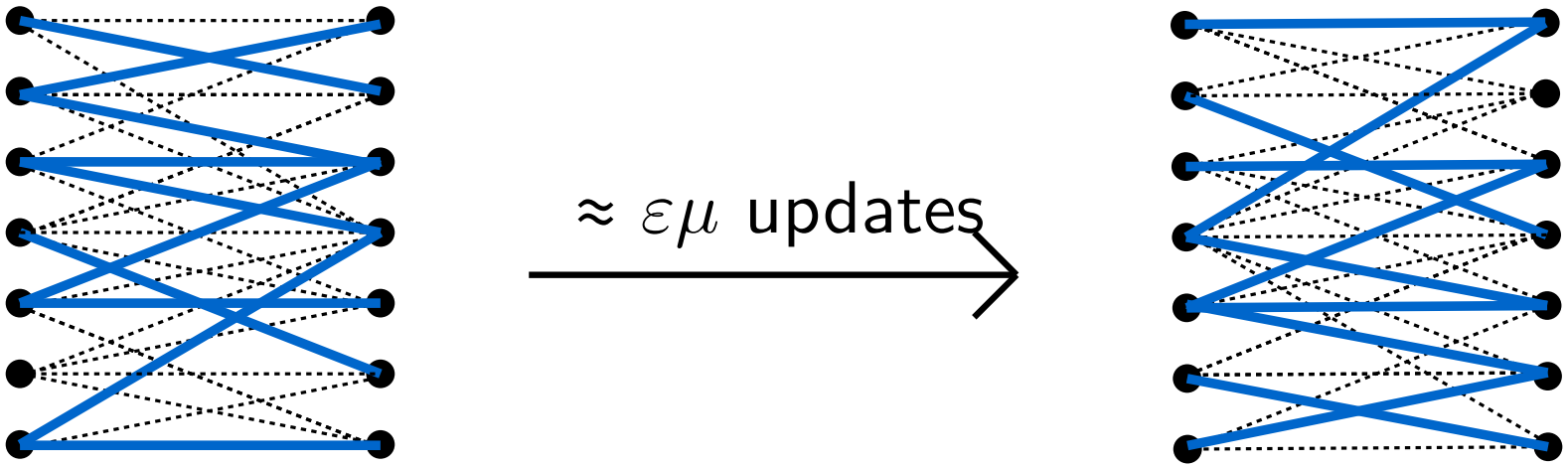

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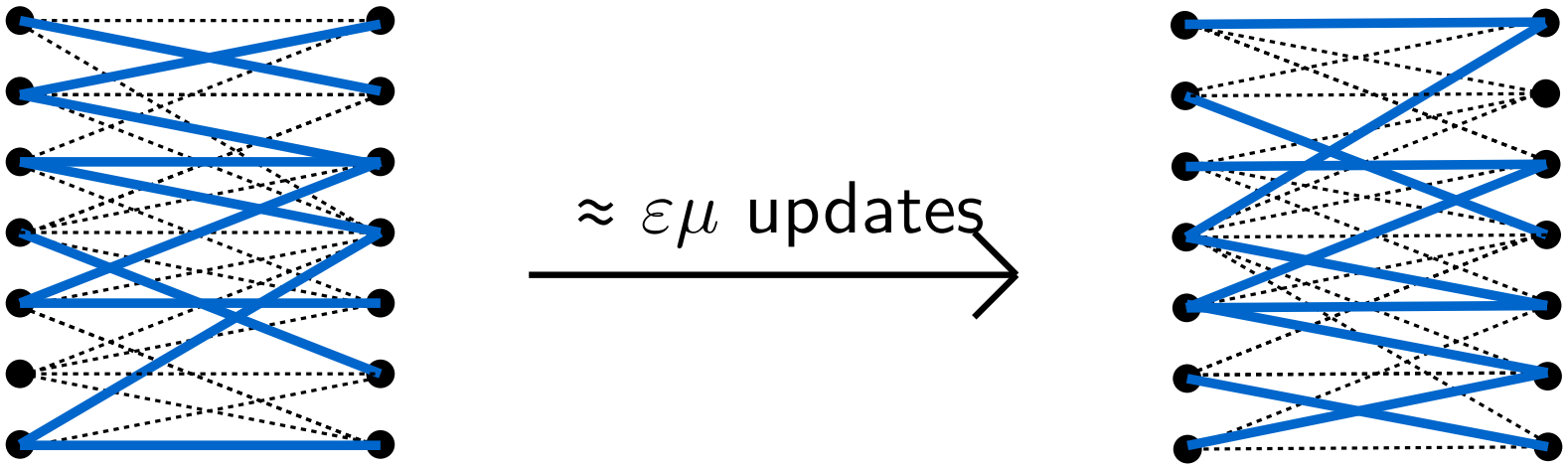

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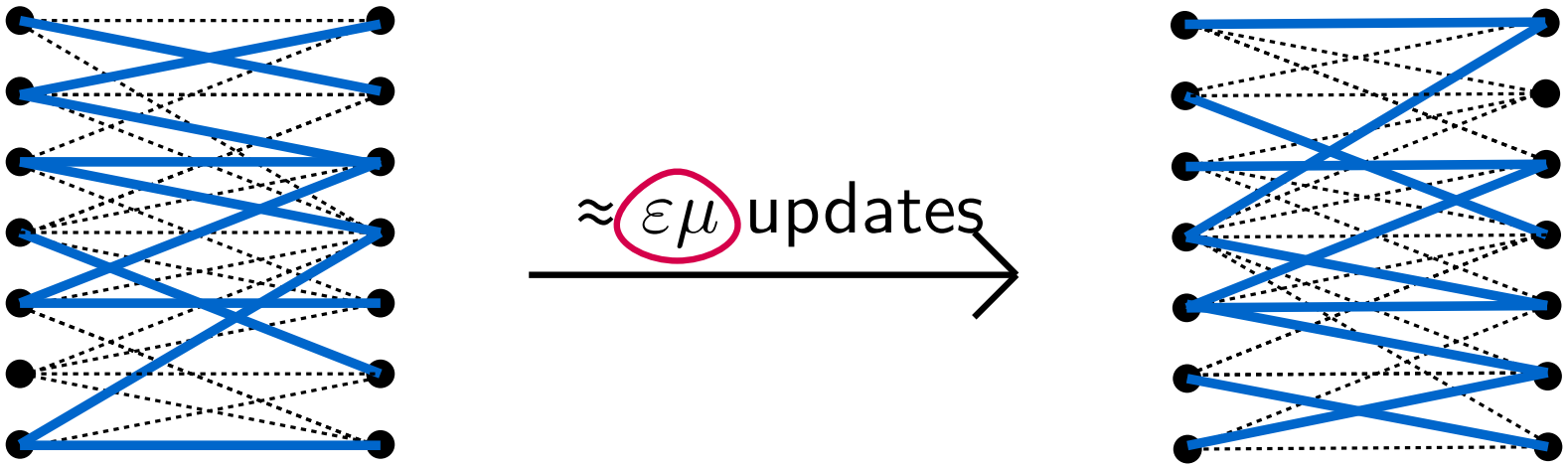

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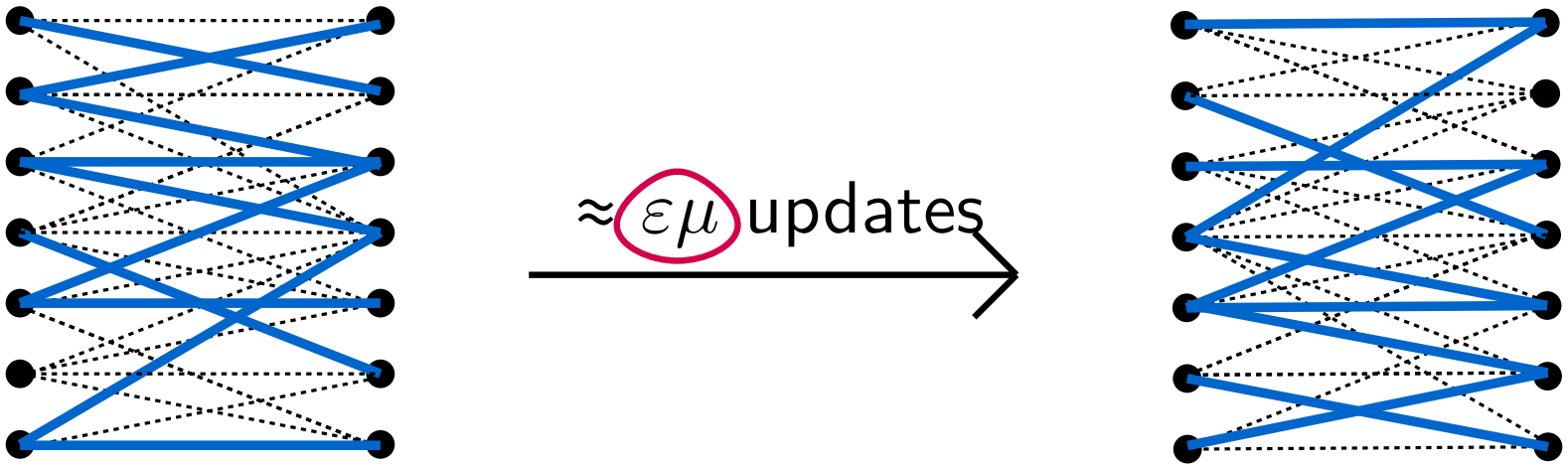

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= poly$(1/\varepsilon)$ (amortized) time per update

Weighted Edge-Degree Constrained Subgraph

[Bernstein-Stein ICALP'15]

Dynamic, Streaming, Sublinear, Communication, Fault Tolerant, ...

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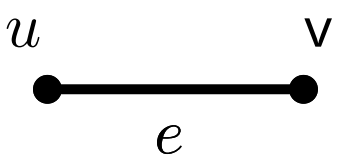

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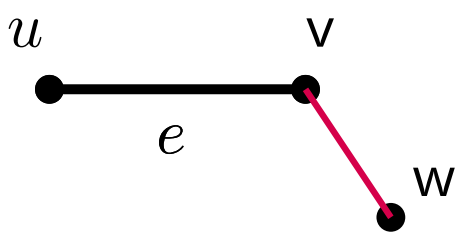

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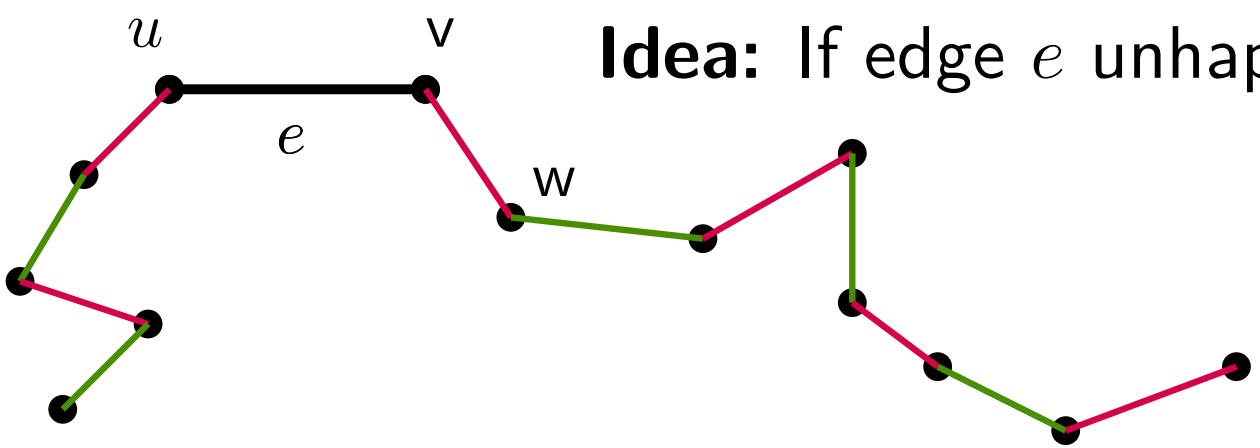

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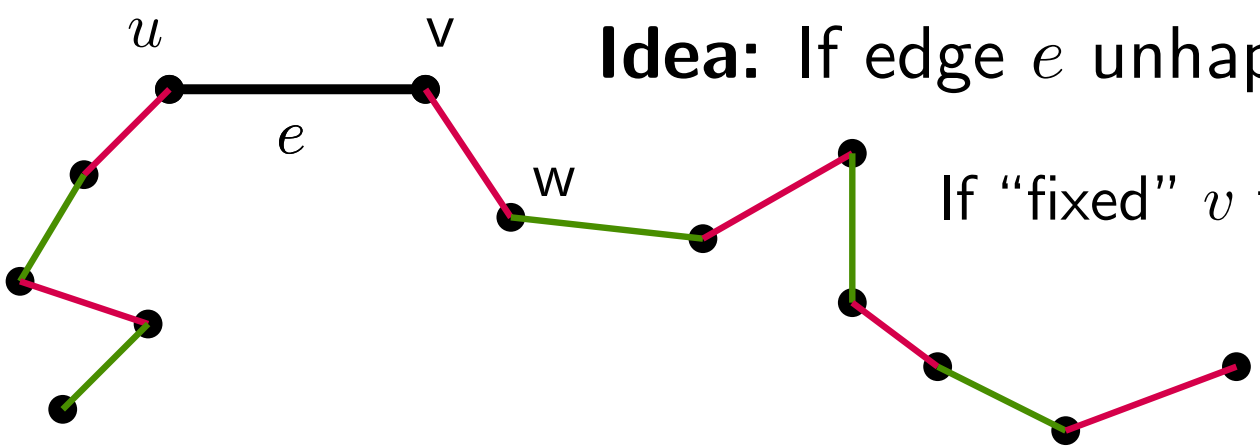

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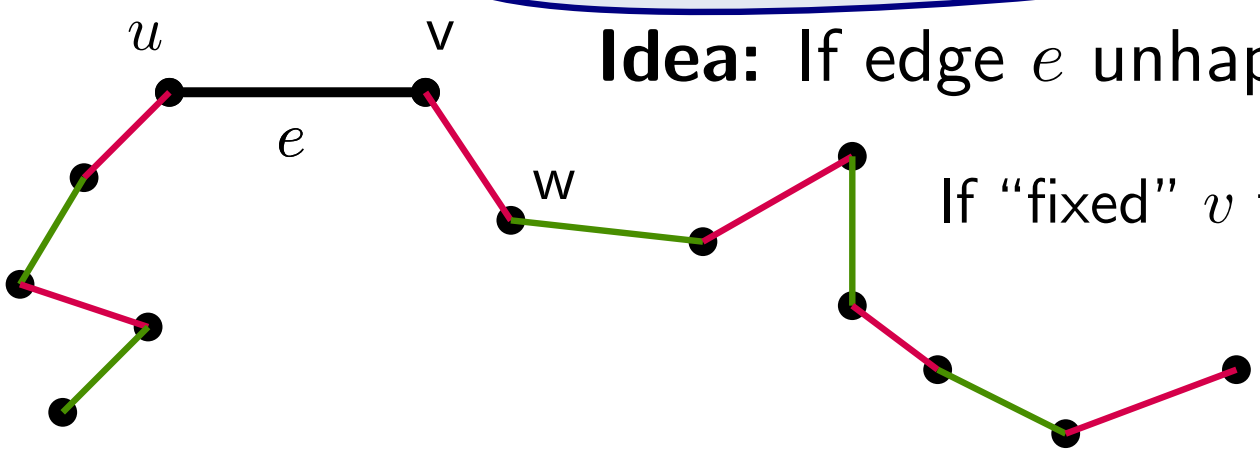

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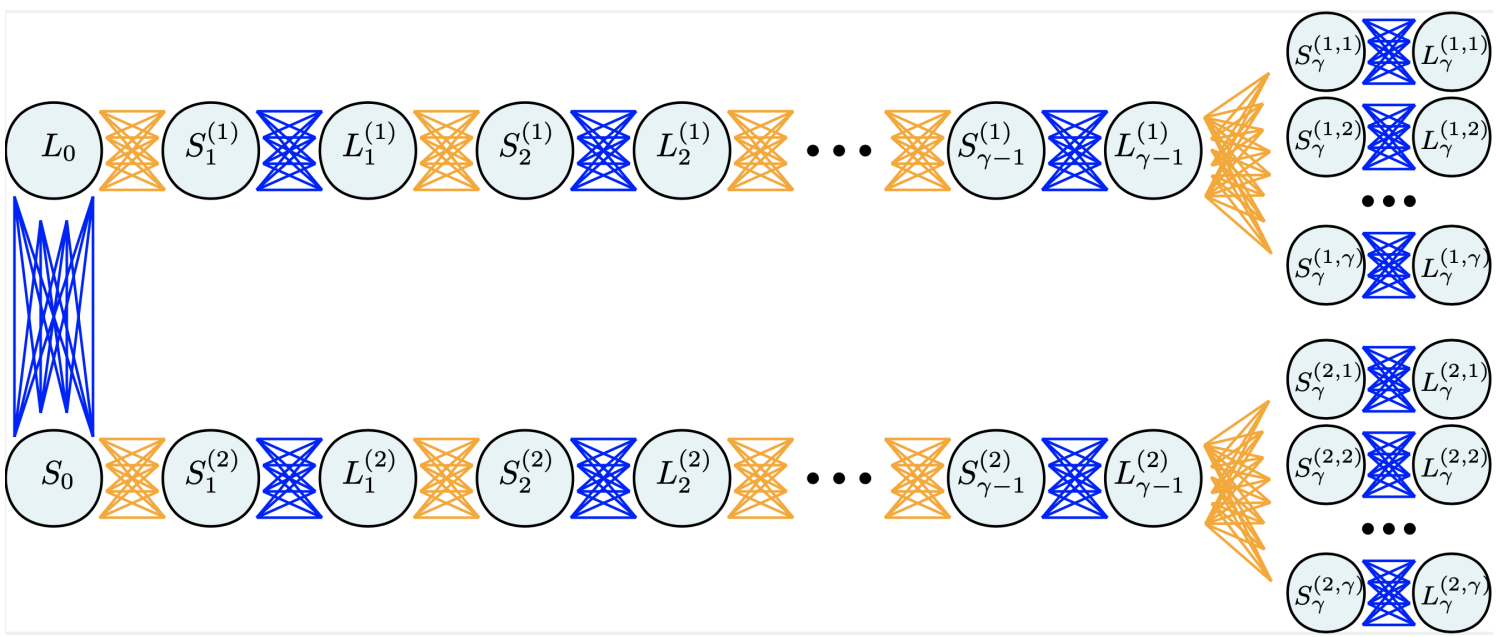
WEDCS Bonus Results

- Simple approximate *fractional matching* & *vertex cover* on H

$$f_{(u,v)} = \min\left(\frac{1}{\deg_H(u)}, \frac{1}{\deg_H(v)}\right)$$

- Setting $\beta = \Theta(1/\varepsilon^2)$ is tight ($=$ maxdegree in H)



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Thanks!