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Bier Spheres and Posets*

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Dedicated to Louis J. Billera on occasion of his 60th birthday

Abstract. In 1992 Thomas Bier presented a strikingly simple method to produce a huge number of simplicial $(n-2)$-spheres on $2n$ vertices, as deleted joins of a simplicial complex on n vertices with its combinatorial Alexander dual.

Here we interpret his construction as giving the poset of all the intervals in a boolean algebra that "cut across an ideal." Thus we arrive at a substantial generalization of Bier's construction: the *Bier posets* $\mathrm{Bier}(P, I)$ of an arbitrary bounded poset P of finite length. In the case of face posets of PL spheres this yields cellular "generalized Bier spheres." In the case of Eulerian or Cohen–Macaulay posets P we show that the Bier posets $\mathrm{Bier}(P, I)$ inherit these properties.

In the boolean case originally considered by Bier, we show that all the spheres produced by his construction are shellable, which yields "many shellable spheres," most of which lack convex realization. Finally, we present simple explicit formulas for the g-vectors of these simplicial spheres and verify that they satisfy a strong form of the g-conjecture for spheres.

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Introduction

In unpublished notes from 1992, Thomas Bier [Bi] described a strikingly simple construction of a large number of simplicial PL spheres. His construction associates a simplicial $(n-2)$-sphere with $2n$ vertices to any simplicial complex $\Delta \subset 2^{[1,n]}$ on n vertices (here $[1, n] := \{1, 2, \ldots, n\}$) by forming the "deleted join" of the complex Δ with its combinatorial Alexander dual, $\Delta^* := \{F \subset [1, n] : [1, n] \backslash F \notin \Delta\}$. Bier proved that this does indeed yield PL spheres by verifying that any addition of a new face to Δ amounts to a bistellar flip on the deleted join of Δ with its Alexander dual Δ^*. A short published account of this proof is given in Section 5.6 of [Ma], to where we also refer for the definition of deleted joins. See [deL] for a simple alternative proof.

In this paper we generalize and further analyze Bier's construction:

- We define more general "Bier posets" $\mathrm{Bier}(P, I)$, where P is an arbitrary bounded poset of finite length and $I \subset P$ is an order ideal.
- We show that in this generality, the order complex of $\mathrm{Bier}(P, I)$ is PL homeomorphic to that of P: it may be obtained by a sequence of stellar subdivisions of edges.
- If P is an Eulerian or Cohen–Macaulay poset or lattice, then $\mathrm{Bier}(P, I)$ will have that property as well.
- If P is the face lattice of a regular CW PL-sphere $\mathcal{S}$, then the lattices $\mathrm{Bier}(P, I)$ are again face lattices of regular CW PL-spheres, the "Bier spheres" of $\mathcal{S}$.
- In the case of the $(n-1)$-simplex, where $P = B_n$ is a boolean algebra, and the ideal in B_n may be interpreted as an abstract simplicial complex Δ, one obtains the "original" Bier spheres as described in [Bi], with face lattice $\mathrm{Bier}(B_n, \Delta)$. We prove that all these simplicial spheres are shellable.
- The number of these spheres is so great that for large n most of the Bier spheres $\mathrm{Bier}(B_n, \Delta)$ are not realizable as polytopes. Thus Bier's construction provides "many shellable spheres" in the sense of Kalai [Ka] and Lee [Le]; see also p. 116 of [Ma]. Similarly, for special choices of the simplicial complex Δ in B_n, and even n, we obtain many nearly neighborly centrally symmetric $(n-2)$-spheres on $2n$ vertices.
- The g-vector of a Bier sphere $\mathrm{Bier}(B_n, \Delta)$ can be expressed explicitly in terms of the f-vector of Δ. We show that these g-vectors are actually K-sequences, and thus they satisfy a strong form of the g-conjecture for spheres. Also, the generalized lower bound conjecture (characterizing the spheres for which $g_k = 0$) is verified for Bier spheres.

1. Basic Definitions and Properties

In this section we introduce our extension of Bier's construction to bounded posets, and present some simple properties. We refer to [St] for background, notation, and terminology relating to posets and lattices. Abstract simplicial complexes, order complexes, and shellability are reviewed in [Bj3]. See [Zi] for polytope theory.

All the posets we consider have finite length. A poset is *bounded* if it has a unique minimal and maximal element; we usually denote these by $\hat{0}$ and $\hat{1}$, respectively. For

$x \le y$, the *length* $\ell(x, y)$ is the length of a longest chain in the *interval* $[x, y] = \{z \in P : x \le z \le y\}$. A bounded poset is *graded* if all maximal chains have the same length. A graded poset is *Eulerian* if every interval $[x, y]$ with $x < y$ has the same number of elements of odd rank and even rank. An *ideal* in P is a subset $I \subseteq P$ such that $x \le y$ with $x \in P$ and $y \in I$ implies that $x \in I$. It is *proper* if neither $I = P$ nor $I = \emptyset$. Our notation in the following is set up in such a way that all elements of P named x, x_i, x'_i are elements of the ideal $I \subset P$, while elements called y, y_j, y'_j are in $P \backslash I$.

Definition 1.1. Let P be a bounded poset of finite length and let $I \subset P$ be a proper ideal. Then the poset $\mathrm{Bier}(P, I)$ is obtained as follows: it consists of all intervals $[x, y] \subseteq P$ such that $x \in I$ and $y \notin I$, ordered by reversed inclusion, together with an additional top element $\hat{1}$.

Here *reversed* inclusion says that $[x', y'] \le [x, y]$ amounts to $x' \le x < y \le y'$. The interval $I = [\hat{0}, \hat{1}]$ is the unique minimal element of $\mathrm{Bier}(P, I)$, so $\mathrm{Bier}(P, I)$ is bounded.

One may observe that the construction of Bier posets has a curious formal similarity to the E_t-construction of Paffenholz and Ziegler as defined in [PZ]. The study of posets of intervals in a given poset, ordered by inclusion, goes back to a problem posed by Lindström [Li]; see [Bj1] and [Bj4] for results on interval posets related to this problem.

Lemma 1.2. *Let P be a poset and let $I \subset P$ be a proper ideal.*

(i) *The posets P and* $\mathrm{Bier}(P, I)$ *have the same length n.*
(ii) $\mathrm{Bier}(P, I)$ *is graded if and only if P is graded.*
In that case, $\mathrm{rk}[x, y] = \mathrm{rk}_P x + (n - \mathrm{rk}_P y)$.
(iii) *The intervals of* $\mathrm{Bier}(P, I)$ *are of the following two kinds*:

$$[[x, y], \hat{1}] \cong \mathrm{Bier}([x, y], I \cap [x, y]),$$
$$[[x', y'], [x, y]] = [x', x] \times [y, y']^{\mathrm{op}},$$

where $[y, y']^{\mathrm{op}}$ *denotes the interval* $[y, y']$ *with the opposite order.*
(iv) *If P is a lattice, then* $\mathrm{Bier}(P, I)$ *is a lattice.*

Proof. $\mathrm{Bier}(P, I)$ is bounded. Thus for (iv) it suffices to show that meets exist in $\mathrm{Bier}(P, I)$. These are given by $[x, y] \wedge [x', y'] = [x \wedge x', y \vee y']$ and $[x, y] \wedge \hat{1} = [x, y]$. The other parts are immediate from the definitions. $\square$

2. Bier Posets via Stellar Subdivisions

For any bounded poset P we denote by $\overline{P} := P \backslash \{\hat{0}, \hat{1}\}$ the proper part of P and by $\Delta(\overline{P})$ the order complex of $\overline{P}$, that is, the abstract simplicial complex of all chains in $\overline{P}$ (see [Bj3]).

In this section we give a geometric interpretation of $\mathrm{Bier}(P, I)$, by specifying how its order complex may be derived from the order complex of P via stellar subdivisions.

For this, we need an explicit description of stellar subdivisions for abstract simplicial complexes. (See, e.g., p. 15 of [RS] for the topological setting.)

Definition 2.1. The *stellar subdivision* $\mathrm{sd}_F(\Delta)$ of a finite-dimensional simplicial complex Δ with respect to a nonempty face F is obtained by removing from Δ all faces that contain F and adding new faces $G \cup \{v_F\}$ (with a new apex vertex v_F) for all faces G that do not contain F, but such that $G \cup F$ is in the original complex.

In the special case of a stellar subdivision of an edge $E = \{v_1, v_2\}$, this means that each face $G \in \Delta$ that contains E is replaced by three new faces, namely $(G\backslash\{v_1\}) \cup \{v_E\}$, $(G\backslash\{v_2\}) \cup \{v_E\}$, and $(G\backslash\{v_1, v_2\}) \cup \{v_E\}$. Note that this replacement does not affect the Euler characteristic.

Remark. The stellar subdivisions in faces $F_1, \dots, F_N$ commute, and thus may be performed in any order—or simultaneously—if and only if no two F_i, F_j are contained in a common face G of the complex, that is, if $F_i \cup F_j$ is not a face for $i \neq j$.

Theorem 2.2. *Let P be a bounded poset of length $\ell(P) = n < \infty$, and let $I \subset P$ be a proper ideal. Then the order complex of $\overline{\mathrm{Bier}(P, I)}$ is obtained from the order complex of $\overline{P}$ by stellar subdivision on all edges of the form $\{x, y\}$, for $x \in \overline{I}$, $y \in \overline{P}\backslash\overline{I}$, $x < y$. These stellar subdivisions of edges $\{x, y\}$ must be performed in order of increasing length $\ell(x, y)$.*

Proof. In the following the elements denoted by x_i or x_i' are vertices of $\overline{P}$ that are contained in $\overline{I} := I\backslash\{\hat{0}\}$, while elements denoted by y_j or y_j' are from $\overline{P}\backslash\overline{I}$. By (x_i', y_i') we denote the new vertex created by subdivision of the edge $\{x_i', y_i'\}$.

We have to verify that subdivision of all the edges of $\Delta(\overline{P})$ collected in the sets

$$E_k := \{\{x, y\} : x < y,\ \ell(x, y) = k,\ x \in \overline{I},\ y \in \overline{P}\backslash\overline{I}\}$$

for $k = 1, \dots, n-2$ (in this order) results in $\Delta(\overline{\mathrm{Bier}(P, I)})$. To prove this, we explicitly describe the simplicial complexes Γ_k that we obtain at intermediate stages, after subdivision of the edges in $E_1 \cup \dots \cup E_k$. (The complexes Γ_k are *not*, in general, order complexes for $0 < k < n-2$.)

Claim. *After stellar subdivision of the edges of $\Delta(\overline{P})$ in the edge sets $E_1, \dots, E_k$ (in this order), the resulting complex Γ_k has the faces*

$$\{x_1, x_2, \dots, x_r, (x_1', y_1'), (x_2', y_2'), \dots, (x_t', y_t'), y_1, y_2, \dots, y_s\}, \tag{1}$$

where

(i) $$x_1 < x_2 < \dots < x_r < y_1 < y_2 < \dots < y_s \qquad (r, s \geq 0)$$

must be a strict chain in $\overline{P}$ that may be empty, but has to satisfy $\ell(x_r, y_1) \geq k+1$ if $r \geq 1$ and $s \geq 1$, while

(ii)
$$[x'_t, y'_t] < \cdots < [x'_2, y'_2] < [x'_1, y'_1] \qquad (t \geq 0)$$

must be a strict chain in $\overline{\mathrm{Bier}(P, I)}$ *that may be empty, but has to satisfy* $\ell(x'_t, y'_t) \leq k$ *if* $t \geq 1$, *and finally*

(iii)
$$x_r \leq x'_t \quad and \quad y'_t \leq y_1$$

must hold if both r *and* t *are positive, resp. if both* s *and* t *are positive.*

Conditions (i)–(ii) together imply that the chains of Γ_k are supported on (weak) chains in $\overline{P}$ of the form

$$\hat{0} < x_1 < x_2 < \cdots < x_r \leq x'_t \leq \cdots \leq x'_2 \leq x'_1 < y'_1 \leq y'_2 \leq \cdots \leq y'_t \leq y_1 < y_2 \cdots < y_s < \hat{1}.$$

In condition (iii) the two inequalities cannot both hold with equality, because of the length requirements for (i) and (ii), which for $r, s, t \geq 1$ mandate that $\ell(x'_t, y'_t) \leq k < \ell(x_r, y_1)$, and thus $[x'_t, y'_t] \subset [x_r, y_1]$.

We verify immediately that for $k = 0$ the description of Γ_0 given in the claim yields $\Gamma_0 = \Delta(\overline{P})$, since for $k = 0$ the length requirement for (ii) does not admit any subdivision vertices.

For $k = n - 2$ the simplices of Γ_{n-2} as given by the claim cannot contain both x_r and y_1, that is, they all satisfy either $r = 0$ or $s = 0$ or both, since otherwise we would get a contradiction between the length requirement for (i) and the fact that any interval $[x_r, y_1] \subseteq \overline{P}$ can have length at most $n-2$. Thus we obtain that $\Gamma_{n-2} = \Delta(\overline{\mathrm{Bier}(P, I)})$, if we identify the subdivision vertices (x'_i, y'_i) with the intervals $[x'_i, y'_i]$ in P, the elements x_i with the intervals $[x_i, \hat{1}]$, and the elements $y_j \in \overline{P} \backslash \overline{I}$ with the intervals $[\hat{0}, y_j]$.

Finally, we prove the claim by verifying the induction step from k to $k+1$. It follows from the description of the complex Γ_k that no two edges in E_{k+1} lie in the same facet. Thus we can stellarly subdivide the edges in E_{k+1} in arbitrary order. Suppose the edge (x_r, y_1) of the simplex

$$\{x_1, \ldots, x_{r-1}, x_r, (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), y_1, y_2, \ldots, y_s\}$$

is contained in E_{k+1}. Then stellar subdivision yields the three new simplices

$$\begin{array}{llll}\{x_1, \ldots, x_{r-1}, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), & y_1, y_2, \ldots, y_s, \}, & \\ \{x_1, \ldots, x_{r-1}, x_r, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), & y_2, \ldots, y_s, \}, & \text{and} \\ \{x_1, \ldots, x_{r-1}, & (x_r, y_1), (x'_1, y'_1), (x'_2, y'_2), \ldots, (x'_t, y'_t), & y_2, \ldots, y_s, \}. & \end{array}$$

All three sets then are simplices of Γ_{k+1}, satisfying all the conditions specified in the claim (with t replaced by $t+1$ and r or s or both reduced by 1). Also all simplices of Γ_{k+1} arise this way. This completes the induction step. $\square$

We can write down the subdivision map of the previous proof explicitly: the map

$$\pi\colon \|\Delta(\overline{\mathrm{Bier}(P, I)})\| \to \|\Delta(\overline{P})\|$$

is given on the vertices of $\Delta(\overline{\mathrm{Bier}(P, I)})$ by

$$[x, y] \mapsto \begin{cases} \frac{1}{2}x + \frac{1}{2}y, & \hat{0} < x < y < \hat{1}, \quad x \in I, \quad y \notin I, \\ x, & \hat{0} < x < y = \hat{1}, \quad x \in I, \quad y \notin I, \\ y, & \hat{0} = x < y < \hat{1}, \quad x \in I, \quad y \notin I, \end{cases}$$

and extends linearly on the simplices of $\Delta(\overline{\mathrm{Bier}(P, I)})$.

Corollary 2.3. $\|\Delta(\overline{\mathrm{Bier}(P, I)})\|$ *and* $\|\Delta(\overline{P})\|$ *are PL homeomorphic.*

In the case where P is the face poset of a regular PL CW-sphere or -manifold, this implies that the barycentric subdivision of $\mathrm{Bier}(P, I)$ may be derived from the barycentric subdivision of P by stellar subdivisions. In particular, in this case $\mathrm{Bier}(P, I)$ is again the face poset of a PL-sphere or manifold.

Corollary 2.4. *If P is the face lattice of a strongly regular PL CW-sphere, then so is* $\mathrm{Bier}(P, I)$.

Corollary 2.5. *If P is Cohen–Macaulay, then so is* $\mathrm{Bier}(P, I)$.

Proof. This follows from the fact that Cohen–Macaulayness (with respect to arbitrary coefficients) is a topological property [Mu]. □

3. Eulerian Posets

From now on we assume that P is a graded poset of length n, and that $I \subset P$ is a proper order ideal, with $\hat{0}_P \in I$ and $\hat{1}_P \notin I$. First we compute the f-vector $f(\mathrm{Bier}(P, I)) := (f_0, f_1, \ldots, f_n)$, where f_i denotes the elements of *rank i* in the poset $\mathrm{Bier}(P, I)$. (This notation is off by 1 from the usual convention in polytope theory, as in [Zi].) By definition we have $f_n(\mathrm{Bier}(P, I)) = 1$ and

$$f_i(\mathrm{Bier}(P, I)) = \#\{[x, y] : x \in I, y \notin I, \mathrm{rk}_P x + n - \mathrm{rk}_P y = i\}$$

for $0 \leq i \leq n - 1$. In particular, $f_0(\mathrm{Bier}(P, I)) = 1$.

Theorem 3.1. *Let P be an Eulerian poset and let $I \subset P$ be a proper ideal. Then* $\mathrm{Bier}(P, I)$ *is also an Eulerian poset.*

Proof. $\mathrm{Bier}(P, I)$ is a graded poset of the same length as P by Lemma 1.2. Thus it remains to prove that all intervals of length ≥ 1 in $\mathrm{Bier}(P, I)$ contain equally many odd and even rank elements.

This can be done by induction. For length $\ell(P) \leq 1$ the claim is true. Proper intervals of the form $[[x, y], \hat{1}]$ are, in view of Lemma 1.2, Eulerian by induction. Proper intervals of the form $[[x', y'], [x, y]]$ are Eulerian, since any product of Eulerian posets is Eulerian.

Finally, the whole poset $\mathrm{Bier}(P, I)$ contains the same number of odd and even rank elements by the following computation:

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^{n-i} f_i(\mathrm{Bier}(P, I)) &= 1+\sum_{i=0}^{n-1}(-1)^{n-i} f_i(\mathrm{Bier}(P, I)) \\
&= 1+\sum_{y\notin I}\sum_{\substack{x\in I\\ x\le y}}(-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} \\
&= 1+\sum_{y\notin I}\sum_{x\le y}(-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} - \sum_{y\notin I}\sum_{\substack{x\notin I\\ x\le y}}(-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} \qquad (2)\\
&= 1+0-\sum_{x\notin I}\sum_{x\le y}(-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} \qquad (3)\\
&= 1+0-1=0,
\end{aligned}$$

where the first double sum in (2) is 0 as $[\hat{0}_P, y]$ is Eulerian and $\mathrm{rk}(y) \ge 1$, and the double sum in (3) is -1 as $[x, \hat{1}_P]$ is Eulerian and trivial only for $x = \hat{1}_P$. $\square$

Alternatively, the result of the computation in this proof also follows from the topological interpretation of $\mathrm{Bier}(P, I)$ in the previous section.

4. Shellability of Bier Spheres

Now we specialize to Bier's original setting, where $P = B_n$ is the boolean lattice of all subsets of the ground set $[1, n] = \{1, \dots, n\}$ (which may be identified with the set of atoms of B_n), ordered by inclusion. We use notation like $[1, n]$ or $(x, n]$ freely to denote closed or half-open sets of integers.

Any nonempty ideal in the boolean algebra B_n can be interpreted as an abstract simplicial complex with at most n vertices, so we denote it by Δ.

We get

$$\mathrm{Bier}(B_n, \Delta)\backslash\{\hat{1}\} = \{(B, C) : \emptyset \subseteq B \subset C \subseteq [1, n], B \in \Delta, C \notin \Delta\}$$

again ordered by reversed inclusion of intervals. We denote the facets of $\mathrm{Bier}(B_n, \Delta)$ by $(A; x) := (A, A \cup \{x\}) \in \mathrm{Bier}(B_n, \Delta)$ and the set of all facets by $\mathcal{F}(\Delta)$.

The poset $\mathrm{Bier}(B_n, \Delta)$ is the face lattice of a simplicial PL $(n-2)$-sphere, by Corollary 2.4. We will now prove a strengthening of this, namely that $\mathrm{Bier}(B_n, \Delta)$ is shellable. (As is known, see [Bj3], shellability implies the PL-sphericity for pseudomanifolds.)

Theorem 4.1. *For every proper ideal $\Delta \subset B_n$, the $(n-2)$-sphere* $\mathrm{Bier}(B_n, \Delta)$ *is shellable.*

Proof. The shellability proof is in two steps. First we show that the rule

$$\begin{aligned}
R\colon \mathcal{F}(\Delta) &\to \mathrm{Bier}(B_n, \Delta),\\
(A; x) &\mapsto (A \cap (x, n], A \cup [x, n]) \qquad (4)
\end{aligned}$$

defines a restriction operator in the sense of [Bj2]; that is, it induces a partition

$$\mathrm{Bier}(B_n, \Delta) = \biguplus_{(A;x)\in\mathcal{F}(\Delta)} [R(A;x), (A;x)].$$

Then we prove that the precedence relation forced by this restriction operator is acyclic. Thus, any linear extension of the precedence relation yields a shelling order.

That the restriction operator indeed defines a partition can be seen as follows: Take any element $(B, C) \in \mathrm{Bier}(B_n, \Delta)$. Set

$$\begin{aligned} x :=&\ \min\{y \in C\backslash B : B \cup (C \cap [1, y]) \notin \Delta\} \\ =&\ \max\{y \in C\backslash B : B \cup (C\backslash[y, n]) \in \Delta\} \end{aligned}$$

and $A := B \cup (C \cap [1, x))$. Then we have

$$A \cap (x, n] \subseteq B \subseteq A \subset A \cup \{x\} \subseteq C \subseteq A \cup [x, n]$$

and thus (B, C) is contained in $[R(A; x), (A; x)]$.

To see that the intervals in the partition do not intersect we have to show that if both $R(A; x) \le (A'; x')$ and $R(A'; x') \le (A; x)$, then $(A; x) = (A'; x')$. This is a special case of a more general fact we establish next, so we do not give the argument here.

For any shelling order "$\prec$" that would induce R as its "unique minimal new face" restriction operator we are forced to require that if $R(A; x) \le (A'; x')$ for two facets $(A; x)$ and $(A'; x')$, then $(A; x) \preceq (A'; x')$. By definition, $R(A; x) \le (A'; x')$ means that

$$A \cap (x, n] \subseteq A' \subset A' \cup \{x'\} \subseteq A \cup [x, n], \tag{5}$$

which may be reformulated as

$$(A \cup \{x\})_{>x} \subseteq A' \quad \text{and} \quad (A' \cup \{x'\})_{<x} \subseteq A. \tag{6}$$

We now *define* the relation $(A; x) \prec (A'; x')$ to hold if and only if (6) holds together with

$$(A \cup \{x\})_{\le x} \nsubseteq A' \quad \text{and} \quad (A' \cup \{x'\})_{\ge x} \nsubseteq A. \tag{7}$$

Note that our sets A, A' belong to an ideal which does not contain $A \cup \{x\}$, $A' \cup \{x'\}$, so (7) applies if (6) does.

By the *support* of $(A; x)$ we mean the set $A \cup \{x\}$. The element x of the support is called its *root element*.

We interpret a relation $(A; x) \prec (A'; x')$ as a *step* from $(A; x)$ to $(A'; x')$. The first conditions of (6) and (7) say that

> In each step the elements that are deleted from the support are $\le x$; moreover, we must either lose some element $\le x$ from the support, or we must choose x' from $(A \cup \{x\})_{\le x}$, or both. (8)

Similarly, the second conditions of (6) and (7) say that

> In each step the elements that are added to the support are $> x$; moreover, we must either add some element $> x$ to the support, or we must keep x in the support, or both. (9)

Now we show that the transitive closure of the relation $\prec$ does not contain any cycles. So, suppose that there is a cycle,

$$(A_0; x_0) \prec (A_1; x_1) \prec \cdots \prec (A_k; x_k) = (A_0; x_0).$$

First assume that not all root elements x_i in this cycle are equal. Then by cyclic permutation we may assume that x_0 is the smallest root element that appears in the cycle, and that $x_1 > x_0$. Thus x_1 is clearly not from $(A \cup \{x_0\})_{\leq x_0}$, so by condition (8) we lose an element $\leq x_0$ from the support of $(A_0; x_0)$ in this step. However, in all later steps the elements we add to the support are $> x_i \geq x_0$, so the lost element will never be retrieved. Hence we cannot have a cycle.

The second possibility is that all root elements in the cycle are equal, that is, $x_0 = x_1 = \cdots = x_k = x$. Then by conditions (8) and (9), in the whole cycle we lose only elements $< x$ from the support, and we add only elements $> x$. The only way this can happen is that, when we traverse the cycle, no elements are lost and none are added, so $A_0 = A_1 = \cdots = A_k$. Consequently, there is no cycle. □

The relation defined on the set of all pairs $(A; x)$ with $A \subset [1, n]$ and $x \in [1, n]\backslash A$ by (6) alone does have cycles, such as

$$(\{1, 4\}, 2) \prec (\{1, 4\}, 3) \prec (\{4\}, 1) \prec (\{1, 4\}, 2).$$

This is the reason why we also require condition (7) in the definition of "$\prec$".

The shelling order implied by the proof of Theorem 4.1 may also be described in terms of a linear ordering. For that we associate with each facet $(A; x)$ a vector $\chi(A; x) \in \mathbb{R}^n$, defined as follows:

$$\chi(A; x)_a := \begin{cases} -1 & \text{for} \quad a \in (A \cup \{x\})_{\leq x}, \\ 0 & \text{for} \quad a \notin A \cup \{x\}, \\ +1 & \text{for} \quad a \in (A \cup \{x\})_{>x}. \end{cases} \quad \text{and}$$

With this assignment, we get that $(A; x) \prec (A'; x')$, as characterized by conditions (8) and (9), implies that $\chi(A; x) <_{\text{lex}} \chi(A'; x')$. Thus we have that lexicographic ordering on the χ-vectors induces a shelling order for every "boolean Bier sphere."

5. g-Vectors

The f-vectors of triangulated spheres are of great combinatorial interest. In this section we derive the basic relationship between the f-vector of a Bier sphere $\text{Bier}(B_n, \Delta)$ and the f-vector of the underlying simplicial complex Δ. (Such an investigation had been begun in Bier's note [Bi].)

In an extension of the notation of Section 3 let $f_i(\Delta)$ denote the number of sets of cardinality i in a complex Δ. The *f-vector* of a proper subcomplex $\Delta \subset B_n$ is $f(\Delta) = (f_0, f_1, \ldots, f_n)$, with $f_0 = 1$ and $f_n = 0$.

Now let Γ be a finite simplicial complex that is pure of dimension $d = n-2$, that is, such that all maximal faces have cardinality $n-1$. (Below we apply this to $\Gamma = \mathrm{Bier}(B_n, \Delta)$.) We define $h_i(\Gamma)$ by

$$h_i(\Gamma) := \sum_{j=0}^{n-1} (-1)^{i+j} \binom{n-1-j}{n-1-i} f_j(\Gamma) \tag{10}$$

for $0 \le i \le n-1$, and $h_i(\Gamma) := 0$ outside this range. Then, conversely,

$$f_i(\Gamma) = \sum_{j=0}^{n-1} \binom{n-1-j}{n-1-i} h_j(\Gamma).$$

Finally, for $0 \le i \le \lfloor (n-1)/2 \rfloor$ let $g_i(\Gamma) := h_i(\Gamma) - h_{i-1}(\Gamma)$, with $g_0(\Gamma) = 1$.

Now we consider the f-, h-, and g-vectors of the sphere $\Gamma = \mathrm{Bier}(B_n, \Delta)$. It is an $(n-2)$-dimensional shellable sphere on $f_1(\Delta) + n - f_{n-1}(\Delta)$ vertices. (So for the usual case of $f_1 = n$ and $f_{n-1} = 0$, when Δ contains all the 1-element subsets but no $(n-1)$-element subset of $[1, n]$, we get a sphere on $2n$ vertices.) In terms of the facets $(A; x) \in \mathcal{F}(\Delta)$ we have the following description of its h-vector:

$$h_i(\mathrm{Bier}(B_n, \Delta)) = \#\{(A; x) \in \mathcal{F}(\Delta) : |\, A \cap (x, n]\,| + |\,[1, x)\backslash A\,| = i\} \tag{11}$$

for $0 \le i \le n-1$. This follows from the interpretation of the h-vector of a shellable complex in terms of the restriction operator as

$$h_i(\mathrm{Bier}(B_n, \Delta)) = \#\{(A; x) \in \mathcal{F}(\Delta) : \mathrm{rk}(R(A; x)) = i\},$$

see p. 229 of [Bj2], together with (4) and Lemma 1.2(ii).

Lemma 5.1 (Dehn–Sommerville Equations). *For* $0 \le i \le n-1$,

$$h_{n-1-i}(\mathrm{Bier}(B_n, \Delta)) = h_i(\mathrm{Bier}(B_n, \Delta)).$$

Proof. It is a nontrivial fact that this relation is true for *any* triangulated $(n-2)$-sphere. However, in our situation it is a direct and elementary consequence of (11).

Namely, neither the definition of the h-vector nor the construction of the Bier sphere depends on the ordering of the ground set. Thus we can reverse the order of the ground set $[1, n]$, to get that

$$h_i(\mathrm{Bier}(B_n, \Delta)) = \#\{(A; x) \in \mathcal{F}(\Delta) : |\, A \cap [1, x)\,| + |\,(x, n]\backslash A\,| = i\}. \tag{12}$$

Thus a set A contributes to $h_i(\mathrm{Bier}(B_n, \Delta))$ according to (11) if and only if the complement of A with respect to the $(n-1)$-element set $[1, n]\backslash\{x\}$ contributes to $h_{n-1-i}(\mathrm{Bier}(B_n, \Delta))$ according to (12). $\square$

The g-vector of $\mathrm{Bier}(B_n, \Delta)$ has the following nice form:

Theorem 5.2. *For all* $i = 0, \ldots, \lfloor (n-1)/2 \rfloor$,

$$g_i(\mathrm{Bier}(B_n, \Delta)) \;=\; f_i(\Delta) - f_{n-i}(\Delta).$$

Proof. Let Δ^{aug} be the same complex as Δ, but viewed as sitting inside the larger boolean lattice B_{n+1}. We claim that

$$h_i(\mathrm{Bier}(B_{n+1}, \Delta^{\mathrm{aug}})) \;=\; h_{i-1}(\mathrm{Bier}(B_n, \Delta)) + f_i(\Delta) \tag{13}$$

for $0 \le i \le n$. This is seen from (12) as follows. The facets $(A; x)$ of $\mathrm{Bier}(B_{n+1}, \Delta^{\mathrm{aug}})$ that contribute to $h_i(\mathrm{Bier}(B_{n+1}, \Delta^{\mathrm{aug}}))$ are of two kinds: either $x \neq n+1$ or $x = n+1$. There are $h_{i-1}(\mathrm{Bier}(B_n, \Delta))$ of the first kind and $f_i(\Delta)$ of the second.

Using both (13) and Lemma 5.1 twice we compute

$$\begin{aligned}
g_i(\mathrm{Bier}(B_n, \Delta)) &= h_i(\mathrm{Bier}(B_n, \Delta)) - h_{i-1}(\mathrm{Bier}(B_n, \Delta)) \\
&= h_{n-1-i}(\mathrm{Bier}(B_n, \Delta)) - h_{i-1}(\mathrm{Bier}(B_n, \Delta)) \\
&= h_{n-i}(\mathrm{Bier}(B_{n+1}, \Delta^{\mathrm{aug}})) - f_{n-i}(\Delta) - h_{i-1}(\mathrm{Bier}(B_n, \Delta)) \\
&= h_i(\mathrm{Bier}(B_{n+1}, \Delta^{\mathrm{aug}})) - f_{n-i}(\Delta) - h_{i-1}(\mathrm{Bier}(B_n, \Delta)) \\
&= f_i(\Delta) - f_{n-i}(\Delta).
\end{aligned}$$ □

Corollary 5.3. *The face numbers* $f_i(\mathrm{Bier}(B_n, \Delta))$ *of the Bier sphere depend only on* n *and the differences* $f_i(\Delta) - f_{n-i}(\Delta)$.

Proof. The g-vector determines the h-vector (via Lemma 5.1), which determines the f-vector. □

For example, if $n = 4$ and $f(\Delta) = (1, 3, 0, 0, 0)$ or $f(\Delta) = (1, 4, 3, 1, 0)$, then we get $g(\mathrm{Bier}(B_4, \Delta)) = (1, 3)$ and $f(\mathrm{Bier}(B_4, \Delta)) = (1, 7, 15, 10)$.

Theorem 5.4. *Every simplicial complex* $\Delta \subseteq B_n$ *has a subcomplex* Δ' *such that*

$$f_i(\Delta') = f_i(\Delta) - f_{n-i}(\Delta)$$

for $0 \le i \le \lfloor n/2 \rfloor$ *and* $f_i(\Delta') = 0$ *for* $i > \lfloor n/2 \rfloor$.

Proof. For any simplicial complex Δ in B_n, define the d-vector by $d_i(\Delta) = f_i(\Delta) - f_{n-i}(\Delta)$ for $0 \le i \le \lfloor n/2 \rfloor$ and $d_i(\Delta) = 0$ for greater i. We shall find a subcomplex $\Delta' \subseteq \Delta$ with $f_i(\Delta') = d_i(\Delta)$ for all i.

Choose Δ' as a minimal subcomplex of Δ with the same d-vector. We must show that $f_i(\Delta') = 0$ for all $\lfloor n/2 \rfloor < i \le n$. Suppose that there is a set $C \in \Delta'$ with $|C| > n/2$. Then there is an involution $\pi : [1, n] \to [1, n]$, i.e., a permutation of the ground set of order two, such that

$$\pi(C) \supseteq [1, n] \backslash C, \tag{14}$$

where $\pi(C)$ is the image of C. Now define $\varphi\colon B_n \to B_n$ by $\varphi(B) = [1,n]\backslash\pi(B)$ for all $B \subseteq [1,n]$. Observe that φ satisfies the following for all $B \subseteq [1,n]$:

(a) $\varphi(\varphi(B)) = B$,
(b) $B' \subseteq B \Rightarrow \varphi(B') \supseteq \varphi(B)$,
(c) $|B| + |\varphi(B)| = n$.

Let $K := \{B \in \Delta' : \varphi(B) \in \Delta'\}$. We claim that $\Delta'\backslash K$ is a simplicial complex with the same d-vector as Δ'.

First, we show that $\Delta'\backslash K$ is a complex. Let $B' \subseteq B \in \Delta'\backslash K$. Then $B' \in \Delta'$ so we must show that $B' \notin K$. Property (b) gives $\varphi(B') \supseteq \varphi(B)$, so we get $B \notin K \Rightarrow \varphi(B) \notin \Delta' \Rightarrow \varphi(B') \notin \Delta' \Rightarrow B' \notin K$.

Let $K_i = \{B \in K : |B| = i\}$ for $0 \le i \le n$. We have $d_i(\Delta'\backslash K) = (f_i(\Delta') - |K_i|) - (f_{n-i}(\Delta') - |K_{n-i}|) = d_i(\Delta') - (|K_i| - |K_{n-i}|)$ for $0 \le i \le \lfloor n/2 \rfloor$. We must show that $|K_i| = |K_{n-i}|$ for all i. Property (a) gives that $B \in K \Leftrightarrow \varphi(B) \in K$. Finally, property (c) gives that φ is a bijection between K_i and K_{n-i} for all i.

Fortunately, $K \neq \emptyset$ since $\varphi(C) = [1,n]\backslash\pi(C) \subseteq C$ by (14), whence $\varphi(C) \in \Delta'$ and $C \in K$. Thus we have found a strictly smaller subcomplex of Δ' with the same d-vector—a contradiction against our choice of Δ'. □

Corollary 5.5. *There is a subcomplex Δ' of Δ such that*

$$g_i(\mathrm{Bier}(B_n, \Delta)) = f_i(\Delta')$$

for $0 \le i \le \lfloor (n-1)/2 \rfloor$ and $f_i(\Delta') = 0$ for $i > \lfloor (n-1)/2 \rfloor$.

It is a consequence of Corollary 5.5 that the g-vector $(g_0, g_1, \ldots, g_{\lfloor (n-1)/2 \rfloor})$ of $\mathrm{Bier}(B_n, \Delta)$ is a *K-sequence*, i.e., it satisfies the Kruskal–Katona theorem. This is of interest in connection with the so-called g-conjecture for spheres, which suggests that g-vectors of spheres are M-sequences (satisfy Macaulay's theorem). K-sequences are a very special subclass of M-sequences, thus g-vectors (and hence f-vectors) of Bier spheres are quite special among those of general triangulated $(n-2)$-spheres on $2n$ vertices. See Chapter 8 of [Zi] for details concerning K- and M-sequences and g-vectors.

What has been shown also implies the following:

Corollary 5.6. *Every K-sequence $(1, n, \ldots, f_k)$ with $k \le \lfloor (n-1)/2 \rfloor$ can be realized as the g-vector of a Bier sphere with $2n$ vertices.*

We need to review the definition of bistellar flips: Let Γ be a simplicial d-manifold. If A is a $(d-i)$-dimensional face of Γ, $0 \le i \le d$, such that $\mathrm{link}_\Gamma(A)$ is the boundary $Bd(B)$ of an i-simplex B that is not a face of Γ, then the operation Φ_A on Γ defined by

$$\Phi_A(\Gamma) := (\Gamma\backslash(A * Bd(B))) \cup (Bd(A) * B)$$

is called a *bistellar i-flip*. Then $\Phi_A(\Gamma)$ is itself a simplicial d-manifold, homeomorphic to Γ, and if $0 \le i \le \lfloor (d-1)/2 \rfloor$, then

$$\begin{aligned} g_{i+1}(\Phi_A(\Gamma)) &= g_{i+1}(\Gamma) + 1, \\ g_j(\Phi_A(\Gamma)) &= g_j(\Gamma) \qquad \text{for all} \quad j \neq i+1. \end{aligned} \tag{15}$$

Furthermore, if d is even and $i = d/2$, then $g_j(\Phi_A(\Gamma)) = g_j(\Gamma)$ for all j. See p. 83 of [Pa].

It follows from Corollary 5.5 that $g_k(\mathrm{Bier}(B_n, \Delta)) \geq 0$. The case of equality is characterized as follows:

Corollary 5.7. *For* $2 \leq k \leq \lfloor (n-1)/2 \rfloor$, *the following are equivalent*:

(1) $g_k(\mathrm{Bier}(B_n, \Delta)) = 0$,
(2) $f_k(\Delta) = 0$ *or* $f_{n-k}(\Delta) = \binom{n}{i}$,
(3) $\mathrm{Bier}(B_n, \Delta)$ *is obtained from the boundary complex of the* $(n-1)$*-simplex via a sequence of bistellar* i*-flips, with* $i \leq k-2$ *at every flip*.

Proof. (1) $\Rightarrow$ (2) Consider the bipartite graph $G_{n,k}$ whose edges are the pairs (A, B) such that A is a k-element subset, B is an $(n-k)$-element subset of $[1, n]$, and $A \subset B$, where the inclusion is strict since $k < n-k$. Then $G_{n,k}$ is a regular bipartite graph (all vertices have the same degree), so by standard matching theory $G_{n,k}$ has a complete matching. The restriction of such a matching to the sets B in Δ gives an injective mapping $\Delta_{n-k} \to \Delta_k$ from Δ's faces of cardinality $n-k$ to those of cardinality k.

Equality $f_{n-k}(\Delta) = f_k(\Delta)$ implies that $G_{n,k}$ consists of two connected components, one of which is induced on $\Delta_{n-k} \cup \Delta_k$. A nontrivial such splitting cannot happen since $G_{n,k}$ is connected, so either Δ_{n-k} and Δ_k are both empty, or they are both the full families of cardinality $\binom{n}{k}$.

(2) $\Rightarrow$ (3) As shown in [Bi] and Section 5.6 of [Ma], adding an i-dimensional face to Δ produces a bistellar i-flip in $\mathrm{Bier}(B_n, \Delta)$. Now, Δ can be obtained from the empty complex by adding i-dimensional faces, and here all $i \leq k-2$ if $f_k(\Delta) = 0$ (meaning that there are no faces of dimension $k-1$ in Δ). The case when $f_{n-k}(\Delta) = \binom{n}{i}$ is the same by symmetry.

(3) $\Rightarrow$ (1) This follows directly from (15), since the boundary of the $(n-1)$-simplex has g-vector $(1, 0, \ldots, 0)$. ☐

A convex polytope whose boundary complex is obtained from the boundary complex of the $(n-1)$-simplex via a sequence of bistellar i-flips, with $i \leq k-2$ at every flip, is called *k-stacked*. The *generalized lower bound conjecture* for polytopes maintains that $g_k = 0$ for a polytope if and only if it is k-stacked. This is still open for general polytopes. See [Mc] for a recent discussion. Corollary 5.7 shows that it is valid for those polytopes that arise via the Bier sphere construction.

6. Further Observations

6.1. *Many Spheres*

In the Introduction we remarked that the (isomorphism classes) of Bier spheres are numerous, in fact so numerous that one concludes that most of them lack convex realization. To show this, it suffices to consider Bier spheres $\mathrm{Bier}(B_n, \Delta)$ for complexes Δ that contain all sets $A \subset [1, n]$ of size $|A| \leq \lfloor (n-1)/2 \rfloor$, a subcollection of the sets of size $|A| = \lfloor (n-1)/2 \rfloor + 1 = \lfloor (n+1)/2 \rfloor$, and no larger faces. Equivalently, Δ is a complex of dimension at most $\lfloor (n-1)/2 \rfloor$ with a complete $(\lfloor (n-1)/2 \rfloor - 1)$-skeleton. There are

$$\binom{n}{\lfloor (n+1)/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor}$$

elements in the $\lfloor (n+1)/2 \rfloor$-level of B_n; thus there are at least

$$\frac{2^{\binom{n}{\lfloor n/2 \rfloor}}}{(2n)!} \sim \frac{2^{2^n/\sqrt{n}}}{(2n/e)^{2n}}$$

combinatorially nonisomorphic such Bier spheres (where our rough approximation ignores polynomial factors). On the other hand, there are at most $2^{8n^3+O(n^2)}$ combinatorially nonisomorphic simplicial polytopes on $2n$ vertices (see [GP], and Theorem 5.1 of [Al]).

It is interesting to contrast this with all the ways in which these "numerous" spheres are very special: they are shellable, their g-vectors are K-sequences, and for even n we even get numerous "nearly neighborly" examples (as discussed below). Another construction of "numerous" shellable spheres is known from the work of Kalai [Ka] and Lee [Le].

Though we have defined the construction of a Bier poset for arbitrary posets and have shown that the construction produces sphere lattices from sphere lattices, it remains an open problem of how to extend the Bier construction to obtain numerous simplicial/shellable $(n-2)$-spheres with more than $2n$ vertices.

6.2. *Centrally Symmetric and Nearly Neighborly Spheres*

Let Γ be a triangulated $(n-2)$-sphere on $2m$ vertices. The sphere Γ is *centrally symmetric* if it has a symmetry of order two which fixes no face; that is, if there is a fixed-point-free involution on its set V of vertices such that (i) for every face A of Γ, $\alpha(A)$ is also a face, and (ii) $\{x, \alpha(x)\}$ is not a face, for all $x \in V$. A subset $A \subseteq V$ is *antipode-free* if it contains no pair $\{x, \alpha(x)\}$, for $x \in V$.

A centrally symmetric sphere Γ with involution α is *k-nearly neighborly* if all antipode-free sets $A \subseteq V$ of size $|A| \leq k$ are faces of Γ. Equivalently, Γ must contain the $(k-1)$-skeleton of the m-dimensional hyperoctahedron (cross-polytope). Γ is *nearly neighborly* if it is $\lfloor (n-1)/2 \rfloor$-nearly neighborly.

Thus k-nearly neighborliness is defined only for centrally symmetric spheres. In the case $k \geq 2$ the involution α is uniquely determined by the condition $\{x, \alpha(x)\} \notin \Gamma$.

The concept of nearly neighborliness for centrally symmetric spheres has been studied for centrally symmetric $(n-1)$-polytopes, where α is of course the map $x \mapsto -x$. For instance, work of Grünbaum, McMullen and Shephard, Schneider, and Burton shows that there are severe restrictions to k-nearly neighborliness in the centrally symmetric polytope case, while the existence of interesting classes of nearly neighborly spheres was proved by Grünbaum, Jockusch, and Lutz; see p. 279 of [Zi] and Chapter 4 of [Lu].

Nearly neighborly Bier spheres arise as follows. (In the following, only the special case $m = n$, of an $(n-2)$-sphere with $2n$ vertices, occurs.)

Proposition 6.1. *If* $A \in \Delta \iff [1,n] \backslash A \notin \Delta$, *then* $\mathrm{Bier}(B_n, \Delta)$ *is centrally symmetric.*

Proof. The involution α is given by the pairing $[\{x\}, \hat{1}] \longleftrightarrow [\hat{0}, [1,n] \backslash \{x\}]$. □

Proposition 6.2. *Let* $1 < k \leq \lfloor (n-1)/2 \rfloor$. *The Bier sphere* $\mathrm{Bier}(B_n, \Delta)$ *is a k-nearly neighborly* $(n-2)$*-sphere with* $2n$ *vertices if and only if*

(i) $A \in \Delta \iff [1,n] \backslash A \notin \Delta$, *for all* $A \subseteq [1,n]$,
(ii) $B \in \Delta$, *for all* $B \subseteq [1,n]$, $|B| \leq k$ (*and thus* $C \notin \Delta$ *for all* $C \subseteq [1,n]$, $|C| \geq n-k$).

Proof. The Bier sphere $\mathrm{Bier}(B_n, \Delta)$ has $2n$ vertices if and only if $\Delta \subset 2^{[1,n]}$ is a complex that contains all subsets of cardinality 1 and no subsets of cardinality $n-1$. The antipode-free vertex sets of cardinality k then correspond to intervals $[B, C] \subseteq B_n$ such that $|B| + (n - |C|) = k$. A set B is the minimal element of such an interval if and only if $|B| \leq k$, while C is a maximal element for $|C| \geq n-k$. □

Combining these two propositions we obtain a large number of even-dimensional nearly neighborly centrally symmetric Bier spheres. Indeed, in the case of even n we get at least

$$\frac{2^{\frac{1}{2}\binom{n}{\lfloor n/2 \rfloor}}}{(2n)!}$$

nonisomorphic spheres, from the complexes Δ which contain all sets of size $A < n/2$, and exactly one set from each pair of sets A and $[1,n] \backslash A$ of size $|A| = n/2$.

On the other hand, for odd n (that is, in the case of an odd-dimensional sphere, or an even-dimensional polytope, where the "nearly neighborliness condition" is stronger and hence more interesting) only one instance of a nearly neighborly centrally symmetric Bier $(n-2)$-sphere with $2n$ vertices is obtained; namely, for $\Delta = \{A \subset [1,n] : |A| \leq \lfloor n/2 \rfloor\}$.

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