Mathematics Department
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Algebraic Combinatorics: Problem set #2

(1) Suppose that $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$. Show that:

$$\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} b_i^k \qquad , \quad 1 \le k \le n$$
$$\Rightarrow a_i = b_{\pi(i)} \quad , \quad 1 \le i \le n, \text{ for some } \pi \in S_n.$$

(2) Show that: $\lambda \le \mu \iff \mu' \le \lambda'$, for $\forall \lambda$, $\mu \vdash m$. (Here $\le$ denotes dominance order and λ' conjugate partition.)

(3) Show that dominance order is a *lattice*. (That is, every pair of elements has a greatest lower bound and a least upper bound.)

(4) Prove that

$$\prod_{1 \le i,j \le n} (1 + x_i y_j) = \sum_{\lambda} e_\lambda(x) m_\lambda(y)$$

with sum over all diagrams λ that fit into an $n \times n$ square.

(5) Show that

$$\omega(p_\lambda) = (-1)^{m-\ell(\lambda)} \omega(p_{\lambda'})$$

where ω is the fundamental involution and $\lambda \vdash m$.

(6) Let A_n denote the set of standard Young tableaux $T = (t_{i,j})$, $i \in [2], j \in [n]$ of shape $\lambda = (n, n)$ and with the following symmetry property: $t_{1,j} = 2n + 1 - t_{2,n+1-j}$, for all $j \in [n]$. Show that A_n has $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements.

(7) Prove Newton's formulas:

(a)

$$m h_m = \sum_{k=1}^{m} p_k h_{m-k} \quad , \quad m \ge 1.$$

(b)

$$h_m = \frac{1}{m!} \det \begin{vmatrix} p_1 & -1 & 0 & & 0 \\ p_2 & p_1 & -2 & & \vdots \\ p_3 & p_2 & p_1 & & 0 \\ & & & \ddots & 1-m \\ p_m & p_{m-1} & p_{m-2} & \cdots & p_1 \end{vmatrix}$$

(c)

$$p_m = (-1)^{m-1} \det \begin{vmatrix} h_1 & 1 & 0 & & 0 \\ 2h_2 & h_1 & 1 & & \\ 3h_3 & h_2 & h_1 & & \\ & & & \ddots & 1 \\ mh_m & h_{m-1} & h_{m-2} & & h_1 \end{vmatrix}$$

(d) Deduce the corresponding formulas for e_m.