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# Complex Analysis for F2

## Projects

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Suggested projects ask you to prove a few important and difficult theorems in complex analysis. Though proofs are elaborate, they can be split into a few fairly easy steps, and below you can find instruction how to do that and hints. And it will be a rewarding experience to reconstruct a difficult proof of a very important theorem mostly by yourself.

It still might be a serious task to do it, but you should have no problems if you work in a small group and consult me a few times. I really urge you to come to me and discuss these projects; if there are severe problems, I can also distribute more detailed instructions.

Briefly the projects are:

*1) The Riemann Mapping Theorem:* show that any bounded simply connected domain can be analytically mapped to a unit disc. This is a very important theorem, which helps to reduce many problems from an arbitrary domain to a disc.

*2) Normal Families:* show that from a sequence of bounded analytic functions one can chose a subsequence converging to another analytic function. Normal families is a very powerful tool, helping to construct new analytic functions with desired properties.

*3) The Prime Number Theorem:* provide its main ingredient by showing that Riemann $\zeta$-function has no zeros in the half-plane $\mathrm{Re}(z) \geq 1$. The resulting Prime Number Theorem (that $n$-th prime number $\approx n \log n$) is perhaps the most spectacular application of complex analysis to pure mathematics. If you are willing (and maybe a bigger group, 2-3 persons, is needed for that), we can go all the way and prove the Prime Number Theorem itself.

If you do not like these 3 projects, or you want to do something more applied, please speak to me, I have a few other ideas in mind.

## 1. The Riemann Mapping Theorem

The Riemann Mapping Theorem states that any simply connected domain is *conformally equivalent* to one of the three basic domains: Disc, Complex plane, Complex sphere. That means that there is a 1-to-1 analytic map from this domain to a disc (or plane, or sphere). Analytic maps are traditionally called *conformal*, because they preserve angles.

This is an extremely important and powerful result: e.g. with its help finding solutions to the steady-state heat equation in an arbitrary domain can be reduced to the same problem in the disc. It is also non-trivial and unexpected: the similar statement in 3-dimensional space is wrong, e.g. a cube cannot be mapped to a ball by a map preserving angles.

The goal of this project is to work out the following case:

**The Riemann Mapping Theorem.** *Let $\Omega$ be a bounded simply connected domain. Show that there is a univalent (i.e. 1-to-1) analytic map of $\Omega$ to the unit disc $\mathbb{D} := \{z : |z| < 1\}$.*

The original proof was difficult (actually Riemann was not able to give a complete proof, the first proof was given by Köbe). We will do a (simpler) modern modification, which can be split in a few short and fairly easy steps. Fix a point $a \in \Omega$ and consider the collection $\mathcal{A}$ of all analytic maps of $\Omega$ *into* $\mathbb{D}$ such that $a$ is mapped to 0. We will look for a map $f \in \mathcal{A}$ which has a maximal possible magnification at $a$.

*Step 1: $\mathcal{A}$ is non-empty.* Show that $\mathcal{A}$ is non-empty.

*Step 2: there is maximal magnification.* Show that supremum of all "magnifications" is finite: $\alpha := \sup\{|g'(a)| \,:\, g \in \mathcal{A}\} < \infty$.

Hint: Cauchy formula for derivatives.

*Step 3: there is a map with maximal magnification.* There is $f \in \mathcal{A}$ which attains the supremum: $|f'(a)| = \alpha$.

Hint: Here you can (and should) use another non-trivial statement, the theorem from the Normal Families project. Pick a sequence of functions $f_j \in \mathcal{A}$ with $|f_j'(a)| \to \alpha$, and by the mentioned theorem find a subsequence converging to some analytic function $f$.

Show that $|f'(a)| = \alpha$. You also must show that $f$ maps $\Omega$ into $\mathbb{D}$, i.e. images of distinct points are distinct. To do that take a contour and using Rouché's theorem (or Argument principle, or anything similar) show that $f(z) - w$ should have the same number of roots inside it as $f_j(z) - w$ for large $j$, i.e. at most 1.

*Step 4: the function with maximal magnification is the desired one.* Show that $f$ maps $\Omega$ onto $\mathbb{D}$ (and not on some smaller subset).

Hint: Assume that $f$ omits some value, say $w$, with $|w| < 1$, and try to show that $f$ does not maximize the magnification. Consider new functions:

$$\begin{aligned} f_1(z) &:= \frac{f(z) - w}{1 - \bar{w} f(z)}\,, \\ f_2(z) &:= \sqrt{f_1(z)}\,, \\ f_3(z) &:= \frac{f_2(z) - f_2(a)}{1 - \overline{f_2(a)} f_2(z)}\,. \end{aligned}$$

Show that $f_1$ is univalent, hence $f_2$ is, hence $f_3$ also.

Show that $|f| < 1 \Rightarrow |f_1| < 1 \Rightarrow |f_2| < 1 \Rightarrow |f_3| < 1$.

So $f_3$ is a univalent map into $\mathbb{D}$, i.e. belongs to $\mathcal{A}$. Calculate that

$$|f_3'(a)| \; = \; \frac{1+|w|}{2\sqrt{|w|}}|f'(a)| \; > \; |f'(a)| \; .$$

Thus $f$ cannot omit any values in $\mathbb{D}$. This ends the proof.

Note that $f_1(z) = m_1(f(z))$ and $f_3(z) = m_3(f_2(z))$, where $m_1$ and $m_3$ are appropriate Möbius maps, i.e. functions of the form:

$$m(z) = \frac{z-v}{1-\bar{v}z} \; .$$

Those will be discussed in the class. You can use that they are 1-to-1 maps of the unit disc $\mathbb{D}$ onto itself.

In the definition of $f_2$ we need the following lemma, which will be perhaps discussed in class for other purposes:

Lemma. *Assume that analytic function $\phi$ is non-zero in simply connected domain $\Omega$. Than there exist a branch of $\sqrt{\phi}$ which is single-valued analytic function in $\Omega$.*

Sketch of the proof. Define $\log \phi$ as an integral (i.e. antiderivative) of $\phi'/\phi$. Now set $\sqrt{\phi} := \exp(\frac{1}{2}\log\phi)$.

Extra question. Our proof would work for *all* simply connected domains in the complex plane, whose boundary has at least 1 point. The only step which has to be modified, is why $\mathcal{A}$ is non-empty. Can you think how to do that? (it is elaborate, the square root trick has to be used again)

## 2. Normal families

Some collection of functions analytic in $\Omega$ is called a *normal family*, if for any sequence of such functions one can find a subsequence which converges (uniformly away from the boundary of $\Omega$) to some analytic function $f$.

Similar property in general setting is called *compactness*, e.g. we say that an interval $[0,1]$ is compact because from any sequence of points in it we can select a converging subsequence.

The notion of normal family is very important, because it allows to construct new analytic functions which sometimes can't be obtained in other ways (i.e. as formulas, sums of series with prescribed coefficients, etc.). The *complex dynamics*, which studies iterations of polynomials, Mandelbrot fractals, and such, is based on the normal families.

The goal of this project is to show that a family of analytic functions with absolute value bounded by 1 is normal (a result seriously used in the proof of the Riemann Mapping Theorem):

**Theorem.** *Let $\{f_j\}$ be a sequence of functions analytic in domain $\Omega$ with $|f_j| \le 1$. Show that there is a subsequence $f_{n_j}$ which converges (pointwise) to an analytic in $\Omega$ function $f$ with $|f| \le 1$. Convergence is uniform on subsets of $\Omega$ away from the boundary.*

We split the proof in a few steps.

*Step 1.* Prove the following simple lemma, which is of independent interest:
Schwartz lemma. *If a function $g : \mathbb{D} \to \mathbb{D}$ is analytic and $g(0) = 0$, then $|g(z)| \le |z|$.*

Hint: Note that domain is $\mathbb{D}$ and $|g| < 1$. Use maximum principle for $g(z)/z$. This result can be improved: if at some point $z \ne 0$ there is an equality, then $g(z)$ is a rotation: $g(z) = e^{i\theta} z$.

*Step 2.* Use the same ideology to prove the following
*Assume that $f_k$ and $f_j$ are analytic in a disc of radius $R$ around $a$, and $|f_j|, |f_k| \le 1$, and $|f_j(a) - f_k(a)| < \epsilon < 1$. Then $|f_j(z) - f_k(z)| < 4\epsilon$ for $z$ close to $a$: $|z - a| < \epsilon R$*

Hint: Consider the function $g(z) := f_j(z) - f_k(z) - f_j(a) + f_k(a)$. It is analytic in the disc of radius $R$ around $a$, also $g(a) = 0$ and $|g| < 3$. Use the same argument as in step 1.

*Step 3.* Take a "dense" sequence $a_1, a_2, \ldots$ of points in $\Omega$, e.g. vertices of grid with mesh $1/2$, plus all vertices of the grid with mesh $1/3$, plus $\ldots$ (it would be also fine to take all points with rational coordinates - they can be enumerated).

Chose a subsequence $f_{n_j}$ which will *converge at each point $a_l$*.

Hint: Since functions $f_j$ are bounded, we can chose a subsequence $f_j^1$ such that $\lim_j f_j^1(a_1) = b_1$ for some number $b_1$.

Now since functions $f_j^1$ are bounded, we can chose a sub-subsequence $f_j^2$ such that $\lim_j f_j^2(a_2) = b_2$ for some number $b_2$. Note that we still have $\lim_j f_j^2(a_1) = b_1$. And so on.

Now we use the *Cantor diagonal argument*, picking from the sequences (written as an infinite matrix) the diagonal to obtain a new sequence: $f_1^1, f_2^2, f_3^3, \ldots$, i.e. $f_{n_j} := f_j^j$.

Observe that $\lim_j f_{n_j}(a_l) = b_l$ for every point $a_l$.

*Step 4.* Take some nice subdomain $\Omega' \subset \Omega$ which is $R$-away from the boundary of $\Omega$. (That means that any ball of radius $R$ with center in $\Omega'$ is contained inside $\Omega$).

Show that functions $f_{n_j}$ form a "Cauchy sequence" in $\Omega'$, i.e. for any $\delta > 0$ we have $\sup_{\Omega'} |f_{n_j} - f_{n_k}| < \delta$ once $j, k$ are sufficiently large.

Hint: Take $\epsilon := \delta/4$. Take those points $a_l$ which are vertices of the grid with mesh approximately $\epsilon R$ inside $\Omega'$.

There are finitely many of those!!! Balls with radii $\epsilon R$ centered at these points cover $\Omega'$.

Show that for $j, k$ larger than some big $M$ one has $|f_{n_j}(a_l) - f_{n_k}(a_l)| < \epsilon$ for chosen points $a_l$ (there are only finitely many of them). Use step 2 to show that $|f_{n_j} - f_{n_k}| < 4\epsilon = \delta$ in $\Omega'$.

*Step 5.* Show that $f_{n_j}$ converge uniformly in $\Omega'$ to an analytic function $f$.

Hint: We have shown in step 4 that for every $z \in \Omega'$ sequence $f_{n_j}(z)$ is Cauchy, hence it has a limit.

We define $f(z) := \lim_j f_{n_j}(z)$. A priori this function is not even continuous! But use step 4 to show that $f_{n_j}$ converges to $f$ uniformly in $\Omega'$, and then use Weierstrass theorem to show that $f$ is analytic in $\Omega'$.

*Step 6.* Observe that $\Omega'$ was arbitrary, so $f$ is analytic in $\Omega$.

Extra question. How does one deduce that a family of functions with real parts bounded by a constant is normal? (same trick as in the similar modification of the maximum principle)

## 3. The Prime Number Theorem.

One of the spectacular applications of complex analysis is the Prime Number Theorem, which states that $n$-th prime is $\approx n \log n$. As was mentioned in class, the main ingredient in its proof is the following

**Hadamard and de la Vallée-Poussin theorem.** *Riemann $\zeta$-function has no zeros in the half-plane* $\mathrm{Re}(z) \geq 1$ *and only a simple pole at point* 1.

Deducing the Prime Number Theorem from the result above is not very difficult, but it is mostly playing with real integrals and sums (and only one serious application of complex analysis – Laplace transforms from the chapter 8 of Trim's book), so we will only indicate connection to counting prime numbers at the end.

*Alternatively, you can assemble a bigger group (3-4 persons) and than we can go all the way to prove the Prime Number Theorem.*

One of the biggest open problems in mathematics (now worth \$ 1,000,000, thanks to the Clay foundation) is whether all zeros of Riemann $\zeta$-function in the right half-plane $\mathrm{Re}(z) > 0$ belong to the vertical line $\mathrm{Re}(z) = \frac{1}{2}$. Solving it would greatly extend our knowledge about the distribution of primes.

We will split the proof into a few steps.

By letter $p$ we denote prime numbers, that is positive integers divisible only by 1 and themselves. We remind that in class Riemann $\zeta$-function was defined by

$$\zeta(z) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n^z} \;\equiv\; \prod_p \left(1 - \frac{1}{p^z}\right)^{-1} .$$

We also introduce another function

$$\Phi(z) := \sum_p \frac{\log p}{p^z} .$$

*Step 1.* Show that both the series and the Euler product for $\zeta(z)$ and the series for $\Phi(z)$ converge uniformly and absolutely when $\mathrm{Re}(z) > 1 + \epsilon$. (Recall discussion in class)

*Step 2. The function* $\zeta(z) - \frac{1}{z-1}$ *extends to a holomorphic function for* $\mathrm{Re}(z) > 0$. *It has certain symmetry:* $\zeta(\bar{z}) = \overline{\zeta(z)}$.

Hint. One way is to recall discussion in class. Another method is to write for $\mathrm{Re}(z) > 1$

$$\zeta(z) - \frac{1}{z-1} \;=\; \sum_{1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{1}{x^z} dx \;=\; \sum_{n=1}^{\infty} \int_n^{n+1} \left( \frac{1}{n^z} - \frac{1}{x^z} \right) dx .$$

Check that (integral) terms in the last series can be estimated by $|z|/n^{\mathrm{Re}(z)+1}$ and deduce that it converges absolutely for $\mathrm{Re}(z) > 0$.

Checking symmetry is easy.

*Step 3. Show that*

$$-\frac{\zeta'(z)}{\zeta(z)} \;=\; \Phi(z) \;+\; \sum_p h_p(z) , \qquad (*)$$

*where the "error terms" are*

$$h_p(z) := \frac{1}{p^z}\frac{1}{p^z-1} \; .$$

<u>Hint.</u> Note that $\log\zeta(z) = -\sum_p \log\left(1-\frac{1}{p^z}\right)$. Differentiate. Play with geometric progressions.

*Step 4. Show that the function* $\Phi(z)$ *is meromorphic (i.e. analytic with isolated poles) for* $\mathrm{Re} z > \frac{1}{2}$. *Show that it has poles only at point* 1 *and at zeros of* $\zeta(z)$.

<u>Hint.</u> Use step 3 and show that series of "error terms" converges absolutely and uniformly for $\mathrm{Re}(z) > \frac{1}{2}+\epsilon$.

Then recall our discussion that $f'/f$ has poles (of order 1) only at zeros and poles of $f$ (it is very easy to check).

*Step 5. Show that* $\zeta(z)$ *has no zeros with* $\mathrm{Re}(z) \geq 1$ *and hence* $\Phi(z)$ *for* $\mathrm{Re}(z) \geq 1$ *has only a simple pole at point* 1.

<u>Hint.</u> For $\mathrm{Re}(z) > 1$ this follows from the product definition of $\zeta(z)$ (as dicussed in class), so only the line $\mathrm{Re}(z) = 1$ remains.

Check that if $f(z)$ has a zero of order $-m$ at point $a$ (or a pole of order $m$) then $f'(z)/f(z)$ has a simple pole with residue $m$ at this point. Actually it was checked in class, you can also consult Trim's book, pages 254 and 258. Apply this to the function $\zeta(z)$ and recall formula (*).

Now assume that $\zeta(z)$ has a zero of order $m \geq 1$ at point $z = 1+ib$ with $b \neq 0$. Let $n \geq 0$ be the order of the zero at $z = 1+i2b$ (we set $n = 0$ if there is no zero at this point). Then (use symmetry observed in step 2) we can calculate residues:

$$\lim_{\epsilon\to 0}\epsilon\Phi(1+\epsilon) = 1 \;,\; \lim_{\epsilon\to 0}\epsilon\Phi(1+\epsilon\pm ib) = -m \;,\; \lim_{\epsilon\to 0}\epsilon\Phi(1+\epsilon\pm i2b) = -n \; .$$

Verify the following funny identity (it is easy):

$$\sum_p \frac{\log p}{p^z}(p^{ib/2}+p^{-ib/2})^4 =$$
$$\Phi(z+i2b)+\Phi(z-i2b)+4\Phi(z+ib)+4\Phi(z-ib)+6\Phi(z)$$

The left hand side is positive for $z = 1+\epsilon > 1$ (check). Using calculations of residues and the funny identity conclude that

$$0 \leq -2n-8m+6 \; ,$$

and whence $m = 0$. This finishes the proof.

<u>Extra: Connection to the prime numbers.</u> Define a new function $\varphi(x)$ on the positive reals, which "counts" the primes before $x$:

$$\varphi(x) := \sum_{p\leq x}\log p \; .$$

Show that for $\mathrm{Re}(z) > 1$ we have

$$\Phi(z) = z\int_1^\infty \frac{\varphi(x)}{x^{z+1}}dx \; .$$

<u>Hint:</u> rewrite the integral as a sum of $\int_1^p \log p/x^{z+1}$ and evaluate.