**Mathematics, KTH**
B.Ek

**Exam TEN1 for the course**
**SF2736, DISCRETE MATHEMATICS**
**Thursday January 11, 2018, at 14.00–19.00.**

**Examiner:** Bengt Ek, tel: 790 6951.

**Allowed aids:** Pen/pencil and rubber (and supplied paper), no electronic devices.

**Minimum scores** for the grades A–E are given in the table:

| Grade | A | B | C | D | E | Fx |
|---|---|---|---|---|---|---|
| Score | 32 | 27 | 22 | 18 | 15 | 13 |

Fx is not a passing grade, but may be improved to an E by passing a supplementary exam.

**To give full points, the solutions must be clear and well explained.** Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

---

# PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

**1)** (3p) For how many $x \in \mathbb{Z}_{4851}$ is it true that $x^{292} = 196$? [$4851 = 3^2 \cdot 7^2 \cdot 11$]

**2)** (3p) Let $X$ be a finite set and $f\colon X \to X$ a bijection.
Decide if the binary relation $\mathcal{R}$ on $X$, defined by, for all $x, y \in X$:

$$x\mathcal{R}y \overset{\text{def}}{\Leftrightarrow} y = f^n(x) \text{ (i.e. } y = f(f(\ldots f(x)\ldots)) \text{ ($n$ '$f$'s)) for some } n \in \mathbb{Z}_+,$$

is an equivalence relation (for all such $f$, $X$).
It must be clear from the solution what the conditions for an equivalence relation are, and which of them are certainly (i.e. for all $f$, $X$) satisfied by $\mathcal{R}$.

**3a)** (2p) Let $A = \mathbb{N}_7 = \{1, 2, \ldots, 7\}$ and $B = \mathbb{N}_{13} = \{1, 2, \ldots, 13\}$.
How many functions $f\colon A \to B$ take exactly 4 different values?
**b)** (1p) How many of the functions in a) take at least one odd value?
Answers may contain integers, factorials, powers and the four elementary arithmetical operations.)

**4)** (3p) Let $(G, \cdot)$ be a group and $\varphi\colon G \to G$ be given by $\varphi(g) = g^{-1}$, all $g \in G$.
Show that $\varphi$ is an isomorphism (an automorphism) if and only if $G$ is abelian.

**5)** The permutations $\pi, \sigma \in S_{13}$ are given by the table:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi(i)$ | 9 | 8 | 12 | 7 | 2 | 6 | 11 | 4 | 1 | 13 | 5 | 3 | 10 |
| $\sigma(i)$ | 11 | 2 | 12 | 3 | 5 | 1 | 4 | 8 | 13 | 9 | 6 | 7 | 10 |

**a)** (1p) Find the parities (even or odd) of $\pi$ and $\sigma$.
**b)** (2p) For which $n \in \mathbb{Z}$ is there a $\tau \in S_{13}$ such that $\tau\pi^n = \sigma^n\tau$?

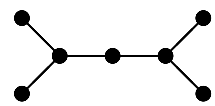
---

*Please turn over*

# PART II

**6)** (4p) $G = (V, E)$ is a plane, connected graph with all degrees greater than 2 (i.e. $\delta(x) \geq 3$ for all $x \in V$) and the number of vertices in the dual graph $G^{\perp} = (V^{\perp}, E^{\perp})$ is less than 12 (i.e. $|V^{\perp}| \leq 11$).
Show that at least one vertex in $G^{\perp}$ has degree less than 5 (i.e. $\delta^{\perp}(x) \leq 4$ for some $x \in V^{\perp}$). (A loop (if one appears) contributes 2 to the degree of its vertex.)

**7)** (4p) Seven beads are connected by strings as in the figure. In how many essentially different ways (i.e. so that they remain different however all or part of the arrangement is rotated) can exactly two beads be coloured red and each of the rest in one of $k$ other colours?



**8)** (4p) Let $G = (V, E)$ be a connected graph.
Show that the edges of $G$ can be directed so that at most one vertex $x$ has odd out-degree $\delta^{+}(x)$ (i.e. an odd number of its edges directed away from $x$).

---

# PART III

For full points on these problems, extra clear and well presented solutions are required.

**9)** (5p) Let (as usual) the Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ be defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$.
Find the value of

$$\sum_{n=0}^{\infty} 2^{-n} \cdot F_n.$$

For full points you have to show that the series is convergent (i.e. (since all terms are $\geq 0$) that there exists $B \in \mathbb{R}$ such that $\sum_{n=0}^{N} 2^{-n} \cdot F_n < B$ for all $N \in \mathbb{N}$).

**10a)** (1p) Show that if $x, y \in \mathbb{Z}$ and $y^3 = x^2 + 2$, then $x$ is odd.
**b)** (2p) Show that $(R, +, \cdot)$, where $R = \mathbb{Z}[\sqrt{2}\, i] = \{a + b\sqrt{2}\, i \mid a, b \in \mathbb{Z}\}$, has unique factorization in "primes" (apart from the order of the factors and factors $\pm 1$).
(You don't have to give the details of the whole proof, but show the necessary property needed, and how it can be used to reason as in the proof of (essentially) unique factorization in $\mathbb{Z}$.)
**c)** (2p) Find all $x, y \in \mathbb{Z}$ such that $y^3 = x^2 + 2$.
(You may use the results in a) and b) in doing c), even if you haven't solved them.)

**Good luck!**

**Suggested solutions will be posted on the course's web site.**