Mathematics, KTH
B.Ek

Exam TEN1 for the course
SF2736, DISCRETE MATHEMATICS
Thursday April 5, 2018, at 8.00–13.00.

Examiner: Bengt Ek, tel: 790 6951.

Allowed aids: Pen/pencil and rubber (and supplied paper), no electronic devices.

Minimum scores for the grades A–E are given in the table:

Grade	A	B	C	D	E	Fx
Score	32	27	22	18	15	13

Fx (not a grade) may be improved to an E by passing a supplementary exam.

To give full points, the solutions must be clear and well explained. Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

1) (3p) Find all integers x satisfying [$3599 = 59 \cdot 61$]

$$x^{1682} + 22\,x \equiv 1652 \pmod{3599}.$$

2) For sets X, Y and $f\colon X \to Y$, let $f``\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ be given by

$$f``(A) = \{f(a) \mid a \in A\} \text{ for all } A \in \mathcal{P}(X) \text{ (sometimes denoted } f[A]).$$

For the following pairs of sets, decide which of '$=$', '$\subseteq$', '$\supseteq$' have to be true (i.e., are true for all such X, Y, f and $A, B \subseteq X$) between them, in their given order:
a) (1p) $f``(A \cup B)$ and $f``(A) \cup f``(B)$,
b) (1p) $f``(A \cap B)$ and $f``(A) \cap f``(B)$,
c) (1p) $f``(A \smallsetminus B)$ and $f``(A) \smallsetminus f``(B)$.
(Notation (as usual): $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, $A \smallsetminus B = \{x \in A \mid x \notin B\}$.)

3) (3p) In how many ways can four girls and five boys sit on three red and six white chairs, one on each chair, so that at least one girl sits on a red chair? All children and chairs are distinguishable.
Answers may contain integers, factorials, powers and the four elementary arithmetical operations.)

4) The permutations $\pi, \sigma \in S_9$ are given by the table on the right.

i	1	2	3	4	5	6	7	8	9
$\pi(i)$	2	3	1	5	6	7	4	9	8
$\sigma(i)$	4	8	1	9	2	3	7	5	6

a) (1p) Give π, σ and $\pi\sigma$ in cycle form.
b) (2p) Let G be a subgroup of S_9 which contains π and σ.
Show that $|G|$, the number of elements of G, is divisable by 180.

5) An RSA cryptosystem has the public parameters (n, e), where $n = 4331$ [$= 61 \cdot 71$].
a) (1p) Which of 205, 206, 207, 208 and 209 are possible values for e?
b) (2p) Find a corresponding value for d for a possible e from a.

Please turn over!

PART II

6) (4p) Let G, G_1, G_2 be graphs, where $G = (V, E)$, $G_i = (V_i, E_i)$, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ and $E_i = \{\{x, y\} \in E \mid x, y \in V_i\}$ $(i = 1, 2)$.
If $P_G(\lambda)$, $P_{G_i}(\lambda)$ are the chromatic polynomials of G and G_i (i.e., for $\lambda \in \mathbb{N}$, the number of vertex colourings of G, G_i using at most λ colours), show that for $\lambda \in \mathbb{N}$:

$$P_G(\lambda) \leq P_{G_1}(\lambda) \cdot P_{G_2}(\lambda).$$

7) Let X be an infinite set and G the set of bijections of X onto X, $G = \{f : X \rightleftarrows X\}$.
a) (1p) Show that $(G, \circ)$ is a group (where $\circ$ is composition of functions).
b) (2p) Show that $H = \{f \in G \mid |\{x \in X \mid f(x) \neq x\}| < \infty\}$ is a subgroup of G.
c) (1p) Is H a normal subgroup of G (i.e., is $fH = Hf$ for all $f \in G$)?

8) (4p) Let $G = (V, E)$ be a graph and $\mu(x)$ for $x \in V$ be the average of the degrees of the neighbours of x ($\mu(x) = 0$ if the degree $\delta(x) = 0$).
Show that

$$\sum_{x \in V} \mu(x) \geq \sum_{x \in V} \delta(x).$$

PART III

For full points on these problems, extra clear and well presented solutions are required.

9) Let $F[\![x]\!]$ (like in Biggs's book) be the set of formal power series with coefficients in the field F. We write $A(x) = \sum a_n x^n$ for $A(x) \in F[\![x]\!]$ with $a_n \in F$ (so $\sum$ means $\sum_{n=0}^{\infty}$). Define $D : F[\![x]\!] \to F[\![x]\!]$ (formal derivation) by

$$D(A(x)) = \sum n a_n x^{n-1} \text{ when } A(x) = \sum a_n x^n \in F[\![x]\!].$$

a) (1p) Show that for all $A(x), B(x) \in F[\![x]\!]$,

$$D(A(x)B(x)) = D(A(x))B(x) + A(x)D(B(x)).$$

b) (2p) Let $E(x) = \sum \frac{1}{n!} x^n \in \mathbb{R}[\![x]\!]$ and $p_n(x) \in \mathbb{R}[\![x]\!]$ for $n \in \mathbb{N}$ be given by

$$p_0(x) = 1 \text{ and } p_{n+1}(x)E(x) = x\, D(p_n(x)E(x)) \text{ for } n \in \mathbb{N}.$$

Express the coefficients of $p_n(x)$ $(n = 1, 2, \ldots)$ as known combinatorial numbers.
c) (2p) The **Bell number $\boldsymbol{B_n}$** is, for $n \in \mathbb{Z}_+$, the number of all equivalence relations on a set with n elements. Express B_n as a convergent infinite series.
(Without proof you may use that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all $x \in \mathbb{R}$ and that if two power series converge, their product converges to the product of their values.)

10) Let $G = (X \cup Y, E)$ be a (finite) bipartite graph ($X \cap Y = \varnothing$, $e \in E \Rightarrow |e \cap X| = 1$).
For $A \subseteq X$, let $r(A)$ be the greatest number of elements of A that can simultaneously be matched with elements of Y (= the greatest $|M|$ when $M \subseteq E$ is such that $m \in M \Rightarrow |m \cap A| = 1$ and $(m, n \in M$ and $m \neq n) \Rightarrow m \cap n = \varnothing$).
a) (3p) Show that for all $A, B \subseteq X$,

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

b) (2p) Find G, A, B such that $<$ holds in a.

Good luck!

Suggested solutions will be posted on the course's web site.