Mathematics, KTH
B.Ek

Exam TEN1 for the course SF2736, DISCRETE MATHEMATICS Thursday January 12, 2017, at 14.00–19.00.

Examiner: Bengt Ek, tel: 790 6951.

Allowed aids: Pen/pencil and rubber (and supplied paper), no electronic devices.

Minimum scores for the grades A–E are given in the table:

Grade	A	B	C	D	E	Fx
Score	32	27	22	18	15	13

Fx is not a passing grade, but may be improved to an E by passing a supplementary exam.

To give full points, the solutions must be clear and well explained. Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

1a) (1p) Find the least $k \in \mathbb{Z}_+$, or show that there is none, satisfying

$$2^k \equiv 1 \pmod{1989}.$$

b) (2p) Find the greatest $k \in \mathbb{Z}_+$ which is for some $x \in \mathbb{Z}$ the least in $\mathbb{Z}_+$ with

$$x^k \equiv 1 \pmod{1989}$$

and give a corresponding x. [$1989 = 3 \cdot 3 \cdot 13 \cdot 17$]

2) (3p) A binary linear code $\mathcal{C}$ has $|\mathcal{C}| = 8$ and $101010,\ 111001,\ 110111 \in \mathcal{C}$. Find a check matrix H for $\mathcal{C}$ and decide whether $\mathcal{C}$ corrects one error.

3) (3p) Didrik wants to distribute the coming 31 days among study of discrete mathematics, reading novels and gaming, exactly one activity each day.
He intends to game in total 9 days, but never two consecutive days, and spend (strictly) more days on discrete mathematics than on novels (obviously).
In how many ways can he distribute the days among the three activities?
Answers may contain integers, factorials, powers and the four elementary arithmetical operations.)

4) Let $(G, \cdot)$ be a group. We want $a, b, c \in G$ with $a = b^2,\ b = c^2,\ c = a^2$.
a) (1p) Find all such $a, b, c \in G$ with at least two of them equal.
b) (2p) Show that if $|G| = 1467$ there are no such $a, b, c \in G$ which are all distinct.

5) (3p) In a plane, connected graph 2 vertices have degree 3 and all the others have degree 4. 4 regions are bounded by 4 edges and all the others are bounded by 3 edges (including the unbounded region).
Find all possible numbers of vertices, edges and regions in the graph.

PART II

6) Let $A \subseteq G$, $A \neq \emptyset$, where $(G, *)$ is a group.
a) (2p) Show that if $G_A = \{g \in G \mid a \in A \Rightarrow g * a \in A\}$ then $|G_A| \leq |A|$.
b) (1p) Show that if A is finite, G_A is a subgroup of G.
c) (1p) Give an example of G and A such that G_A is not a group.
(You may use the resultat in a) in doing b), even if you haven't solved a).)

7) (4p) For $m, n \in \mathbb{Z}_+$ and $d = \gcd(m, n)$, show that

$$\phi(d)\phi(mn) = d\phi(m)\phi(n),$$

where ϕ is Euler's function ϕ.

8) (4p) The permutation $\sigma \in S_{13}$ is given by the table:

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sigma(i)$	11	8	1	9	4	12	5	13	7	6	3	10	2

For how many $\pi \in S_{13}$ is $\pi\sigma = \sigma^{-1}\pi$?

PART III

For full points on these problems, extra clear and well presented solutions are required.

9) (5p) For $G = (V_G, E_G)$ a graph and $\mathcal{R}$ an equivalence relation on V_G, with equivalence classes $\mathcal{V}_i$, $i \in I$, define the **quotient graph** $G/\mathcal{R} = (V_{G/\mathcal{R}}, E_{G/\mathcal{R}})$ by $E_{G/\mathcal{R}} = \{\{\mathcal{V}_i, \mathcal{V}_j\} \mid \text{ there are } v_i \in \mathcal{V}_i, v_j \in \mathcal{V}_j \text{ with } \{v_i, v_j\} \in E_G\}$ and $V_{G/\mathcal{R}} = \{\mathcal{V}_i \mid i \in I\}$. ($G/\mathcal{R}$ is not necessarily a simple graph, even if G is).
Show that for each connected graph $H = (V_H, E_H)$ there is a **tree** $T = (V_T, E_T)$ and an equvivalence relation $\mathcal{R}$ on V_T, such that $H \approx T/\mathcal{R}$ (i.e. H and $T/\mathcal{R}$ are isomorphic) and $|E_H| = |E_T|$.

10) (5p) We want to colour the sides of a cube (one colour on each side, of course) and we have red, blue and k other colours at our disposal.
In how many essentially different ways (i.e. so that they remain different however they are rotated) can it be done, if the numbers of red and blue sides are to be equal?

Good luck!

Suggested solutions will be posted on the course's web site.