Mathematics, KTH
B.Ek

Exam TEN1 for the course
SF2736, DISCRETE MATHEMATICS
Wednesday April 12, 2017, at 8.00–13.00.

Examiner: Bengt Ek, tel: 790 6951.

Allowed aids: Pen/pencil and rubber (and supplied paper), no electronic devices.

Minimum scores for the grades A–E are given in the table:

Grade	A	B	C	D	E	Fx
Score	32	27	22	18	15	13

Fx is not a passing grade, but may be improved to an E by passing a supplementary exam.

To give full points, the solutions must be clear and well explained. Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

1) (3p) Find all $x \in \mathbb{Z}$ such that

$$x^{31} + 158x \equiv 191 \pmod{385}.$$

2) (3p) Let (for $n \in \mathbb{N}$) $\mathcal{A}_n = \{B \subseteq \{1, 2, \ldots, n\} \mid x, y \in B \Rightarrow y \neq x \pm 1\}$ and

$$f(n) = \textstyle\sum_{B \in \mathcal{A}_n} \prod_{k \in B} k^2,$$

where (as usual) "the empty product" is 1, $\Pi_{x \in \varnothing} x = 1$.

Eg. $A_4 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}\}$,
thus $f(4) = 1 + 1^2 + 2^2 + 3^2 + 4^2 + 1^2 \cdot 3^2 + 1^2 \cdot 4^2 + 2^2 \cdot 4^2 = 120$.

Find a (simple) expression for $f(n)$ and show that it is correct for all $n \in \mathbb{N}$.

3) (3p) In how many ways can 9 different books and 17 identical donuts be distributed among 6 children, so that each child gets at least one book and at least one donut?
Children and books are considered distinct, donuts are considered identical.
Answers may contain integers, factorials, powers and the four elementary arithmetical operations.)

4) Let $(G, \cdot)$ be a group and H, K finite subgroups of G.
a) (1p) For given $|H|$ and $|K|$, what values are possible for $|H \cap K|$?
b) (2p) What values are possible for $|aH \cap bK|$, $a, b \in G$ (given $|H|$ and $|K|$)?
(That a value is possible means that there exist G, H, K giving that value.)

5) (3p) Find a way to order the numbers $1, 2, \ldots, 9$ around a circle, so that the sum of two neighbouring numbers is never divisable by 3, 5 or 7.
Eg. $5 \mid (2+8)$, so 2 and 8 may not be neighbours.
[Hint: Find a suitable cycle in a suitable graph.]

PART II

6a) (2p) A bracelet consists of 6 coloured (k colours available) beads threaded on a loop of string. Find the number of essentially different such bracelets.
b) (2p) How many of them have at least one red and at least one blue bead?
(Red and blue are among the k colours. The beads look the same from both directions and the bracelets can rotate freely. In doing b) but not a), one may call the answer in a) $f(k)$.)

7) (4p) The permutation $\pi \in S_{10}$ is given by the table:

$$\begin{array}{c|cccccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \sigma(i) & 9 & 7 & 10 & 4 & 8 & 1 & 2 & 5 & 6 & 3 \end{array}.$$

For how many functions $f\colon \mathbb{N}_{10} \to \mathbb{N}_{10}$ are $f(\pi(i)) = \pi(f(i))$ for all $i \in \mathbb{N}_{10}$?

8) (4p) Find a set $A \subseteq \mathbb{Z}_+$, such that the number of partitions of each $n \in \mathbb{Z}_+$ in an arbitrary number of pieces of sizes multiples of 3 and at most two for every other size, is the same as the number of partitons of n with the size of every part in A. I.e., with Biggs' notation:

$$p\big(n \mid \text{for all } k \in \mathbb{Z}_+ : \begin{cases} \text{if } 3 \mid k\text{: an arbitrary number of parts of size } k, \\ \text{else: at most two parts of size } k \end{cases}\big) =$$

$$= p(n \mid \text{the size of every part is in } A).$$

PART III

For full points on these problems, extra clear and well presented solutions are required.

9) Let $(G, *)$ be a group with identity I and $(A, \circ)$ be an abelian gruop with identity e. The group G acts on the set A so that (for all $g \in G$, $a, b \in A$) $g(a \circ b) = g(a) \circ g(b)$ (and, as usual, $I(a) = a$ for all $a \in A$).
Now let $K = G \times A$ and introduce $\odot$ by $(g_1, a_1) \odot (g_2, a_2) = (g_1 * g_2, g_1(a_2) \circ a_1)$.
a) (3p) Show that $(K, \odot)$ is a group. Explain properly.
b) (2p) With $H_1 = G \times \{e\}$ and $H_2 = \{I\} \times A$, decide (for $i = 1, 2$) if $(H_i, \odot)$ must be a subgroup and if it must be a normal subgroup (i.e. if left cosets and right cosets must coincide) of $(K, \odot)$.

10) (5p) Let $G = (V, E)$ be an infinite graph with E a countable set.
Show that the following two conditions are equivalent, $i) \Leftrightarrow ii)$,

i) For every finite $X \subset V$ the number of edges with exactly one vertex in X is either infinite or an even number,

ii) There exists a set of cycles and two-way infinite paths such that each $e \in E$ belongs to exactly one of these.

(An infinite graph is defined in the same way as a finite one (except that V and E may be infinite sets), cycles and paths are defined as in the finite case. Vertices may have infinite degree.)

Good luck!

Suggested solutions will be posted on the course's web site.