Mathematics, KTH
B.Ek

Exam TEN1 for the course
SF2736, DISCRETE MATHEMATICS
Wednesday January 13, 2016, at 14.00–19.00.

Examiner: Bengt Ek, tel: 790 6951.

Allowed aids: Pen/pencil and rubber (and supplied paper), no electronic devices.

Minimum scores for the grades A–E are given in the table:

Grade	A	B	C	D	E	Fx
Score	32	28	22	18	15	13

Fx is not a passing grade, but may be improved to an E by passing a supplementary exam.

To give full points, the solutions must be clear and well explained. Theorems from the course may (unless otherwise stated) be used without proof, if it is clearly written what they say.

PART I

The score on this part is the least of 15 and the sum of the points given for problems 1–5 and bonus points from the homework assignments (at most 4p).

1) (3p) Does there exist an $n \in \mathbb{Z}_+$ such that

$$233^n \equiv 1 \pmod{2310}?$$

Find the least such n or show that there is none.

2) A linear binary code is given by the check matrix $H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$.
a) (2p) How many words are there in the code?
b) (1p) How many words (i.e. sequences of length 5 from the alphabet $\{0, 1\}$) can not be corrected by changing at most one bit, i.e. how many words can not result from at most one error in a codeword?

3) (3p) 10 persons are to share 15 identical white cards and 15 different coloured cards. Each person shall have exactly 3 cards, at least one white and one coloured.
In how many ways can the cards be distributed?
(The persons are considered distinguishable. Your answer may contain integers, factorials, powers and the four elementary arithmetical operations.)

4) (3p) $x^2a = bxc^{-1}$ and $acx = xac$, where $a, b, c, x \in G$, a group.
Find x (expressed in a, b, c).

5) (3p) Show that if the graph $G = (V, E)$ is Hamiltonian (i.e. it has a Hamiltonian cycle) and one deletes $k \in \mathbb{Z}_+$ edges from E, the resulting graph will contain at most k components.

PART II

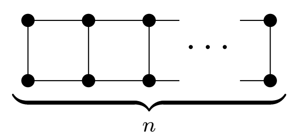
6) $\pi, \sigma \in S_8$ (the group of permutations of $\{1, 2, \ldots, 8\}$) are $\pi = (1\,7\,5\,6\,8)(2\,3\,4)$ and $\sigma = (1\,4\,7\,8\,6\,5\,3)$. Let H be the set of all products of π:s and σ:s, recursively defined as the least set such that 1. $id \in H$ and 2. for all $\tau \in H$: $\tau\pi, \tau\sigma \in H$.
a) (2p) Show that H is a subgroup of S_8.
b) (2p) Show that $|H| \geq 420$.

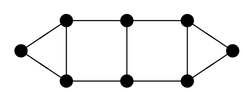
7) (4p) Find a set $A \subseteq \mathbb{Z}_+$ such that for any $n \in \mathbb{Z}_+$, the number of partitons of n such that for all odd $k \in \mathbb{Z}_+$ there is at most one part of size k (and for even $k \in \mathbb{Z}_+$ there can be any number of parts of size k), is equal to the number of partitions of n with every part of a size which is an element of A.
I.e., in Biggs' notation:

$$p\big(n \mid \text{for all } k \in \mathbb{Z}_+ : \begin{cases} \text{if } k \text{ is odd: at most one part of size } k, \\ \text{if } k \text{ is even: any number of parts of size } k \end{cases}\big) =$$

$$= p(n \mid \text{the size of every part is in } A).$$

8) Let G_n be the upper graph on the right (in total $2n$ vertices) and $P_n(\lambda)$ its chromatic polynomial (i.e. $P_n(\lambda)$ for $\lambda \in \mathbb{N}$ is the number of vertex colourings of G_n using at most λ colours).
a) (1p) Find $P_1(\lambda)$ and $P_2(\lambda)$.
b) (2p) Find $P_n(\lambda)$ for all $n = 1, 2, \ldots$
c) (1p) How many vertex colourings of the lower graph on the right using at most 4 colours are there?
The answer in c) may contain integers, powers, and products.





PART III

9) 20 children are to stand in a line. How many orders between them are possible in the following cases?
a) (2p) Harry and Draco may not have exactly one child between them (but they may stand next to each other).
b) (3p) Neither Harry and Draco nor Ron and Draco may have exactly one child between them.
(The answers may contain integers, powers, factorials and the four elementary arithmetical operations.)

10) Like Biggs, we let a digraph be a $G = (V, A)$, where V is a set and A is any subset of $V \times V$ (so we allow loops, but not multiple arcs).
a) (3p) Find the number of non-isomorphic digraphs $G = (V, A)$ with $|V| = 3$.
b) (2p) The same problem with $|V| = 4$.
(Answer a) with an integer, b) with (maybe) the four elementary arithmetic operations on integers.)

Good luck!

Suggested solutions will be posted on the course's web site.