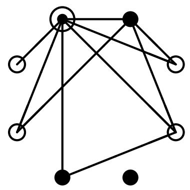
(SF2736, Discrete maths, ht15: L20, Thu 3 Dec 2015)

Graph colourings

A **vertex colouring** of the graph $G = (V, E)$:
a function $c : V \to \mathbb{N}$
such that $xy \in E \Rightarrow c(x) \neq c(y)$



The chromatic number $\chi(G)$ of G:
the least number of colours with which one can vertex colour G

A **greedy algorithm** to find upper bounds of $\chi(G)$:
1. order the vertices: $v_1, v_2, \ldots, v_n$
2. choose in order $c(v_1), c(v_2), \ldots$ as the least number ("colour") possible, given the $c(v_i)$ already chosen

Theorem: G has maximum degree $k \Rightarrow$
· $\chi(G) \leq k + 1$
· if G is connected and not regular, $\chi(G) \leq k$

(In fact the stronger **Brooks' theorem** (1941) (not in Biggs' book) holds:
If G is connected and not isomorphic to K_m for some m or to C_n for some odd n,
$\chi(G) \leq k$, the maximum degree of G.)

The chromatic polynomial $P_G(\lambda)$ of a graph G

The **number of ways** to vertex colour the graph $G = (V, E)$ with (at most) λ colours is given by the **chromatic polynomial** $P_G(\lambda)$.
The chromatic number $\chi(G)$ is the **least** $\lambda = 0, 1, 2, \ldots$ with $\boldsymbol{P_G(\lambda) \neq 0}$.
$P_G(\lambda)$ is, as the name suggests, a polynomial.
Its highest degree term is $\lambda^{|V|}$.
The next to highest degree term is $-|E|\lambda^{|V|-1}$.
The coefficients are alternating ≥ 0 and ≤ 0.
The constant term is 0 (if $V \neq \varnothing$).

Recursion:

$$\boldsymbol{P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda),}$$

where $G - e$ is the graph G with the edge e removed and G/e is G with the edge e contracted (i.e. the vertices in e are merged). Since the graphs in the right hand side have strictly fewer edges than G, this together with the base $\boldsymbol{P_{(V,\varnothing)}(\lambda) = \lambda^{|V|}}$ gives the polynomial $P_G(\lambda)$ uniquely for all (finite) graphs.
With induction corresponding to the recursion the properties for $P_G(\lambda)$ mentioned above are shown.

Using the recursion we found for the cycle graph C_n

$$P_{C_n}(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

As an application of the sieve principle, it can be shown that the number of ways to vertex colour G with **exactly** λ colours (i.e. using all λ colours) is

$$\textstyle\sum_{j=\chi(G)}^{\lambda} (-1)^{\lambda-j} \binom{\lambda}{j} P_G(j).$$

Planar graphs

A **plane graph** (or a plane drawing of a graph): A "concrete graph" in the plane (equivalently: on a sphere) without crossing edges

A **planar graph**: A graph which is isomorphic to a plane graph,
i.e., it **can be drawn** in the plane without crossing edges

The **dual graph** (not necessarily a simple graph, even if G is) $\boldsymbol{G^\perp}$ of a plane graph G:
(isomorphic plane graphs can have non-isomorphic dual graphs.)

one vertex in each of the regions of G (the regions formed by the G-edges)
one edge through each of G's edges (connecting the $G^\perp$-vertices on either side)

Then $v^\perp = r,\ e^\perp = e$ (with v, e, r as below).

If G is (plane and) connected, $(G^\perp)^\perp$ is isomorphic to G and $r^\perp = v$.

Theorem: (Euler's polyhedron formula): If a connected plane graph has v vertices, e edges and r regions,

$$\boldsymbol{v - e + r = 2}$$

More generally: If a plane graph has c components,

$$\boldsymbol{v - e + r - c = 1}$$

It gives (since every region (if $e \geq 2$) has at least 3 edges)

Theorem: For a connected planar graph with $e \geq 2$, $\boldsymbol{3v \geq e + 6}$
If the graph is also bipartite, $\boldsymbol{2v \geq e + 4}$

Theorem: A planar graph has vertices with degree ≤ 5.

The complete graphs $K_n,\ n \geq 5$ and $K_{p,q},\ p, q \geq 3$ are **not planar**

If a graph is non-planar, it "contains" (at least) one of K_5 and $K_{3,3}$:

Kuratowski's (1930) and Wagner's (1937) theorems:
The graph G is non-planar **iff** a graph which is isomorphic to K_5 or $K_{3,3}$ can be obtained from G by a sequence (0 or more) of the operations:

- delete a vertex (and its edges)
- delete an edge
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  - Kuratowski: "erase" a vertex of degree 2
    (keeping an edge between its neighbours)
  - Wagner: contract an edge
    (merging the vertices in the edge to a new vertex)

(The condition in Wagner's theorem is expressed by saying that K_5 or $K_{3,3}$ is a **minor** of G.)

The **four colour theorem**: G planar $\Rightarrow \chi(G) \leq 4$

In the lecture the six and five colour theorems were proved.

Not treated in the lectures (will not be needed for the exam):

A **regular polyhedron** (a platonic solid):
all sides regular m-gons, all vertices congruent, degree n.

That corresponds to a "doubly regular" plane, connected graph:
all regions have $\boldsymbol{m}$ **edges** (i.e. $G^\perp$ is m-regular), all vertices have **degree** $\boldsymbol{n}$.

The only such graphs:

m	n	v	e	r	corresponding polyhedron
3	5	12	30	20	icosahedron (dual to the dodecahedron)
3	4	6	12	8	octahedron (dual to the hexahedron)
3	3	4	6	4	tetrahedron (self-dual)
4	3	8	12	6	hexahedron (cube) (dual to the octahedron)
5	3	20	30	12	dodecahedron (dual to the icosahedron)