Inference rules for natural deduction

Rule of premises

j	(j)	p	premise	

Rule of assumptions

j	(j)	p	assumption	

Rule of $\wedge$E:

$a_1, \ldots, a_n$	(j)	$p \wedge q$		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	p	j	$\wedge$E
		(**or** q)		

Rule of $\rightarrow$E:

$a_1, \ldots, a_m$	(j)	$p \rightarrow q$		
	$\vdots$			
$b_1, \ldots, b_n$	(k)	p		
	$\vdots$			
$a_1, .., a_m, b_1, .., b_n$	(l)	q	j,k	$\rightarrow$E

Rule of $\neg$E:

$a_1, \ldots, a_m$	(j)	$\neg p$		
	$\vdots$			
$b_1, \ldots, b_n$	(k)	p		
	$\vdots$			
$a_1, .., a_m, b_1, .., b_n$	(l)	$\bot$	j,k	$\neg$E

Rule of $\vee$E:

$a_1, \ldots, a_m$	(g)	$p \vee q$		
	$\vdots$			
h	(h)	p	assumption	
	$\vdots$			
$b_1, \ldots, b_n$	(i)	r		
	$\vdots$			
j	(j)	q	assumption	
	$\vdots$			
$c_1, \ldots, c_u$	(k)	r		
	$\vdots$			
X	(l)	r	g,h,i,j,k	$\vee$E

where $X = \{a_1, \ldots, a_m\} \cup$
$\cup \{b_1, \ldots, b_n\}/\text{h} \cup \{c_1, \ldots, c_u\}/\text{j}$

Rule of DN

$a_1, \ldots, a_n$	(j)	$\neg\neg p$		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	p	j	DN

Rule of $\wedge$I:

$a_1, \ldots, a_m$	(j)	p		
	$\vdots$			
$b_1, \ldots, b_n$	(k)	q		
	$\vdots$			
$a_1, .., a_m, b_1, .., b_n$	(l)	$p \wedge q$	j,k	$\wedge$I

Rule of $\rightarrow$I:

j	(j)	p	assumption	
	$\vdots$			
$a_1, \ldots, a_n$	(k)	q		
	$\vdots$			
$\{a_1, \ldots, a_n\}/j$	(l)	$p \rightarrow q$	j,k	$\rightarrow$I

Rule of $\neg$I:

j	(j)	p	assumption	
	$\vdots$			
$a_1, \ldots, a_n$	(k)	$\bot$		
	$\vdots$			
$\{a_1, \ldots, a_n\}/j$	(l)	$\neg p$	j,k	$\neg$I

Rule of $\vee$I:

$a_1, \ldots, a_n$	(j)	p		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	$p \vee q$	j	$\vee$I
		(**or** $q \vee p$)		

Rule of $\leftrightarrow$E:

$a_1, \ldots, a_m$	(j)	$p \leftrightarrow q$		
(**or** $a_1, \ldots, a_m$	(j)	$q \leftrightarrow p$)		
	$\vdots$			
$b_1, \ldots, b_n$	(k)	p		
	$\vdots$			
$a_1, .., a_m, b_1, .., b_n$	(l)	q	j,k	$\leftrightarrow$E

Rule of $\leftrightarrow$I:

g	(g)	p	assumption	
	$\vdots$			
$a_1, \ldots, a_m$	(h)	q		
	$\vdots$			
j	(j)	q	assumption	
	$\vdots$			
$b_1, \ldots, b_n$	(k)	p		
	$\vdots$			
X	(l)	$p \leftrightarrow q$	g,h,j,k	$\leftrightarrow$I

where $X = \{a_1, \ldots, a_m\}/\text{g} \cup \{b_1, \ldots, b_n\}/\text{j}$

Rule of $\forall$E:

$a_1, \ldots, a_n$	(j)	$\forall \nu \phi \nu$		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	ϕt	j	$\forall$E

where ϕt is obtained from $\phi \nu$ by replacing **every** occurence of ν in $\phi \nu$ with t

Rule of $\forall$I:

$a_1, \ldots, a_n$	(j)	ϕt		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	$\forall \nu \phi \nu$	j	$\forall$I

where t is not in any of the lines $a_1, \ldots a_n$
$\phi \nu$ is ϕt with **all** t replaced by ν
the variable ν is not in ϕt

Rule of $\exists$E:

$a_1, \ldots, a_m$	(j)	$\exists \nu \phi \nu$		
	$\vdots$			
k	(k)	ϕt	assumption	
	$\vdots$			
$b_1, \ldots, b_n$	(l)	ψ		
	$\vdots$			
X	(m)	ψ	j,k,l	$\exists$E

where $X = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_n\}/\text{k}$
ϕt is $\phi \nu$ with **all** ν replaced by t
the name t is not in
any of the lines j, l, $\{b_1, \ldots, b_n\}/\text{k}$

Rule of $\exists$I:

$a_1, \ldots, a_n$	(j)	ϕt		
	$\vdots$			
$a_1, \ldots, a_n$	(k)	$\exists \nu \phi \nu$	j	$\exists$I

$\phi \nu$ is ϕt with an **arbitrary number** of t replaced by ν

Rule of $=$E:

$a_1, \ldots, a_m$	(j)	$t_1 = t_2$		
	$\vdots$			
$b_1, \ldots, b_n$	(k)	ϕt_1		
	$\vdots$			
X	(l)	ϕt_2	j,k	$=$E

ϕt_2 is ϕt_1 with an **arbitrary number** of t_1 replaced by t_2

Rule of $=$I:

(j)	$t = t$		=I

Axioms for (partial) orders

Language $\mathcal{L}_{\text{order}} = \{\preccurlyeq\}$

$\preccurlyeq$ a binary relation symbol

Axioms:

$\preccurlyeq$1	$\forall x\, x \preccurlyeq x$	reflexivity
$\preccurlyeq$2	$\forall x\, \forall y\, ((x \preccurlyeq y \wedge y \preccurlyeq x) \to x = y)$	antisymmetry
$\preccurlyeq$3	$\forall x\, \forall y\, \forall z\, ((x \preccurlyeq y \wedge y \preccurlyeq z) \to x \preccurlyeq z)$	transitivity

Axioms for groups

Language $\mathcal{L}_{\text{grp}} = \{e, ^{-1}, *\}$

e a constant (a 0-ary function symbol)

$^{-1}$ a unary function symbol (we write x^{-1} instead of $^{-1}(x)$)

$*$ a binary function symbol (we write $x * y$ instead of $*(x, y)$)

Axioms:

G1	$\forall x \forall y \forall z (x * y) * z = x * (y * z)$	$*$ is associative
G2	$\forall x (x * e = x \wedge e * x = x)$	e is an identity element
G3	$\forall x (x * x^{-1} = e \wedge x^{-1} * x = e)$	$^{-1}$ gives an inverse

Peano's axioms for $\mathbb{N}$

Language $\mathcal{L}_{\text{Peano}} = \{0, S, +, *\}$ (the intended interpretation in [ ]):

0 a constant (a 0-ary function symbol) [the number 0]

S a unary function symbol [the next number]

$+, *$ binary function symbols [addition and multiplication]

(we write, for instance, $x + y$ and $x * y$ in place of $+(x, y)$ and $*(x, y)$:)

Axioms:

P1	$\forall x\, \forall y\, (S(x) = S(y) \to x = y)$	successor function injective
P2	$\forall x\, S(x) \neq 0$	0 is not a successor
P3	$\forall x\, x + 0 = x$	+ defined, base
P4	$\forall x\, \forall y\, x + S(y) = S(x + y)$	+ defined, step
P5	$\forall x\, x * 0 = 0$	$*$ defined, base
P6	$\forall x\, \forall y\, x * S(y) = (x * y) + x$	$*$ defined, step
P7	$\forall z_1 \ldots \forall z_n ((\phi 0 \wedge \forall x\, (\phi x \to \phi S(x))) \to \forall x\, \phi x)$	induction axiom

In P7 ϕx is an arbitrary $\mathcal{L}_{\text{Peano}}$-formula with all free variables among $x, z_1, \ldots, z_n$. So P7 is really an infinite number of axioms, a so-called **axiom schema**.

With P7 (usually with $n = 0$) we can perform **proofs by induction** in $\mathbb{N}$.

ZERMELO-FRAENKEL'S AXIOMS for set theory

Language $\boldsymbol{\mathcal{L}_{\text{sets}}} = \{\in\}$

$\in$ a binary relation symbol

(Although the intended interpretations have sets as elements, we can not be sure that every "real" set of elements from a model also corresponds to an element in the model.)

1. **Extensionality**

If x and y have the same elements, then $x = y$,

$$\forall \boldsymbol{x}\, \forall \boldsymbol{y}\, (\forall \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{x} \leftrightarrow \boldsymbol{z} \in \boldsymbol{y}) \rightarrow \boldsymbol{x} = \boldsymbol{y})$$

2. **Pairing**

For all x and y there is a set $\{x, y\}$ containing exactly x and y,

$$\forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \exists \boldsymbol{z}\, \forall \boldsymbol{u}\, (\boldsymbol{u} \in \boldsymbol{z} \leftrightarrow (\boldsymbol{u} = \boldsymbol{x} \vee \boldsymbol{u} = \boldsymbol{y}))$$

3. **Separation**

If P is a property (with a parameter q), there is for every x and q a set $y = \{z \in x \mid P(z, q)\}$ of all $z \in x$ with the property P,

$$\forall \boldsymbol{x}\, \forall \boldsymbol{q}\, \exists \boldsymbol{y}\, \forall \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{y} \leftrightarrow (\boldsymbol{z} \in \boldsymbol{x} \wedge \boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{q})))$$

($\phi(z, q)$ an arbitrary $\mathcal{L}_{\text{sets}}$-formula (defining P))

4. **Union**

For every set x, there is a set $y = \bigcup x$, the union of the elements of x,

$$\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{y} \leftrightarrow \exists \boldsymbol{u}(\boldsymbol{z} \in \boldsymbol{u} \wedge \boldsymbol{u} \in \boldsymbol{x}))$$

5. **Power set**

For every set x, there is a set $y = \mathcal{P}(x)$, the set of all subsets of x,

$$\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{y} \leftrightarrow \forall \boldsymbol{u}\, (\boldsymbol{u} \in \boldsymbol{z} \rightarrow \boldsymbol{u} \in \boldsymbol{x}))$$

6. **Infinity**

There exists an infinite set (a set containing $\mathbb{N}$),

$$\exists \boldsymbol{x}\, (\varnothing \in \boldsymbol{x} \wedge \forall \boldsymbol{y}\, (\boldsymbol{y} \in \boldsymbol{x} \rightarrow \bigcup \{\boldsymbol{y}, \{\boldsymbol{y}\}\} \in \boldsymbol{x}))$$

($\varnothing$ exists by 3. with $z \neq z$ as $\phi(z)$. $\{y\} = \{y, y\}$ exists by 2.)

7. **Replacement**

If f is a function, then for any set x there exists a set $y = \{f(y) \mid y \in x\}$

$$\forall \boldsymbol{p}\, (\forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \forall \boldsymbol{z}\, ((\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}) \wedge \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{p})) \rightarrow \boldsymbol{y} = \boldsymbol{z}) \rightarrow$$
$$\forall \boldsymbol{x}\, \exists \boldsymbol{y}\, \forall \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{y} \leftrightarrow \exists \boldsymbol{u}\, (\boldsymbol{u} \in \boldsymbol{x} \wedge \boldsymbol{\varphi}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{p}))))$$

(p a parameter, the $\mathcal{L}_{\text{sets}}$-formula φ defines f)

8. **Regularity (well-foundedness)**

$\in$ is a well-founded relation,

$$\forall \boldsymbol{x}\, (\boldsymbol{x} \neq \varnothing \rightarrow \exists \boldsymbol{y}\, (\boldsymbol{y} \in \boldsymbol{x} \wedge \neg \exists \boldsymbol{z}\, (\boldsymbol{z} \in \boldsymbol{x} \wedge \boldsymbol{z} \in \boldsymbol{y})))$$

9. **Choice**

If $\varnothing \notin x$, there is a function f on x which chooses an element from each $y \in x$,

$$\forall \boldsymbol{x}\, (\neg \varnothing \in \boldsymbol{x} \rightarrow \exists \boldsymbol{f}\, (\boldsymbol{funk}(\boldsymbol{f}) \wedge \forall \boldsymbol{y}\, (\boldsymbol{y} \in \boldsymbol{x} \rightarrow \boldsymbol{f}(\boldsymbol{y}) \in \boldsymbol{y})))$$

($funk(f)$ is an $\mathcal{L}_{\text{sets}}$-formula saying that f is a function, i.e. a set of ordered pairs with $\forall x\, \forall y\, \forall z\, ((\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f) \rightarrow y = z)$. $f(z)$ is given by $\langle z, f(z)\rangle \in f$. $\langle a, b\rangle$ is $\{\{a\}, \{a, b\}\}$.)