Mathematics, KTH **SF2736 Discrete mathematics**
B.Ek **autumn 2015**

Problem session 6, 14 December 2015

1. What numbers of regions are possible for a plane, regular, Eulerian, simple graph with v vertices?

2. Show that in an arbitrary (not necessarily simple) graph $G = (V, E)$ it is possible to introduce directions on the edges so that in the resulting digraph $G' = (V, A)$ the difference between the in- and the out-degrees at each vertex is at most 1. (The in-degree $\delta^-(v)$ at the vertex v is the number of arcs ending in v and the out-degree $\delta^+(v)$ the number starting in it.)

3. Show that if a connected, planar graph $G = (V, E)$ has at least one cycle and all cycles have length at least $k \geq 3$, then

$$e \leq \tfrac{k}{k-2}(v - 2).$$

Use this to show that Petersen's graph is not planar. That graph (see also Biggs 15.2:2 and 15.8:3) has the adjacency list

a	b	c	d	e	f	g	h	i	j
b	a	b	c	a	a	b	c	d	e
e	c	d	e	d	h	i	f	f	g
f	g	h	i	j	i	j	j	g	h

(A common way to draw it is as a (regular) pentagon with vertices in the order *abcde* and inside them their respective neighbours *fghij* (forming the vertices of a pentagonal star).)

Also use each of Kuratowski's and Wagner's theorems to show that Petersen's graph is not planar.

4. Show that if the graph G is bipartite and k-regular, G has at least k different complete matchings.
How many are there if $k = 2$?

5. Ten students have taken an exam with ten problems. Each problem was solved by at least six students. Each student solved at least four problems.
Show that the problems can be given to the students with one per student, so that each problem was solved by its student.

6. Start from the matching $M = \{a5, c4, d1\}$ and use augmenting alternating paths to find a maximum matching in the bipartite graph $G = (X \cup Y, E)$ with $X = \{a, b, \ldots, e\}$, $Y = \{1, 2, \ldots, 5\}$, $E = \{a1, a5, b1, b4, c2, c4, d1, d3, e4, e5\}$.

7. Let $X = A_1 \cup \ldots \cup A_n = B_1 \cup \ldots \cup B_n$ be two partitions of a set X.
Show that there is a simultaneous transversal for the two partitions (a set $\{x_1, x_2, \ldots, x_n\}$ of n distinct elements with, for some $\pi \in S_n$, $x_i \in A_i \cap B_{\pi(i)}$ for $i = 1, 2, \ldots$) iff for each $k = 1, 2, \ldots, n-1$, no k of the A_i are contained in the union of $k - 1$ of the B_j.

8a. If all 52 cards from a deck of cards (containing 4 cards of each rank $(1, 2, \ldots, 13)$) are dealt into 13 piles with 4 cards in each pile, is it always possible to choose one card from each pile so that the 13 chosen cards contain one of each rank?
b. What if the piles contain $2, 3, 4, 4, \ldots, 4, 5, 6$ cards?

9. Let $M_0 = \{1, 2, \ldots\}$, $M_1 = \{1\}$, $M_2 = \{1, 2\}$, $M_3 = \{1, 2, 3\}, \ldots$
a. Does $\{M_i\}_{i=0}^{\infty}$ have a transversal?
b. Does $\{M_i\}_{i=0}^{\infty}$ satisfy Hall's condition $|\bigcup_{i \in H} M_i| \geq |H|$ for all $H \subseteq \mathbb{N}$?

10. A network and a flow are defined as below

(x,y)	(s,a)	(s,b)	(s,c)	(a,b)	(a,d)	(b,c)	(b,d)	(b,e)	(c,e)	(d,t)	(e,t)
$c(x,y)$	5	6	8	4	10	2	3	11	6	9	4
$f(x,y)$	5	6	0	0	5	1	2	3	1	7	4

a. What is the value of f?
b. Find an f-augmenting path and compute the value of the augmented flow.
c. Find a cut with capacity 12.

11. On a certain island far, far away, every inhabitant is either a knight (and always tells the truth) or a knave (and always lies). A, B and C are inhabitants of this island and once made the following statements.
A: "C is a knave if B is a knight."
B: "A is a knave if C is a knight."
C: "Exactly one of us three is a knight."
What is each of A, B and C, a knight or a knave?

12. On the island of knights and knaves, I met the two inhabitants D and E.
I asked D: "What would E answer to the question 'Are you both knights?'?"
D:s answer was enough for me to know what they were.
What were they?