**Mathematics, KTH** | **SF2736 Discrete mathematics**
**B.Ek** | **autumn 2015**

# Problem session 5, 7 December 2015

**1.** Solve the recursion

$$\begin{cases} u_{n+2} + 8u_{n+1} - 9u_n = 8 \cdot 3^{n+1}, & n = 0, 1, 2, \ldots \\ u_0 = 2, \; u_1 = -6. \end{cases}$$

**2.** For a convex polygonal region $R$ with $n+1$ sides ($n = 2, 3, \ldots$), let $h_n$ be the number of ways to divide $R$ into triangular regions by drawing diagonals which do not intersect in the interior of $R$ (for instance, $h_2 = 1$, $h_3 = 2$, $h_4 = 5$).
Also, let $h_1 = 1$.
Find the generating function $H(x) = \sum_{n=1}^{\infty} h_n x^n$ and use it to find the $h_n$.

**3.** Find the number of partitions of 16 in which each part is an odd prime.

**4.** Prove that the number of partitions of $n$ in which each part is 1 or 2 is equal to the number of partitions of $n+3$ which have exactly two distinct parts.

**5.** Show that for $k, n \in \mathbb{Z}_+$, the number of partitions of $n$ in which there are at most $k$ parts of each size is equal to the number of partitions of $n$ into parts, none of which is of size a multiple of $k+1$.

**6.** A family of two grown-ups and two children want to cross a river, but their boat can carry at most the weight of one grown-up. The weight of each child is half that of a grown-up.
Use a directed graph to find a way for them to cross.

**7.** Are the graphs given by the following adjacency lists isomorphic?

| a | b | c | d | e | f |
|---|---|---|---|---|---|
| b | a | a | a | b | c |
| c | c | b | e | d | d |
| d | e | f | f | f | e |

and

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | 1 | 2 | 3 | 2 | 1 |
| 4 | 3 | 4 | 5 | 4 | 3 |
| 6 | 5 | 6 | 1 | 6 | 5 |

**8.** Show that for a graph $G = (V, E)$ with $|V| \geq 2$ there are $x, y \in V$, $x \neq y$ with $\delta(x) = \delta(y)$.

**9.** In each case, is there a simple graph with seven vertices and degrees,
i) $0, 2, 3, 3, 4, 4, 5$, ii) $2, 3, 3, 3, 3, 3, 3$, iii) $2, 2, 3, 5, 5, 5, 6$ ?

**10.** Show that if the graph $G = (V, E)$ is not connected, then the complement graph $\overline{G} = (V, E^c)$ (where $E^c = \{\{x, y\} \mid x, y \in V, x \neq y, \{x, y\} \notin E\}$) is connected.

**11.** Show that if the graph $G = (V, E)$ with $|V| = n$ satisfies $\delta(x) + \delta(y) \geq n - 1$ for all $x, y \in V$, $x \neq y$, then $G$ is connected.

**12.** Find the maximum number of vertices in a graph with 28 edges if the degree of every vertex is at least 3.

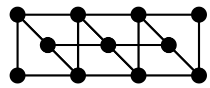
**13.** Find the number of (non-isomorphic) 4-regular graphs with seven vertices.

**14.** Show that in a simple graph $G = (V, E)$ there are at least

$$\sum_{v \in V} \frac{1}{\delta(v) + 1}$$

independent vertices (i.e. vertices, no two of which are adjacent).

**15.** Is the graph on the right Hamiltonian?



**16a.** Show that if $m, n \in \mathbb{Z}_+$ are both odd (and not both $= 1$), there is no closed knight's tour (i.e. a sequence of moves of a knight on a chessboard such that the knight visits every square exactly once and returns to the beginning square) on an $m \times n$-"chessboard". (In one move, a knight moves two squares ahead and one to the side.)

**b.** Show the same for a board of dimension $4 \times n$, $n \in \mathbb{Z}_+$.

**17.** An acyclic graph has 316 vertices and 302 edges.
How many components are there in the graph?

**18.** Let $G = (V, E)$ be a graph. Show that its vertices can be coloured with at most two colours, so that each vertex has the same colour as at most half of its neighbours.

**19.** Find orderings of the vertices of the cube such that the greedy algorithm for colouring its vertices requires 2, 3 and 4 colours, respectively.

**20.** Show that if $\overline{G} = (V, E^c)$ is the complement graph of $G = (V, E)$, then $\chi(G)\chi(\overline{G}) \geq |V|$.

**21.** Let $G = (V, E)$ be a graph with $|V| = n$.
Show that the coefficient of $\lambda^{n-1}$ in the chromatic polynomial $P_G(\lambda)$ is $-|E|$.

**22.** Find the chromatic polynomial $P_{C_5}(\lambda)$ of the cycle graph $C_5$, by combinatorial reasoning ("case by case").
In a lecture we found the general $P_{C_n}(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

**23.** Let $G = (V, E)$ be a connected, 4-regular graph which is also planar. How many regions are there in a plane drawing of $G$ if $|E| = 16$?

**24.** In a connected, plane graph each vertex has degree 3 or 5. The number of vertices is 12 and the number of regions is 11. How many vertices have degree 3 and how many have degree 5?

**25.** In a 3-regular, plane, connected graph all regions have either 4 or 6 edges (including the unbounded region). How many regions have 4 edges?

**26.** Let $G_i = (V_i, E_i)$, $i = 1, 2$ be (simple) graphs and define their **lexicographic product $G_1[G_2]$** to be the graph $(V_L, E_L)$ with

$$V_L = V_1 \times V_2 \text{ and } \{(u_1, u_2), (v_1, v_2)\} \in E_L \Leftrightarrow \begin{cases} \{u_1, v_1\} \in E_1 \text{ or} \\ u_1 = v_1 \text{ and } \{u_2, v_2\} \in E_2. \end{cases}$$

Show that $\overline{G_1[G_2]} = \overline{G}_1[\overline{G}_2]$ (where $\overline{G}$ is the complement graph of $G$).