Mathematics, KTH
B.Ek

SF2736 Discrete mathematics
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Problem session 4, 30 November 2015

1. It is possible to complete the table on the right to form the group table of a group. Do that.
a. Is the group abelian?
b. Which element is the identity element?
c. Find inverses of all elements.
d. Find the order of all elements and all cyclic subgroups of the group.
e. Find $a * b * c * d * f * g$. (No brackets needed?)

$*$	a	b	c	d	f	g
a			c			
b					d	c
c		f		g		
d				a		b
f		c	d			
g					a	

2. Show that this table is not the group table of any group:

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	d	e	c
b	b	e	c	d	a
c	c	d	a	b	e
d	d	c	e	a	b

3. Let G be a group with identity element 1 and $a, b, c \in G$.

a. Given that there is an $x \in G$ satisfying $\begin{cases} ax^2 = b \\ x^3 = 1, \end{cases}$ find all such x.

b. Given that there is an $x \in G$ satisfying $\begin{cases} (xax)^3 = bx \\ x^2a = (xa)^{-1}, \end{cases}$ find all such x.

c. Show that $bac = a^{-1} \Rightarrow cab = a^{-1}$.
d. Show that $(abc)^{-1} = abc \Rightarrow (bca)^{-1} = bca$.
e. Show that $a^3 = 1 \Rightarrow a$ has a square root, i.e. $a = r^2$ for some $r \in G$.
f. Show that $b^2ab = a^{-1} \Rightarrow a$ has a cube root, i.e. $a = s^3$ for some $s \in G$.

4. Can this table be completed to the group table of a group?

$\circ$	e	a	b	c	d
e	e				
a		e			
b					
c					
d					

5a. Find the smallest subgroup of $(\mathbb{Z}_{18}, +)$ containing the elements 3 and 7.
b. Find the smallest coset of some subgroup of $(\mathbb{Z}_{18}, +)$ containing the elements 3 and 7.

6a. For any two subgroups H and K of a group G show that $H \cap K$ is a subgroup of G.
b. Show that there does not exist a group G with two subgroups H and K, such that neither $H \not\subseteq K$ nor $K \not\subseteq H$, and such that $H \cup K$ is a subgroup of G.
c. H and K are subgroups of G with $|H| = 52$, $|K| = 151$. Find $|H \cap K|$.

7. Find a non-abelian group of size 66.

8. Is the group $(\mathbb{Z}_{19} \setminus \{0\}, \cdot\,)$ a cyclic group?

9. Let S_4 denote the group consisting of all permutations on the set $\{1, 2, 3, 4\}$. Find the smallest subgroup of S_4 containing the permutations $(1\ 2\ 3)$ and $(3\ 4)$.

10. Assume that the two elements a and b in a group G commute, i.e., $ab = ba$. Is it then always true that the order $o(ab)$ of the element ab satisfies $o(ab) = \mathrm{lcm}(o(a), o(b))$?

11. Show that every subgroup of a cyclic group is cyclic.

12. Show that every group with 55 elements contains at least one element of order 5 and at least one element of order 11.

13. Find a group with 64 elements, which all have order either 1 or 2.

14. Let G be S_4, the group of permutations of $\{1, 2, 3, 4\}$, and also let (in cycle notation) $N = \{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \subset G$.
a. Call the vertices of a regular tetrahedron 1, 2, 3, 4. The elements of G then correspond to rigid (i.e. distance-preserving) transformations of three-dimensional space, which leave the tetrahedron invariant but permute the vertices. For each conjugacy class in G, find the sort of transformation its elements correspond to.
b. Which types of transformations correspond to even permutations of G and which to odd ones?
c. Show that N is a **normal** subgroup of G and describe the quotient group G/N (i.e. give a well-known group which is isomorphic to G/N).

15. Peter wants to produce square trays to sell in his interior decoration shop. The trays are to have coloured (yellow, red or blue) beads in the four corners (the vertices of the square). How many essentially different (i.e. so that they can not be made identical by rotation) such trays are there in the following cases?
a. The upper sides and the undersides of the trays are alike, so the trays look the same if they are turned upside-down.
b. The upper sides and the undersides of the trays are not alike.
c. What are the answers in a. and b. if the number of different colours of the beads is k?

16. From six drinking straws of equal length a nice decoration in the form of a tetrahedron (with the straws as edges) is made. How many essentially different (i.e. so that they can not be made identical by rotation) such decorations are there, if one can use straws of k different colours?