Problem session 3, 23 November 2015

1. If n, r, $k \in \mathbb{N}$ with $n \geq r \geq k$, prove that
$$\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k} .$$

2. Show that if $n \in \mathbb{Z},\ n > 2$,
$$\sum_{k=2}^{n} (-1)^k\, k(k-1) \binom{n}{k} = 0.$$

3. Find the coefficient of x^{12} in the polynomial $(4 + 3x^2)^{10}$.

4. How many odd numbers between 1000 and 10 000 have four distinct digits?

5. Find the number of ways we can form words using each letter in the word DISKRET exactly once, if none of the words RET, SEK or DIS may appear as subwords.

6. Find the number of positive integers d that divide the integer 129 600.

7. In how many ways can fifteen children in a class be placed into three (unlabeled) rows?

8. Find the number of ways to divide the set $\{1, 2, 3, 4, 5, 6\}$ into three non-empty subsets in such a way that the elements 1 and 2 will belong to distinct subsets.

9. If eight dice are rolled, what is the probability that all six numbers appear?

10. Find the number of solutions to the Diophantine problem
$$x_1 + x_2 + x_3 \leq 15$$
if we require that $0 \leq x_1 \leq 5,\ -3 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 9$.

11. In how many ways can five girls and five boys be divided into three groups in such a way that each group will contain at least one boy and one girl?

12. In a lecture it was shown that if you start from 6 points and between each pair of points draw either a red or a blue line, there will be at least one triangle with all sides of the same colour.
a. Show that there will in fact be at least two such triangles (maybe of different colours).
b. Show that if you start from 10 points and proceed in the same way, there will be either a red triangle or a blue tetrahedron (i.e., four points, all pairwise connected by blue lines).
c. Show that 9 points suffice in **b**.

13. How many results are possible when you roll n identical dice?
(By a "result" is meant a list of the number of dice showing each number of dots.)

14. Find the area of a spherical triangle with angles α, β, γ on a sphere of radius R. A spherical triangle has parts of three great circles as sides and a sphere of radius R has area $4\pi R^2$.

15. In how many ways can 7 marbles be distributed in 4 boxes in the following cases?
a. Distinct marbles, distinct boxes, boxes may be empty.
b. Distinct marbles, distinct boxes, boxes must not be empty.
c. Identical marbles, distinct boxes, boxes may be empty.
d. Identical marbles, distinct boxes, boxes must not be empty.
e. Distinct marbles, identical boxes, boxes may be empty.
f. Distinct marbles, identical boxes, boxes must not be empty.

16. Let $\sigma(n)$, for $n \in \mathbb{Z}_+$, be the sum of all positive divisors of n. What is

$$\sum_{d|n} \mu(d)\sigma(\tfrac{n}{d}) \ ?$$

($\sum_{d|n}$ means the sum over all **positive** divisors of n.)

17. Let $\pi \in S_7$ be given by $\pi = (1\,4)(2\,6\,3)(5\,7)$.
a. How many $\sigma \in S_7$ commute with π, that is, satisfy $\sigma\pi = \pi\sigma$?
b. How many of those σ are odd permutations?

18. In a very small village, consisting just of a row of six houses (numbered, in order, 1–6), there live six married couples (each consisting of one woman and one man), one pair in each house.
Each one of the women also has (exactly) one brother among the six men and the men each have one sister. No one lives next door to, or is married to, his sister or her brother.
Anders only has one next door house and only one brother-in-law (Anders lives in number 1 and his sister is married to his wife's brother). Anders' neighbour Börje, on the other hand, has two next door houses (obviously) and two brothers-in-law (both in the village).
a. In which house does Anders' sister Anna live?
b. In the other end house, number 6, Cecilia lives with her husband. Börje's sister's name is Birgitta.
For each house, decide where the brother of that house's wife lives.

19. If $n \in \mathbb{N}$, what is

$$\binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{n-\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \ ?$$

Show it. ($\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$, the greatest integer $\leq \frac{n}{2}$.)