Mathematics, KTH
B.Ek

SF2736 Discrete mathematics
autumn 2015

Problem session 2, 16 November 2015

1. Is the following information sufficient to find the relation $\mathcal{R}$?
1. $\mathcal{R}$ is an equivalence relation on $\mathcal{M} = \{1, 2, 3, 4, 5, 6\}$.
2. $\{(1, 2), (2, 3), (2, 4), (5, 6)\} \subseteq \mathcal{R}$.
3. $(2, 6) \notin \mathcal{R}$.

2. Describe all equivalence relations $\mathcal{R}$ on $\mathcal{M} = \{1, 2, 3, 4, 5, 6, 7\}$ such that

$$\{(1, 5), (1, 4), (2, 3), (3, 6)\} \subseteq \mathcal{R}.$$

3. Is divisibility on $\mathbb{Z}$ (i.e. the relation $\mathcal{R}$ defined by $m\mathcal{R}n \Leftrightarrow m \mid n$ for all $m, n \in \mathbb{Z}$) a partial order?

4. Let $f : X \to X$ be a function on a set X and the corresponding binary relation on X be $\mathcal{Q}$ (i.e. for all $x, y \in X$, $x\mathcal{Q}y \Leftrightarrow y = f(x)$). What are the conditions on f that make $\mathcal{Q}$ reflexive, symmetric, antisymmetric, and transitive?

5a. Give an example of sets X, Y, Z and functions $f : X \to Y$, $g : Y \to Z$, such that the composite $gf : X \to Z$ is a bijection, although neither f nor g is. Is either of f and g necessarily injective or surjective?
b. The same question, but with the further condition $X = Y = Z$.

6. Assume that $f : A \to B$ and $g : B \to A$ are such that

$$(g \circ f)(x) = x \quad \text{for all} \quad x \in A.$$

Is either of f and g necessarily injective, surjective or bijective?

7. Show that to any sequence $a_1, a_2, \dots, a_n$ of n positive integers there exists at least one non-empty subsequence

$$a_{i_1}, a_{i_2}, \dots, a_{i_t} \quad \text{such that} \quad a_{i_1} + a_{i_2} + \ldots + a_{i_t} \equiv 0 \pmod{n}.$$

8. Let $m, n \in \mathbb{Z}_+$ and $a_1, a_2, \dots, a_{mn+1}$ be a sequence of distinct real numbers. Show that there is at least one of
a subsequence $a_{i_1} < a_{i_2} < \ldots < a_{i_{m+1}}$ (where $i_1 < i_2 < \ldots < i_{m+1}$) and
a subsequence $a_{j_1} > a_{j_2} > \ldots > a_{j_{n+1}}$ (where $j_1 < j_2 < \ldots < j_{n+1}$).

9a. Show that a union of finitely many countable sets is countable.
b. Is a union of countably many countable sets always countable?
c. Is the set of bijective maps $\mathbb{N} \to \mathbb{N}$ countable?

10. Show that the set of all **finite** sets of natural numbers is countable, i.e.

$$|\mathcal{P}_{fin}(\mathbb{N})| = |\mathbb{N}|,$$

where $\mathcal{P}_{fin}(\mathbb{N}) = \{A \mid A \subset \mathbb{N}, |A| < \infty\}$.

11. Let $A \subset \mathbb{R}$ be countable and B be the set of all $x \in \mathbb{R}$ such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0,$$

where $a_i \in A$ for $i = 0, 1, \dots, n$. Is B necessarily countable?

12a. Are there $f : \mathbb{R} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{R}$, such that $gf : \mathbb{R} \to \mathbb{R}$ is a bijection?

b. Are there $f : \mathbb{Q} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{Q}$, such that $gf : \mathbb{Q} \to \mathbb{Q}$ is a bijection?

13. Recall the **Fibonacci numbers** $F_n : 0, 1, 1, 2, 3, 5, 8, 13, \ldots$, recursively defined by

$$\begin{cases} F_0 = 0, \ F_1 = 1 \\ F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, \ldots \end{cases}$$

a. Show that $F_0^2 + F_1^2 + \cdots + F_n^2 = \sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$ for all $n = 0, 1, 2, \ldots$

b. Show that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for all $n = 1, 2, 3, \ldots$

c. Show that for any $k \in \mathbb{Z}_+$ there is an $n \in \mathbb{Z}_+$ such that $k \mid F_n$.

d. Show that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ for all $m, n = 0, 1, 2, \ldots$

14. Show that if x, y, $z \in \mathbb{Z}$ satisfy $x^3 + 3y^3 = 9z^3$, then $x = y = z = 0$.

15. Two lighthouses emit (very short) flashes of light at regular intervals of α and β (minutes), respectively. α and β are irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ (so on average, there is one flash per minute).

Show that if they both emit a flash at time $t = 0$, there will be exactly one flash between $t = n - 1$ and $t = n$, for all integer $n > 1$.