Mathematics, KTH	SF2736 Discrete mathematics
B.Ek	autumn 2015

Answers and hints for homework assignment set 2, (for Monday 21 December)

1. (0.4p) We shall show that for all $m, n \in \mathbb{Z}_+$

$$\sum_{k=1}^{m} S(m,k) \cdot (n)_k = n^m.$$

(With $S(m,k)$ the Stirling number (of the second kind). $(n)_k = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$.)
Hint: Both sides give the total number of functions from an m-set to an n-set, in the left hand side partitioned by k, the number of values taken.

2. (0.4p) With, for all $n \in \mathbb{Z}_+$, $\sigma(n)$ the sum of all positive divisors of n and $\mu(n)$ the Möbius function, we want $\sum_{d|n} \mu(d)\sigma(\frac{n}{d})$ for $n \in \mathbb{Z}_+$ (sum over all positive divisors of n).
Answer: It is n.
Hint: Möbius inversion: $\sigma = 1 * id \Leftrightarrow id = \mu * \sigma$ (or as on p118 in Biggs (Thm 11.5.2 and exercise 11.5.5)).
(By mistake, this was also a problem for problem session 3.)

3. (0.4p) We want the number of essentially different (i.e. so that none of them can be rotated to look exactly like any other) ways to colour the edges of a cube with four red, four blue and four yellow edges.
Answer: There are 1 479 such colourings.
Hints: We want the number of orbits when G (the group of symmetry rotations of the cube) acts on the set X of colourings of the edges of the cube with four edges each red, blue and yellow. By "Burnside's lemma" (Thm 21.4 in Biggs) it is $\frac{1}{|G|}\sum_{g\in G}|F(g)|$, with $|F(g)|$ the number of such colourings which are not changed when g rotates the cube.
Taking x as a vertex of the cube, one finds $|G| = |Gx| \cdot |G_x| = 8 \cdot 3 = 24$ and $|F(g)|$ for the different types of g (ordered by angle of rotation and where the axis of rotation meets the cube; all edges in the same orbit must have the same colour):

angle	axis	number of such g	orbits of edges	$\lvert F(g)\rvert$
0	—	1	$[1^{12}]$	$\binom{12}{4,4,4} = \frac{12!}{(4!)^3} = 34\,650$
$\frac{2\pi}{3}$	vertex	8	$[3^4]$	0
π	edge	6	$[1^2 2^5]$	$\binom{6}{2,2,2} = 90$
$\frac{\pi}{2}$	side	6	$[4^3]$	$\binom{3}{1,1,1} = 6$
π	side	3	$[2^6]$	$\binom{6}{2,2,2} = 90$

The number of colourings: $\frac{1}{24}(34\,650 + 8 \cdot 0 + 6 \cdot 90 + 6 \cdot 6 + 3 \cdot 90) = 1\,479$.

4. (0.4p) We shall show that, for any $n \in \mathbb{Z}_+$, the number of partitions of n with (for $k \in \mathbb{Z}_+$) at most k parts of size k is equal to the number of partitions of n with no part of size k^2+k (for $k \in \mathbb{Z}_+$), i.e.

$$p(n \mid \text{for } k \in \mathbb{Z}_+\text{: at most } k \text{ parts of size } k) =$$
$$= p(n \mid \text{for } k \in \mathbb{Z}_+\text{: no part of size } k^2+k).$$

Hint: The generating function of the numbers on the left hand side is
$$(1+x)\cdot(1+x^2+x^4)\cdot\ldots\cdot(1+x^k+x^{2k}+\ldots+x^{k^2})\cdot\ldots =$$
$$= \tfrac{1-x^2}{1-x}\cdot\tfrac{1-x^6}{1-x^2}\cdot\ldots\cdot\tfrac{1-x^{k^2+k}}{1-x^k}\cdot\ldots =$$
$$= \tfrac{1}{1-x}\cdot\cancel{\tfrac{1}{1-x^2}}\cdot\ldots\cdot\cancel{\tfrac{1}{1-x^6}}\cdot\ldots\cdot\cancel{\tfrac{1}{1-x^{k^2+k}}}\cdot\ldots,$$
the generating function of the numbers on the right hand side.

5. (0.4p) A (finite) set A of people, each with a list of books from the set B he or she wants to borrow, is given. We want (for any $k \in \mathbb{Z}_+$) a necessary and sufficient condition on the lists for it to be possible for each person to borrow k books from her or his list.

Answer: The condition is $|J(S)| \geq |S|$ for all $S \subseteq A$, where $J(S)$ is the set of books on the list of at least one of the people in S.

Hint: Consider the bipartite graph $G = (A \cup B, E)$, where $\{a,b\} \in E$ iff b is on a's list. We want the condition for there to exist an $N \subseteq E$ such that each $a \in A$ is in exactly k and each $b \in B$ in at most one of the $n \in N$.
"Splitting" each person into k copies gives the bipartite graph $G' = (A' \cup B, E')$, where each $a \in A$ corresponds to k elements $a_1, \ldots, a_k \in A'$ (different a_i for different a, no other elements in A') and $\{a,b\} \in E \Leftrightarrow \{a_i, b\} \in E'$ for $i = 1, \ldots, k$. Then an $N \subseteq E$ as above exists iff there is a complete (of A') matching $M \subseteq E'$ in G' (a borrows the k books $a_1, \ldots, a_k$ borrow) and by Hall's theorem such an M exists iff $|J'(T)| \geq |T|$ for all $T \subseteq A'$ ($J'(T) = \{b \in B \mid \{a', b\} \in E' \text{ for some } a' \in T\}$). Given $T \subseteq A'$, let $T' = \{a' \in A' \mid a', a'' \text{ correspond to the same } a \in A \text{ for some } a'' \in T\}$. Then $J'(T) = J'(T')$ and $|T'| \geq |T|$, $|T'| = k|S|$ for $S = \{a \in A \mid a_i \in T \text{ for some } i\}$. The condition $|J'(T)| \geq |T|$ for all $T \subseteq A'$ is then satisfied iff $|J(S)| \geq k|S|$ for all $S \subseteq A$.