Mathematics, KTH — SF2736 Discrete mathematics
B.Ek — autumn 2015

Homework assignment set number 1, autumn 2015

Solutions (handwritten, in English or in Swedish) must be submitted not later than **Monday 23 November** at 17.00 (then in hall L52). Don't forget to enter **your name and your e-mail address** on your solutions.

It is allowed to discuss the problems with classmates, but not to obtain help from others (also not from fora on the internet). Everyone must write his/her own solutions, so copying is not allowed (and would be considered as cheating).

To give full points, the solutions must be clear and well explained.

The points given for the solutions will be added to the points on the exam, as explained on the course's web page (the total number of points on homework will be rounded to the nearest integer; $x + 0.5$ ($x \in \mathbb{Z}$) is rounded to $x + 1$).

1. (0.2p) Let X, Y, Z be finite sets with $Z \subseteq Y$, $|Z| = k$, $|X| = m$, $|Y| = n$, where $k \leq m \leq n$.
How many injections $f : X \to Y$ that satisfy $Z \subseteq f[X]$ are there?
($f[X]$ is the image of X under f, $f[X] = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$.)

2. (0.3p) Find all $x \in \mathbb{Z}$ such that $1632x \equiv 3377^{2595} \pmod{481}$ [$481 = 13 \cdot 37$].

3. Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$ and define the binary relation $\mathcal{R}$ on $\mathbb{N}^{\mathbb{N}}$ by, for all $f, g \in \mathbb{N}^{\mathbb{N}}$:

$$f\mathcal{R}g \text{ iff } \{x \in \mathbb{N} \mid f(x) \neq g(x)\} \text{ is a finite set.}$$

a. (0.2p) Is $\mathcal{R}$ an equivalence relation?
b. (0.2p) If $f_1, f_2, g_1, g_2 \in \mathbb{N}^{\mathbb{N}}$ and $f_1\mathcal{R}f_2$, $g_1\mathcal{R}g_2$, which (if any) of the following are necessarily true?

$$(f_1 + g_1)\mathcal{R}(f_2 + g_2), \quad (f_1 \cdot g_1)\mathcal{R}(f_2 \cdot g_2), \quad (f_1 \circ g_1)\mathcal{R}(f_2 \circ g_2)$$

(Where, as expected, $(f + g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $(f \circ g)(x) = f(g(x))$.)

4. (0.5p) Let, as usual, S_{11} be the set of all permutations of $\{1, 2, \ldots, 11\}$ and $\sigma, \tau \in S_{11}$ be given by (in cycle notation)

$$\sigma = (1\ 3\ 7\ 11\ 6)(2\ 9\ 5)(4\ 8\ 10),\ \tau = (1\ 10\ 6\ 5\ 11\ 3\ 7)(2\ 9\ 8).$$

Find (in cycle notation) all $\pi \in S_{11}$ such that $\pi\sigma\pi = \tau$.

5a. (0.2p) Show that if $a_0, a_1, \ldots, a_n, m, r \in \mathbb{Z}$, $n, s \in \mathbb{Z}_+$, p is a prime, and

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0,$$

then

$$f(m + rp^s) \equiv f(m) + rp^s f'(m) \pmod{p^{s+1}},$$

where f' is the (formal) derivative of f,

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1.$$

b. (0.4p) Find all $x \in \mathbb{Z}$ such that

$$x^3 - x^2 + 4x + 1 \equiv 0 \pmod{125}.$$