Interpolating sequences on analytic Besov type spaces

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(Joint work with Nicola Arcozzi and Jordi Pau)

Outline

Motivation

Let $\mathcal{B}$ denote the set of analytic functions from the unit disc $\mathbb{D}$ to $\overline{\mathbb{D}}$.

Question

Given $\{z_1, \dots, z_N\} \subset \mathbb{D}$, for which $\{w_1, \dots, w_N\}$ the interpolation

$$f(z_n) = w_n, \quad n = 1, 2, \dots, n, \tag{1}$$

has a solution $f \in \mathcal{B}$?

Theorem Pick'17

There exists $f \in \mathcal{B}$ satisfying (1) if and only if the quadratic form

$$Q_n(t_1, \dots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w}_k}{1 - z_j \overline{z}_k} t_j \overline{t}_k$$

is nonnegative, $Q_n \geq 0$. When $Q_n \geq 0$ there is a Blaschke product of degree at most n which solves (1).

$H^\infty \equiv$ bounded analytic functions in $\mathbb{D}$

Definition

$\{z_n\}$ is an interpolating sequence for H^∞ if for any sequence $\{w_n\} \in \ell^\infty$, the interpolation problem

$$f(z_n) = w_n, \quad n = 1, 2, \ldots$$

has a solution $f \in H^\infty$.

Theorem [Carleson'58]

The following conditions are equivalent

(a) $\{z_n\}$ is an interpolating sequence for H^∞

(b) $\inf_{n \neq m} \beta(z_n, z_m) > 0$ and $\mu = \sum (1 - |z_n|)\delta_{z_n}$ is a Carleson measure.

Let H be a Hilbert space of functions, and let

$$< f, g > \text{ be the associated inner product, for } f, g \in H.$$

Claim

If the point evaluation functional

$$\begin{array}{rccc} T_z : & H & \longrightarrow & \mathbb{C} \\ & f & \longrightarrow & f(z) \end{array}$$

is bounded, then there exists a unique function $k_z \in H$ with

$$< f, k_z >= f(z) \quad \forall f \in H$$

called reproducing kernel, and it satisfies $\|T_z\| = \|k_z\|_H$.

Interpolating Sequence

A sequence of unimodular functions $\{u_n\} \subset H$ is an Interpolating Sequence (IS) if the operator

$$\begin{array}{ccc} H & \longrightarrow & l^2 \\ f & \longrightarrow & \{< f, u_n >\} \end{array} \quad \text{is onto.}$$

Interpolating Sequence

A sequence of points $\{z_n\}$ is an Interpolating Sequence for H if $\left\{\frac{k_{z_n}}{\|k_{z_n}\|}\right\}$ is an Interpolating Sequence. Ie,

$$\begin{array}{ccc} H & \longrightarrow & l^2 \\ f & \longrightarrow & \left\{\frac{f(z_n)}{\|k_{z_n}\|}\right\} \end{array} \quad \text{is onto.}$$

$\forall \{w_n\} \subset l^2$, there exists $f \in H$ such that $\frac{f(z_n)}{\|k_{z_n}\|} = w_n, \; n = 1, 2, \ldots$

Let D be the Dirichlet space of analytic functions f with

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

Interpolating Sequence for D

A sequence $\{z_n\} \subset \mathbb{D}$ is an Interpolating Sequence for D if for any $\{w_n\} \subset l^2$ there exists $f \in D$ with $\frac{f(z_n)}{\beta(0,z_n)^{1/2}} = w_n$, for $n = 1, 2, \ldots$

Theorem (Marshall-Sundberg'90s)

$\{z_n\} \subset \mathbb{D}$ is an interpolating sequence for D if and only if

- $\inf\limits_{n \neq m} \beta(z_n, z_m) \geq C\beta(0, z_n)$, for $n, m = 1, 2, \ldots$
- $\sum \frac{1}{\beta(0,z_n)} \delta_{z_n}$ is a Carleson Measure for D.

The spaces $B_p(s)$

$B_p(s) \equiv$ Analytic functions on $\mathbb{D}$ with

$$\|f\|^p_{B_p(s)} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty$$

for $1 < p < \infty$ and $0 \leq s < 1$.

Special cases

$p = 2,\ s = 0$ corresponds to the Dirichlet space $\mathcal{D}$.

$p \neq 2,\ s = 0$ corresponds to the Besov space B_p.

Questions

1- What is an interpolating sequence for $B_p(s)$?

2- How we can characterize these sequences?

Carleson measure

A positive measure μ on $\mathbb{D}$ is a Carleson measure for $B_p(s)$ if

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leq C \, \|f\|^p_{B_p(s)}$$

whenever f is in $B_p(s)$.

A Geometric Description of Carleson measures for $B_p(s)$ was given by [*Arcozzi, Rochberg and Sawyer,02*] and [*Stegenga, 80*].

Multiplier Space

$$\mathcal{M}(B_p(s)) = \{f \text{ such that } fg \in B_p(s) \text{ whenever } g \in B_p(s)\}$$

$$f \in \mathcal{M}(B_p(s)) \text{ if and only if } \begin{cases} f \in H^\infty \\ |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) \text{ is a CM for } B_p(s) \end{cases}$$

The point evaluation functional $T_z : \begin{array}{ccc} B_p(s) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(z) \end{array}$ yields a bounded linear functional at each point $z \in \mathbb{D}$ with norm

$$\|T_z\| \approx \frac{1}{(1-|z|^2)^{s/p}} \qquad \text{for } s > 0$$

$$\|T_z\| \approx \beta(0,z)^{(p-1)/p} \qquad \text{for } s = 0$$

Main result

Interpolating sequences for $B_p(s)$

$\{z_n\}$ is an interpolating sequence for $B_p(s)$ if the map

$$f \mapsto \left\{ \frac{f(z_n)}{\|T_{z_n}\|} \right\} \text{ maps } B_p(s) \text{ onto } \ell^p$$

Interpolating Sequences for $\mathcal{M}(B_p(s))$

$\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p(s))$ if the map

$$f \mapsto \{f(z_n)\} \text{ transforms the multipliers of } B_p(s) \text{ onto } \ell^\infty$$

The interpolating sequences for $\mathcal{D}$ were simultaneously characterized by Marshall-Sundberg and Bishop.

Theorem [Böe, '02]

Let $1 < p < \infty$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for B_p.

(ii) $\inf\limits_{n \neq m} \beta(z_n, z_m) \geq C\beta(z_n, 0)$ and $\sum \frac{1}{\beta(0,z_n)^{p-1}} \delta_{z_n}$ is a Carleson measure for B_p.

(iii) $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p)$.

Theorem [Cohn, '93]

Let $1 < p < \infty$, $0 < s$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for $B_p(s)$.

(ii) $\inf\limits_{n \neq m} \beta(z_n, z_m) \geq C$ and $\sum (1 - |z_n|^2)^s \delta_{z_n}$ is a Carleson measure for $B_p(s)$.

Theorem [Arcozzi, B, Pau '07]

Let $1 < p < \infty,\ 0 < s < 1$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for $B_p(s)$.

(ii) $\inf\limits_{n \neq m} \beta(z_n, z_m) \geq C$ and $\sum (1 - |z_n|^2)^s \delta_{z_n}$ is a Carleson measure for $B_p(s)$.

(iii) $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p(s))$.

Remark

- If $s > 1$ then $\mathcal{M}(B_p(s)) = H^\infty$
- If $s = 1$?

Proof of the main result

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Interp. for $\mathcal{M}(B_p(s)) \Rightarrow$ Separation + Carleson Measure

Separation is trivial

$$\mathcal{M}(B_p(s)) \subset H^\infty$$

To show the Carleson Measure Condition

$$\sum |g(z_n)|^p (1-|z_n|^2)^s \leq C \|g\|^p_{B_p(s)} \quad \text{for all } g \in B_p(s),$$

we use Khinchine's inequality and a Reproducing formula for $B_p(s)$.

Separation + Carleson Measure $\Rightarrow$ Interp. for $\mathcal{M}(B_p(s))$

Non analytic solution

Given $\{w_n\} \in l^\infty$, we can find φ such that

i) $\varphi(z) = w_n$ for $z \in D_h(z_n, \varepsilon)$

ii) $\varphi(z) \equiv 0$ for $z \in \mathbb{D} \setminus \bigcup D_h(z_n, 2\varepsilon)$

iii) $d\mu_\varphi = |\nabla\varphi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is a Carleson measure for $B_p(s)$

Observe that $\varphi(z_n) = w_n$ but is not analytic.

Analytic solution

Consider $f = \varphi - Bu$ where

i) $B(z)$ is the Blaschke product with zeros $\{z_n\}$

ii) $u(z)$ is a solution of the $\overline{\partial}-$problem

$$\overline{\partial} u = \frac{1}{B} \overline{\partial} \varphi$$

We want a solution u such that $f \in \mathcal{M}(B_p(s))$

Now, $f(z_n) = w_n$ and $f \in Hol(\mathbb{D})$.

How to check that $f \in \mathcal{M}(B_p(s))$?

Let L^p_s be the space of functions $f \in L^p(\mathbb{T})$ such that

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\xi})|^p}{|e^{it} - e^{i\xi}|^{2-s}} d\xi dt < \infty$$

Theorem

Let $1 < p < \infty$, $0 < s < 1$, and let $f \in H^\infty(\mathbb{D})$, then

$$f \in \mathcal{M}(B_p(s)) \text{ if and only if } f_{|\mathbb{T}} \in \mathcal{M}(L^p_s).$$

So, it is enough to show that $f = \varphi - Bu \in \mathcal{M}(L^p_s)$

Lemma

Let $\{z_n\}$ be a separated sequence in $\mathbb{D}$ such that $\sum(1-|z_n|^2)^s\delta_{z_n}$ is a Carleson measure for $B_p(s)$, then $B \in \mathcal{M}(L^p_s)$, where B is the Blaschke product with zeros $\{z_n\}$.

Solution of the $\overline{\partial}-$problem

Theorem

Suppose that $|g(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is a Carleson measure for $B_p(s)$ (and $|g(z)|(1-|z|) \leq C$ for $1 < p < 2$). Then there is u defined on $\overline{\mathbb{D}}$ such that

$$\frac{\partial u}{\partial \overline{z}} = g(z) \quad \text{for all } z \in \mathbb{D},$$

and such that the boundary value function u belongs to $\mathcal{M}(L^p_s)$

Open Problems

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Problem 1

It is well known that the Dirichlet space D is conformally invariant. Ie, if $\varphi \in$Möbius map on $\mathbb{D}$, then

$$\int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

If $\{z_n\}$ is an IS for D then $\{\tau(z_n)\}$ is an IS for D?

NO.

K. Seip'04

Perhaps there is a conformally invariant interpolation problem for the Dirichlet space yet to be studied.

Idea

Observe that if $f \in D$, then there exists a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C\beta(z, w)^{1/2} \text{ for all } z, w \in \mathbb{D}.$$

Interpolating Sequence for D

A sequence of points $\{z_n\} \subset \mathbb{D}$ is an interpolating sequence for D if there exists a constant $C > 0$ such that for any $\{w_n\} \subset \mathbb{C}$ with

$$|w_n - w_m| \leq C\beta(z_n, z_m)^{1/2} \quad n, m = 1, 2, ...$$

then there exists a function $f \in D$ with $f(z_n) = w_n$ for $n = 1, 2, ...$

In this case the conformally invariance is for free.

Problem 2

Consider the space D_ρ of analytic functions f such that

$$\|f\|^2_{D_\rho} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \rho(z) dA(z) < \infty,$$

where ρ is a regular weight satisfying the Bekollé-Bonami condition

$$\int_{S(a)} \rho(z) dA(z) \int_{S(a)} \rho^{-1}(z) dA(z) \leq C\, m(S(a))^2.$$

Carleson measures for D_ρ

Geometric description due to Arcozzi, Rochberg and Sawyer'02.

Question

Characterize the interpolating sequences for the Dirichlet type spaces D_ρ.

Problem 3

A Hilbert space H has the Nevanlinna-Pick property when the matrix

$$(1 - w_n \overline{w}_m) < k_{z_i}, k_{z_j} >$$

being positive semi-definite is necessary and sufficient for the existence of $\varphi \in M_H$ satisfying $\varphi(z_n) = w_n$, $\|\varphi\|_{M_H} \leq 1$.

Conjecture (Seip)

Let H be a Hilbert space of analytic functions with the Pick property, then a sequence of points $\{z_n\}$ is an IS if and only if $\{z_n\}$ is $H-$separated and $\sum_n \|k_{z_n}\|_H^{-2} \delta_{z_n}$ is a Carleson measure for H.

Theorem (Böe'05)

Under some assumptions on H, the conjecture is true.

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