Interpolating sequences on analytic Besov type spaces

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Outline

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Motivation

Let $\mathcal{B}$ denote the set of analytic functions from the unit disc $\mathbb{D}$ to $\overline{\mathbb{D}}$.

Question

Given $\{z_1, \ldots, z_N\} \subset \mathbb{D}$, for which $\{w_1, \ldots, w_N\}$ the interpolation

$$f(z_n) = w_n, \quad n = 1, 2, \ldots, n,$$

has a solution $f \in \mathcal{B}$?

Theorem Pick'17

There exists $f \in \mathcal{B}$ satisfying (1) if and only if the quadratic form

$$Q_n(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w}_k}{1 - z_j \overline{z}_k} t_j t_k$$

is nonnegative, $Q_n \geq 0$. When $Q_n \geq 0$ there is a Blaschke product of degree at most $n$ which solves (1).
Motivation

$H^\infty \equiv$ bounded analytic functions in $\mathbb{D}$

Definition

$\{z_n\}$ is an interpolating sequence for $H^\infty$ if for any sequence $\{w_n\} \in \ell^\infty$, the interpolation problem

$$f(z_n) = w_n, \quad n = 1, 2, \ldots$$

has a solution $f \in H^\infty$.

Theorem [Carleson’58]

The following conditions are equivalent

(a) $\{z_n\}$ is an interpolating sequence for $H^\infty$

(b) $\inf_{n \neq m} \beta(z_n, z_m) > 0$ and $\mu = \sum (1 - |z_n|) \delta_{z_n}$ is a Carleson measure.
Let $H$ be a Hilbert space of functions, and let

$$\langle f, g \rangle$$

be the associated inner product, for $f, g \in H$.

**Claim**

If the point evaluation functional

$$T_z : H \rightarrow \mathbb{C}$$

$$f \rightarrow f(z)$$

is bounded, then there exists a unique function $k_z \in H$ with

$$\langle f, k_z \rangle = f(z) \quad \forall f \in H$$

called reproducing kernel, and it satisfies $\| T_z \| = \| k_z \|_H$. 
Interpolating Sequence

A sequence of unimodular functions $\{u_n\} \subset H$ is an Interpolating Sequence (IS) if the operator

$$
H \longrightarrow l^2
f \longrightarrow \{< f, u_n >\}
$$

is onto.

Interpolating Sequence

A sequence of points $\{z_n\}$ is an Interpolating Sequence for $H$ if $\left\{ \frac{k_{z_n}}{\|k_{z_n}\|} \right\}$ is an Interpolating Sequence. I.e,

$$
H \longrightarrow l^2
f \longrightarrow \left\{ \frac{f(z_n)}{\|k_{z_n}\|} \right\}
$$

is onto.

$\forall \{w_n\} \subset l^2$, there exists $f \in H$ such that $\frac{f(z_n)}{\|k_{z_n}\|} = w_n$, $n = 1, 2, ...$
Let $D$ be the Dirichlet space of analytic functions $f$ with

$$
\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.
$$

**Interpolating Sequence for $D$**

A sequence $\{z_n\} \subset \mathbb{D}$ is an Interpolating Sequence for $D$ if for any $\{w_n\} \subset l^2$ there exists $f \in D$ with $\frac{f(z_n)}{\beta(0,z_n)^{1/2}} = w_n$, for $n = 1, 2, \ldots$

**Theorem (Marshall-Sundberg’90s)**

$\{z_n\} \subset \mathbb{D}$ is an interpolating sequence for $D$ if and only if

1. $\inf_{n \neq m} \beta(z_n, z_m) \geq C \beta(0, z_n)$, for $n, m = 1, 2, \ldots$
2. $\sum \frac{1}{\beta(0,z_n)} \delta_{z_n}$ is a Carleson Measure for $D$. 
The spaces $B_p(s)$

$B_p(s) \equiv$ Analytic functions on $\mathbb{D}$ with

$$
\|f\|_{B_p(s)}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2+s} \, dA(z) < \infty
$$

for $1 < p < \infty$ and $0 \leq s < 1$.

**Special cases**

$p = 2, s = 0$ corresponds to the Dirichlet space $\mathcal{D}$.

$p \neq 2, s = 0$ corresponds to the Besov space $B_p$.

**Questions**

1- What is an interpolating sequence for $B_p(s)$?

2- How we can characterize these sequences?
Carleson measure

A positive measure $\mu$ on $\mathbb{D}$ is a Carleson measure for $B_p(s)$ if

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \|f\|^p_{B_p(s)}$$

whenever $f$ is in $B_p(s)$.

A Geometric Description of Carleson measures for $B_p(s)$ was given by [Arcozzi, Rochberg and Sawyer, 02] and [Stegenga, 80].

Multiplier Space

$$\mathcal{M}(B_p(s)) = \{f \text{ such that } fg \in B_p(s) \text{ whenever } g \in B_p(s)\}$$

$f \in \mathcal{M}(B_p(s))$ if and only if

$$f \in H^\infty, \quad |f'(z)|^p(1 - |z|^2)^{p-2+s}dA(z) \text{ is a CM for } B_p(s)$$
The point evaluation functional $T_z : B_p(s) \rightarrow \mathbb{C}$ yields a bounded linear functional at each point $z \in \mathbb{D}$ with norm

$$\|T_z\| \approx \frac{1}{(1 - |z|^2)^{s/p}}$$

for $s > 0$

$$\|T_z\| \approx \beta(0, z)^{(p-1)/p}$$

for $s = 0$
Interpolating sequences for $B_p(s)$

{$z_n$} is an interpolating sequence for $B_p(s)$ if the map

$$f \mapsto \left\{ \frac{f(z_n)}{\|T_{z_n}\|} \right\}$$

maps $B_p(s)$ onto $\ell^p$.

Interpolating Sequences for $\mathcal{M}(B_p(s))$

{$z_n$} is an interpolating sequence for $\mathcal{M}(B_p(s))$ if the map

$$f \mapsto \{f(z_n)\}$$

transforms the multipliers of $B_p(s)$ onto $\ell^\infty$. 
Main result

The interpolating sequences for $\mathcal{D}$ were simultaneously characterized by Marshall-Sundberg and Bishop.

**Theorem [Böe, ’02]**

Let $1 < p < \infty$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for $B_p$.

(ii) $\inf_{n \neq m} \beta(z_n, z_m) \geq C \beta(z_n, 0)$ and $\sum \frac{1}{\beta(0, z_n)^{p-1}} \delta_{z_n}$ is a Carleson measure for $B_p$.

(iii) $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p)$.

**Theorem [Cohn, ’93]**

Let $1 < p < \infty$, $0 < s$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for $B_p(s)$.

(ii) $\inf_{n \neq m} \beta(z_n, z_m) \geq C$ and $\sum (1 - |z_n|^2)^s \delta_{z_n}$ is a Carleson measure for $B_p(s)$.
Theorem [Arcozzi, B, Pau ’07]

Let $1 < p < \infty$, $0 < s < 1$. The following conditions are equivalent

(i) $\{z_n\}$ is an interpolating sequence for $B_p(s)$.

(ii) $\inf_{n \neq m} \beta(z_n, z_m) \geq C$ and $\sum (1 - |z_n|^2)^s \delta_{z_n}$ is a Carleson measure for $B_p(s)$.

(iii) $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p(s))$.

Remark

- If $s > 1$ then $\mathcal{M}(B_p(s)) = H^\infty$
- If $s = 1$?
Proof of the main result

**Interp. for** $\mathcal{M}(B_p(s)) \Rightarrow \text{Separation} + \text{Carleson Measure}$$

Separation is trivial

$$\mathcal{M}(B_p(s)) \subset H^\infty$$

To show the **Carleson Measure** Condition

$$\sum |g(z_n)|^p (1 - |z_n|^2)^s \leq C\|g\|_{B_p(s)}^p$$

for all $g \in B_p(s)$, we use Khinchine’s inequality and a Reproducing formula for $B_p(s)$. 
**Proof of the main result**

**Separation + Carleson Measure ⇒ Interp. for \( \mathcal{M}(B_p(s)) \)**

**Non analytic solution**

Given \( \{w_n\} \in l^\infty \), we can find \( \varphi \) such that

i) \( \varphi(z) = w_n \) for \( z \in D_h(z_n, \varepsilon) \)

ii) \( \varphi(z) \equiv 0 \) for \( z \in \mathbb{D} \setminus \bigcup D_h(z_n, 2\varepsilon) \)

iii) \( d\mu_\varphi = |\nabla \varphi(z)|^p (1 - |z|^2)^{p-2+s} dA(z) \) is a Carleson measure for \( B_p(s) \)

Observe that \( \varphi(z_n) = w_n \) but is not analytic.
Analytic solution

Consider $f = \varphi - Bu$ where

i) $B(z)$ is the Blaschke product with zeros $\{z_n\}$

ii) $u(z)$ is a solution of the $\overline{\partial}$–problem

$$\overline{\partial} u = \frac{1}{B} \overline{\partial} \varphi$$

We want a solution $u$ such that $f \in \mathcal{M}(B_p(s))$

Now, $f(z_n) = w_n$ and $f \in Hol(D)$. 
How to check that \( f \in \mathcal{M}(B_p(s)) \)?

Let \( L^p_s \) be the space of functions \( f \in L^p(\mathbb{T}) \) such that

\[
\int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\xi})|^p}{|e^{it} - e^{i\xi}|^{2-s}} d\xi dt < \infty
\]

**Theorem**

Let \( 1 < p < \infty \), \( 0 < s < 1 \), and let \( f \in H^\infty(\mathbb{D}) \), then

\[
f \in \mathcal{M}(B_p(s)) \text{ if and only if } f|_{\mathbb{T}} \in \mathcal{M}(L^p_s).
\]

So, it is enough to show that \( f = \varphi - Bu \in \mathcal{M}(L^p_s) \).
Lemma

Let \( \{z_n\} \) be a separated sequence in \( \mathbb{D} \) such that \( \sum (1 - |z_n|^2)^s \delta_{z_n} \) is a Carleson measure for \( B_p(s) \), then \( B \in \mathcal{M}(L^p_s) \), where \( B \) is the Blaschke product with zeros \( \{z_n\} \).

Solution of the \( \overline{\partial} \)-problem

Theorem

Suppose that \( |g(z)|^p (1 - |z|^2)^{p-2+s} dA(z) \) is a Carleson measure for \( B_p(s) \) (and \( |g(z)|(1 - |z|) \leq C \) for \( 1 < p < 2 \)). Then there is \( u \) defined on \( \overline{\mathbb{D}} \) such that

\[
\frac{\partial u}{\partial z} = g(z) \quad \text{for all } z \in \mathbb{D},
\]

and such that the boundary value function \( u \) belongs to \( \mathcal{M}(L^p_s) \).
Open Problems

**Problem 1**

It is well known that the Dirichlet space $D$ is conformally invariant. I.e, if $\varphi \in \text{M"obius map on } \mathbb{D}$, then

$$\int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

If $\{z_n\}$ is an IS for $D$ then $\{\tau(z_n)\}$ is an IS for $D$?

**NO.**

K. Seip’04

Perhaps there is a conformally invariant interpolation problem for the Dirichlet space yet to be studied.
**Idea**

Observe that if \( f \in D \), then there exists a constant \( C > 0 \) such that

\[
|f(z) - f(w)| \leq C \beta(z, w)^{1/2} \quad \text{for all } z, w \in \mathbb{D}.
\]

**Interpolating Sequence for \( D \)**

A sequence of points \( \{z_n\} \subset \mathbb{D} \) is an interpolating sequence for \( D \) if there exists a constant \( C > 0 \) such that for any \( \{w_n\} \subset \mathbb{C} \) with

\[
|w_n - w_m| \leq C \beta(z_n, z_m)^{1/2} \quad n, m = 1, 2, ...
\]

then there exists a function \( f \in D \) with \( f(z_n) = w_n \) for \( n = 1, 2, .... \)

In this case the conformally invariance is for free.
**Problem 2**

Consider the space \( D_\rho \) of analytic functions \( f \) such that

\[
\|f\|_{D_\rho}^2 = |f(0)|^2 + \int_\mathbb{D} |f'(z)|^2 \rho(z) dA(z) < \infty,
\]

where \( \rho \) is a regular weight satisfying the Bekollé-Bonami condition

\[
\int_{S(a)} \rho(z) dA(z) \int_{S(a)} \rho^{-1}(z) dA(z) \leq C m(S(a))^2.
\]

**Carleson measures for \( D_\rho \)**

Geometric description due to Arcozzi, Rochberg and Sawyer’02.

**Question**

Characterize the interpolating sequences for the Dirichlet type spaces \( D_\rho \).
**Problem 3**

A Hilbert space $H$ has the Nevanlinna-Pick property when the matrix

$$(1 - w_n \overline{w}_m) < k_{z_i, k_{z_j}}$$

being positive semi-definite is necessary and sufficient for the existence of $\varphi \in M_H$ satisfying $\varphi(z_n) = w_n$, $\|\varphi\|_{M_H} \leq 1$.

**Conjecture (Seip)**

Let $H$ be a Hilbert space of analytic functions with the Pick property, then a sequence of points $\{z_n\}$ is an IS if and only if $\{z_n\}$ is $H$–separated and $\sum_n \|k_{z_n}\|_H^{-2} \delta_{z_n}$ is a Carleson measure for $H$.

**Theorem (Böe’05)**

Under some assumptions on $H$, the conjecture is true.
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