Lecture 20

Risk theory and premium principles

Introduction

In risk theory one studies the probability that an insurance company does not have enough money to cover the claims.

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We will mainly study the most basic model, but will also give some directions of extensions.

The basic risk theory model

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Using these we construct the surplus process:

$$X_t = x + pt - \sum_{i=1}^{N_t} Y_i.$$

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denote the probability of being ruined before time T, and let

$$\psi(x) = P\left(\inf_{t \ge 0} X_t < 0\right)$$

denote the probability of ultimate ruin.

The Cramér-Lundberg model (1)

When we add the assumptions

- The Y_i's are independent and identically distributed with mean $\mu > 0$

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Goal

Calculate $\psi(x)$ in this model.

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Note that p is the income per time unit for the insurance company, and $\lambda\mu$ is the cost per time unit for the company; it follows that $p - \lambda\mu$ is the net profit.

The Cramér-Lundberg model (4)

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The idea is to study the stochastic process X_t under the interval $[0, \min(T_1, h)]$, where T_1 is the first jump time of the Poisson process and $h > 0$.

The Cramér-Lundberg model (5)

The key observaton is that on $[0, \min(T_1, h)]$

$$\begin{aligned}\psi_c(x) &= \psi_c(x+ph)\cdot e^{-\lambda h} \\ &\quad + \int_0^h \left[\int_0^{x+ph} \psi_c(x+pt-y)\frac{1}{\mu}e^{-y/\mu}dy\right]\lambda e^{-\lambda t}dt\end{aligned}$$

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- The first term represents no jumps on $[0, h]$.
- The second term represents the fact that the first jump will not ruin us.

The Cramér-Lundberg model (6)

We can rewrite this equation as

$$p\frac{\psi_c(x+ph)-\psi_c(x)}{ph} = \frac{1-e^{-\lambda h}}{h}\psi_c(x+ph) - \frac{\lambda}{\mu}\cdot\frac{1}{h}\int_0^h\left[\int_0^{x+pt}\psi_c(x+pt-y)e^{-y/\mu}dye^{-\lambda t}\right]$$

(We have reshuffled and then divided by h.)

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Now let $h \downarrow 0$.

The Cramér-Lundberg model (7)

This yields

$$p\psi_c'(x) = \lambda\psi_c(x) - \frac{\lambda}{\mu}\int_0^x \psi_c(x-y)e^{-y/\mu}dy.$$

Here we used that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^h f(t)dt = f(0)$$

if f is nice.

The Cramér-Lundberg model (8)

In order to solve the equation for $\psi_c(x)$ we start by rewriting the integral.

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The equation for $\psi_c(x)$ now becomes

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Differenting with respect to x yields (after simplification!)

$$p\psi_c''(x) = \left(\lambda - \frac{p}{\mu}\right)\psi_c'(x).$$

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This is a first-order ODE in $\psi_c'(x)$,

$$[\psi_c'(x)]' + \left(\frac{1}{\mu} - \frac{\lambda}{p}\right)\psi_c'(x) = 0.$$

The Cramér-Lundberg model (9)

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Integrating once more yields

$$\psi_c(x) = B_1 + B_2 e^{-(1/\mu - \lambda/p)x},$$

where B_1, B_2 are two constants.

The Cramér-Lundberg model (10)

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Hence

$$R := \frac{1}{\mu} - \frac{\lambda}{p} > 0.$$

This constant is called the Lundberg exponent.

The Cramér-Lundberg model (11)

Now

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Using $\psi(x) = 1 - \psi_c(x)$ we get

$$\psi(x) = 1 - \left(1 + B_2 e^{-Rx}\right) = -B_2 e^{-Rx}.$$

The Cramér-Lundberg model (12)

Note that

$$\psi(0) = -B_2 \quad \Leftrightarrow \quad B_2 = -\psi(0),$$

so we can write

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Using the integro-differential equation for $\psi_c(x)$ one can show that

$$\psi_c(0) = 1 - \frac{\lambda}{\mu p}.$$

It follows that

$$\psi(0) = 1 - \psi_c(0) = \frac{\lambda}{\mu p}.$$

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Theorem

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is satisfied, then the ultimate ruin probability is given by

$$\psi(x) = \begin{cases} 1 & \text{when } x < 0 \\ \frac{\lambda}{\mu p} e^{-Rx} & \text{when } x \geq 0, \end{cases}$$

where the Lundberg exponent R is given by

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If the NPC is not satsified, then

$$\psi(x) = 1 \textit{ for every } x \in \mathbb{R}.$$

Extensions of the Cramér-Lundberg model

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- Use another claim size distribution than the exponential distribution. In order to model the possibility of extreme claims, a subexponential distribution can be used. In this case the Lundberg exponent does not exist.
- Use a more general claim arrival process than the Poisson process. If a renewal process is used, then the model is often referred to as the Sparre Andersen model.

Premium principles (1)

The premium rate p is set by the insurance company. We know from the NPC that in the Cramér-Lundberg model we must have

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But the question is how large should p be, and how should we set it?

Premium principles (2)

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We will also normalise and set $\lambda = 1$. This implies that we need to ensure that

$$p > \mu.$$

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- The standard deviation principle

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Assume that the insurance company has utility function U and wealth W. If the insurance company sets the premium p accodring to

$$\underbrace{U(W)}_{\text{utility without claim}} = \underbrace{E[U(W + p - Y)]}_{\text{utility with claim}},$$

then the company is indifferent between not taking the claim Y and taking the claim.

Reference

I have used

- *Lecture notes on Risk theory* by Hanspeter Schmidli

for this lecture.