Lecture 20
Risk theory and premium principles
In risk theory one studies the probability that an insurance company does not have enough money to cover the claims.
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We will mainly study the most basic model, but will also give some directions of extensions.
The basic risk theory model

The following are the basic ingredients in the model:

- The initial capital $x$

The surplus process is constructed as:

$$X_t = x + pt - \sum_{i=1}^{N_t} Y_i$$
The basic risk theory model

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- The **premium rate** $p$
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denote the probability of being ruined before time $T$, and let

$$
\psi(x) = P \left( \inf_{t \geq 0} X_t < 0 \right)
$$

denote the probability of ultimate ruin.
The Cramér-Lundberg model (1)

When we add the assumptions

- The $Y_i$’s are independent and identically distributed with mean $\mu > 0$

The premium rate $p > 0$ then we get the Cramér-Lundberg model.

Goal
Calculate $\psi(x)$ in this model.
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Calculate $\psi(x)$ in this model.
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\[
= x + (p - \lambda \mu) t.
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This is called the **net profit condition (NPC)** and it means that we have a strictly positive drift in the expected value of $X_t$. 

Note that $p$ is the income per time unit for the insurance company, and $\lambda \mu$ is the cost per time unit for the company; it follows that $p - \lambda \mu$ is the net profit.
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The idea is to study the stochastic process $X_t$ under the interval $[0, \min(T_1, h)]$, where $T_1$ is the first jump time of the Poisson process and $h > 0$. 
The Cramér-Lundberg model (5)

The key observation is that on \([0, \min(T_1, h)]\)

\[
\psi_c(x) = \psi_c(x + ph) \cdot e^{-\lambda h}
+ \int_0^h \left[ \int_0^{x + ph} \psi_c(x + pt - y) \frac{1}{\mu} e^{-y/\mu} dy \right] \lambda e^{-\lambda t} dt
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- The first term represents no jumps on \([0, h]\).
- The second term represents the fact that the first jump will not ruin us.
We can rewrite this equation as

$$\frac{\psi_c(x + ph) - \psi_c(x)}{ph} = \frac{1 - e^{-\lambda h}}{h} \psi_c(x + ph)$$

$$-\frac{\lambda}{\mu} \cdot \frac{1}{h} \int_0^h \left[ \int_0^{x+pt} \psi_c(x + pt - y) e^{-y/\mu} dy \right] e^{-\lambda t} dt$$

(We have reshuffled and then divided by $h$.)
The Cramér-Lundberg model (6)

We can rewrite this equation as

\[ p \frac{\psi_c(x + ph) - \psi_c(x)}{ph} = \frac{1 - e^{-\lambda h}}{h} \psi_c(x + ph) \]

\[ -\frac{\lambda}{\mu} \cdot \frac{1}{h} \int_0^h \left[ \int_0^{x+pt} \psi_c(x + pt - y) e^{-y/\mu} dy \right] e^{-\lambda t} dt \]

(We have reshuffled and then divided by \( h \).)

Now let \( h \downarrow 0 \).
This yields

\[ p\psi'_c(x) = \lambda \psi_c(x) - \frac{\lambda}{\mu} \int_0^x \psi_c(x - y)e^{-y/\mu} dy. \]

Here we used that

\[ \lim_{h \downarrow 0} \frac{1}{h} \int_0^h f(t)dt = f(0) \]

if \( f \) is nice.
In order to solve the equation for $\psi_c(x)$ we start by rewriting the integral.

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The Cramér-Lundberg model (8)

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$$= \int_x^0 \psi_c(u) e^{-(x-u)/\mu} (-du)$$

$$= e^{-x/\mu} \int_0^x \psi_c(u) e^{u/\mu} du.$$

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$$p \psi'_c(x) = \lambda \psi_c(x) - \frac{\lambda}{\mu} e^{-x/\mu} \int_0^x \psi_c(u) e^{u/\mu} du.$$
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Differenting with respect to $x$ yields (after simplification!)

$$p\psi''_c(x) = \left( \lambda - \frac{p}{\mu} \right) \psi'_c(x).$$
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Differenting with respect to $x$ yields (after simplification!)

$$p\psi''_c(x) = \left(\lambda - \frac{p}{\mu}\right) \psi'_c(x).$$

This is a first-order ODE in $\psi'_c(x)$,

$$[\psi'_c(x)]' + \left(\frac{1}{\mu} - \frac{\lambda}{p}\right) \psi'_c(x) = 0.$$
The solution is

$$\psi'_c(x) = Ae^{-(1/\mu - \lambda/p)x},$$

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Integrating once more yields

\[ \psi_c(x) = B_1 + B_2 e^{-(1/\mu - \lambda/p)x}, \]

where \( B_1, B_2 \) are two constants.
How can we find $B_1$ and $B_2$?
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Recall the NPC:

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Recall the NPC:

$$p - \lambda \mu > 0 \Leftrightarrow \frac{1}{\mu} - \frac{\lambda}{p} > 0.$$ 

Hence

$$R := \frac{1}{\mu} - \frac{\lambda}{p} > 0.$$ 

This constant is called the Lundberg exponent.
The Cramér-Lundberg model (11)

Now

$$\psi_c(x) \to 1 \text{ as } x \to \infty,$$

i.e. the probability of not going into ruin goes to one as the initial capital goes to infinity.
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\[ 1 = \lim_{x \to \infty} \psi_c(x) = \lim_{x \to \infty} \left( B_1 + B_2 e^{-Rx} \right) = B_1. \]
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$$1 = \lim_{x \to \infty} \psi_c(x) = \lim_{x \to \infty} \left( B_1 + B_2 e^{-Rx} \right) = B_1.$$

Using $\psi(x) = 1 - \psi_c(x)$ we get

$$\psi(x) = 1 - \left( 1 + B_2 e^{-Rx} \right) = -B_2 e^{-Rx}.$$
The Cramér-Lundberg model (12)

Note that

\[ \psi(0) = -B_2 \iff B_2 = -\psi(0), \]

so we can write

\[ \psi(x) = \psi(0)e^{-Rx}. \]
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This is the qualitative behavior of the ultimate ruin probability in the Cramér-Lundberg model.

Using the integro-differential equation for \( \psi_c(x) \) one can show that
\[ \psi_c(0) = 1 - \frac{\lambda}{\mu p}. \]

It follows that
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Theorem

In the Cramér-Lundberg model, if the NPC

\[ p > \lambda \mu \]

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is satisfied, then the ultimate ruin probability is given by

\[ \psi(x) = \begin{cases} 
1 & \text{when } x < 0 \\
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\end{cases} \]

where the Lundberg exponent \( R \) is given by

\[ R = \frac{1}{\mu} - \frac{\lambda}{p}. \]
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**Theorem**

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is satisfied, then the ultimate ruin probability is given by

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If the NPC is not satisfied, then

\[ \psi(x) = 1 \text{ for every } x \in \mathbb{R}. \]
Extensions of the Cramér-Lundberg model

- Use another claim size distribution than the exponential distribution. In order to model the possibility of extreme claims, a subexponential distribution can be used. In this case the Lundberg exponent does not exist.
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- Use another claim size distribution than the exponential distribution. In order to model the possibility of extreme claims, a subexponential distribution can be used. In this case the Lundberg exponent does not exist.

- Use a more general claim arrival process than the Poisson process. If a renewal process is used, then the model is often referred to as the Sparre Andersen model.
The premium rate $p$ is set by the insurance company. We know from the NPC that in the Cramér-Lundberg model we must have

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in order to rule out $\psi(x) = 1$. 
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in order to rule out $\psi(x) = 1$.

But the question is how large should $p$ be, and how should we set it?
In some way the premium $p$ must reflect the claim’s properties.
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Here we will consider a general claim size distribution (i.e. not confine us to only the exponential distribution). We let $\sigma$ denote the standard deviation of any of the $Y_i$’s.
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We will also normalise and set $\lambda = 1$. This implies that we need to ensure that

$$p > \mu.$$
Premium principles (3)

The following are three classical examples of premium principles. Here \( \theta > 0 \) is a given constant.

\[
\text{The expected value principle} \\
p = (1 + \theta) \mu
\]

In this case \( \theta > 0 \) is known as the safety loading.

\[
\text{The variance principle} \\
p = \mu + \theta \sigma^2
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Assume that the insurance company has utility function $U$ and wealth $W$. If the insurance company sets the premium $p$ according to

$$U(W)$$

utility without claim

utility with claim,

then the company is indifferent between not taking the claim $Y$ and taking the claim.
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Assume that the insurance company has utility function $U$ and wealth $W$. If the insurance company sets the premium $p$ according to

$$U(W) = E[U(W + p - Y)],$$

utility without claim

utility with claim

then the company is indifferent between not taking the claim $Y$ and taking the claim.
I have used

- *Lecture notes on Risk theory* by Hanspeter Schmidli

for this lecture.