Lecture 10

More on mean-variance analysis

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We assume that the covariance matrix $V = [\sigma_{ij}]$ is invertable and that not all the expected returns are equal.

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The Markowitz model (2)

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Fix a goal level of expected rate of return and find the portfolio with the minimum variance having this expected rate of return.

The Markowitz model (3)

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Here $\bar{r}$ is the wanted level of expected rate of return.

The factor $\frac{1}{2}$ is a scaling in order to get nicer formulas.

The Markowitz model (4)

To solve this problem we use Lagrange multipliers to form the Lagrangian L:

$$L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right).$$

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Here λ and μ are the Lagrange multipliers for the first and second constraint respectively.

The first order conditions are

$$\frac{\partial L}{\partial w_i} = 0,\ \ i = 1, 2, \ldots, n, \quad \frac{\partial L}{\partial \lambda} = 0 \text{ and } \frac{\partial L}{\partial \mu} = 0.$$

The Markowitz model (5)

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$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu \quad = \quad 0, \;\; i = 1, 2, \ldots, n$$

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$$\begin{aligned}\frac{\partial L}{\partial w_i} = \sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu &= 0, \ \ i = 1, 2, \dots, n \\ \frac{\partial L}{\partial \lambda} = -\sum_{i=1}^{n} w_i \bar{r}_i + \bar{r} &= 0\end{aligned}$$

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These equations determine the optimal vector **w** of portfolio weights.

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The optimal weights can now be found from the first equation:

$$\mathbf{w} = V^{-1}\left(\lambda\bar{\mathbf{r}} + \mu\mathbf{1}\right) = \lambda V^{-1}\bar{\mathbf{r}} + \mu V^{-1}\mathbf{1}.$$

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This is a 2-dimensional linear system of equations:

$$\begin{bmatrix} \bar{\mathbf{r}}^T V^{-1}\bar{\mathbf{r}} & \bar{\mathbf{r}}^T V^{-1}\mathbf{1} \\ \mathbf{1}^T V^{-1}\bar{\mathbf{r}} & \mathbf{1}^T V^{-1}\mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \bar{r} \\ 1 \end{bmatrix}$$

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Let

$$a = \bar{\mathbf{r}}^T V^{-1} \bar{\mathbf{r}}, \;\; b = \bar{\mathbf{r}}^T V^{-1} \mathbf{1} \text{ and } c = \mathbf{1}^T V^{-1} \mathbf{1}.$$

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with solution

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}\begin{bmatrix} \bar{r} \\ 1 \end{bmatrix} = \frac{1}{ac-b^2}\begin{bmatrix} c\bar{r}-b \\ a-b\bar{r} \end{bmatrix}.$$

Here we have used that

$$\mathbf{1}^T V^{-1}\bar{\mathbf{r}} = \bar{\mathbf{r}}^T V^{-1}\mathbf{1}.$$

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$\bar{\mathbf{r}}$ is the vector of expected returns of the basic assets.

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As in the two-asset case we can insert the optimal weights in the expression for the standard deviation:

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&= \sqrt{\frac{a - 2b\bar{r} + c\bar{r}^2}{ac - b^2}}
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Again we see that

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The curve called the minimum-variance set and it is symmetric around the minimum-variance point.

The Markowitz model (12)

Since we want to maximise the expected return while minimising the standard deviation, it is never optimal to hold a portfolio on the part of the parabola below the minimum-variance point.

Since we want to maximise the expected return while minimising the standard deviation, it is never optimal to hold a portfolio on the part of the parabola below the minimum-variance point.

Hence, we only hold portoflios on the upper part of the minimum-variance set. This part is known as the efficient frontier.

Portfolio constraints

Recall the n-asset Markowitz problem:

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \end{aligned}$$

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Here we allow the portfolio weight vector w to take on any real value, but somtimes we want to restrict the allows portfolio weights. We do this by adding more constraints to the problem above.

- If we demand $w_i \geq 0$, $i = 1, 2, \ldots, n$, then we do not allow short-selling.
- If we demand $w_i \in [l_i, u_i]$, $i = 1, 2, \ldots, n$, then we require the portfolio weight w_i to lie between the lower and upper boundary l_i and u_i respectively.

The two-fund theorem (1)

Let us return to the first order conditions

$$\begin{aligned}
\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu &= 0, \; i = 1, 2, \ldots, n \\
\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} &= 0 \\
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Assume that we have solved the problem for two different expected rate of return levels $\bar{r}^1$ and $\bar{r}^2$, and let the optimal weights be denoted $\mathbf{w}^1$ and $\mathbf{w}^2$ respectively.

The two-fund theorem (2)

Now let $\bar{r}$ be any expected rate of return level. Then there exists a unique number α such that

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$$\alpha = \frac{\bar{r} - \bar{r}^2}{\bar{r}^1 - \bar{r}^2}.$$

By using the first order conditions above, we can check that

$$\mathbf{w} = \alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$$

is the optimal portfolio weight vector when the expected rate of return level is $\bar{r}$.

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We can now formulate this as follows.

Theorem

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Any portfolio on the minimum-variance set can be written as a linear combination of two fixed minimum-variance optimal portfolios.

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Any portfolio on the minimum-variance set can be written as a linear combination of two fixed minimum-variance optimal portfolios.

Any portfolio on the efficient frontier can be written as a linear combination of two fixed efficient portfolios.

Introducing a risk-free asset (1)

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By risk-free we mean an asset whose return has standard deviation $\sigma_f = 0$.

Introducing a risk-free asset (1)

An important version of the Markowitz model is when we assume that there exists a risk-free asset with rate of return r_f.

By risk-free we mean an asset whose return has standard deviation $\sigma_f = 0$.

Since the fact that $\sigma_f = 0$ implies that the covariance matrix with the risk-free included is non-invertable, we can not use the same analysis as above.

Introducing a risk-free asset (2)

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The mean rate of return is given by

$$E\left(\sum_{i=1}^{n} w_i r_i + w_0 r_f\right) = \sum_{i=1}^{n} w_i \bar{r}_i + w_0 r_f$$

and the variance by

$$\mathrm{Var}\left(\sum_{i=1}^{n} w_i r_i + w_0 r_f\right) = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}.$$

Introducing a risk-free asset (3)

The optimization problem is

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} w_i \bar{r}_i + w_0 r_f = \bar{r} \\ & \textstyle\sum_{i=1}^{n} w_i + w_0 = 1 \end{aligned}$$

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Now we use the fact that

$$\sum_{i=1}^{n} w_i + w_0 = 1 \quad \Leftrightarrow \quad w_0 = 1 - \sum_{i=1}^{n} w_i$$

and replace w_0 with this in the expression for the expected value.

Introducing a risk-free asset (4)

We then arrive at the problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i(\bar{r}_i - r_f) = \bar{r} - r_f. \end{aligned}$$

Solving this problem using Lagrange multipliers yields the optimal weights in the risky assets

$$\mathbf{w} = \frac{(\bar{r} - r_f)V^{-1}(\bar{\mathbf{r}} - r_f\mathbf{1})}{(\bar{\mathbf{r}} - r_f\mathbf{1})V^{-1}(\bar{\mathbf{r}} - r_f\mathbf{1})}.$$

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$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} w_i (\bar{r}_i - r_f) = \bar{r} - r_f . \end{aligned}$$

Solving this problem using Lagrange multipliers yields the optimal weights in the risky assets

$$\mathbf{w} = \frac{(\bar{r} - r_f) V^{-1} (\bar{\mathbf{r}} - r_f \mathbf{1})}{(\bar{\mathbf{r}} - r_f \mathbf{1}) V^{-1} (\bar{\mathbf{r}} - r_f \mathbf{1})}.$$

The standard deviation is given by

$$\sigma(\bar{r}) = \frac{|\bar{r} - r_f|}{\sqrt{(\bar{\mathbf{r}} - r_f \mathbf{1}) V^{-1} (\bar{\mathbf{r}} - r_f \mathbf{1})}}.$$

Introducing a risk-free asset (5)

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Remark

This conclusion needs that $r_f < \bar{r}_{\text{mvp}}$.

The one-fund theorem

When we have a risk-free asset, it is enough to have one non-risk-free asset to span the minimu-variance set and efficient frontier respectively.

Theorem

(The one-fund theorem)

Any portfolio on the minimum-variance set can be written as a linear combination of one non-risk-free fixed minimum-variance optimal portfolio and the risk-free asset.

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The tangent portfolio

An important portfolio when we we have a risk-free asset is the tangent portfolio.

This is the portfolio that has 100% in risky assets, i.e. its weights fulfills

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By inserting this condition in the general expression for optimal weights we get

$$\mathbf{w}_{\tan} = \frac{V^{-1}(\bar{\mathbf{r}} - r_f \mathbf{1})}{\mathbf{1}^T V^{-1}(\bar{\mathbf{r}} - r_f \mathbf{1})}.$$