The multidimensional rational covariance extension problem

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INRIA, Sophia-Antipolis, France

Acknowledgements

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[2] A. Ringh, J. Karlsson, and A. Lindquist. Multidimensional rational covariance extension with applications to spectral estimation and image compression. SIAM Journal on Control and Optimization 54(4), 1950-1982, 2016.

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Outline

- Background
  - Spectral estimation
  - Identification of stochastic linear systems
  - Rational covariance extension
- Multidimensional rational covariance extension
  - A multidimensional trigonometric moment problem
  - Results for exact covariance matching
  - Results for approximate covariance matching
- Example
  - Texture generation by Wiener system identification

Background

Consider a time series $\{x_t \in \mathbb{C}\}_{t\in\mathbb{Z}}$. The interest is to:

- understand the information content of $\{x_t\}_{t\in\mathbb{Z}}$ $\rightsquigarrow$ spectral estimation.
- construct a model for how $\{x_t\}_{t\in\mathbb{Z}}$ was generated $\rightsquigarrow$ linear stochastic systems.

Background

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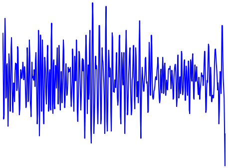
Assumptions on the stochastic process $\{x_t\}_{t\in\mathbb{Z}}$:

- zero-mean: $\mathbb{E}(x_t) = 0$ for $t \in \mathbb{Z}$
- second-order stationery:
  - mean $\mathbb{E}(x_t)$ independent of t
  - covariances $c_k = \mathbb{E}(x_t x_{t-k}^*)$ only depend on the time lag k.
    Note that $c_{-k} = c_k^*$.
- ergodic: time-average is the same as average over probability space.
  "One (sufficiently long) realization is enough to do estimation of statistical properties".

The power spectrum of a stochastic process is the energy distribution across frequencies of the process.



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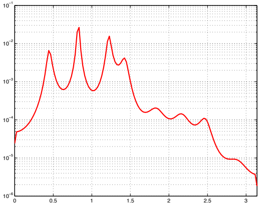

[1] P. Stoica, and R.L. Moses. Spectral analysis of signals. Prentice Hall, 2005.

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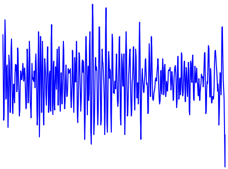

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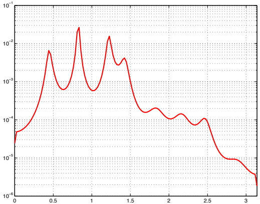

For our process $\{x_t\}_{t\in\mathbb{Z}}$ it is defined as the positive function $\Phi(e^{i\theta})$ on $(-\pi,\pi] \sim \mathbb{T}$,

$$c_k := \frac{1}{2\pi}\int_{\mathbb{T}} e^{ik\theta}\Phi(e^{i\theta})d\theta,\ k\in\mathbb{Z} \quad \Longleftrightarrow \quad \Phi(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta}$$

i.e., with the covariances as Fourier coefficients [1].

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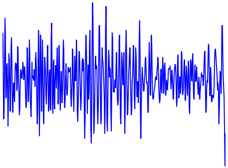
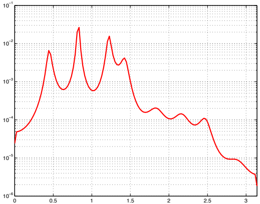


$\Rightarrow$



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Estimation Methods:

- Periodogram
- Burg's method/maximum entropy
- Sparse methods

Applications:

- Radar/Sonar
- Speech analysis
- Communications

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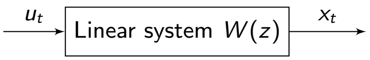

Background
Linear stochastic systems

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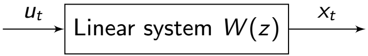

- $\{u_t\}$ is a Gaussian white noise process
  - Gaussian: u_t is Gaussian for all t.
  - White: flat power spectrum $\Phi_u(e^{i\theta}) \equiv 1$.

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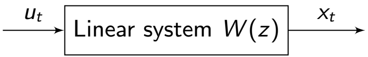

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- $W(z)$ is an autoregressive-moving-average (ARMA) filter

$$x_t + \sum_{k=1}^{n} a_k\, x_{t-k} = \sum_{k=0}^{m} b_k\, u_{t-k} \;\Leftrightarrow\; W(z) = \sum_{k\in\mathbb{Z}} w_k z^{-k} = \frac{\sum_{k=0}^{m} b_k\, z^{-k}}{\sum_{k=0}^{n} a_k\, z^{-k}} = \frac{b(z)}{a(z)}$$

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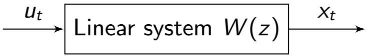

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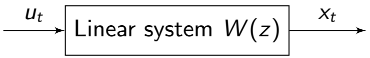
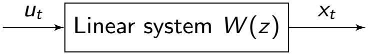
We want to identify $W(z)$ from the data $\{x_t\}$.



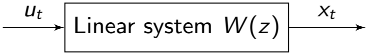
- The knowledge of that $\{x_t\}$ is generated by a linear systems puts constraints on the spectrum.



- The knowledge of that $\{x_t\}$ is generated by a linear systems puts constraints on the spectrum.
- Let $\tilde{c}_k$ be the covariances of u_t. Since $W(z)$ is a linear filter we have $x_t = \sum_{k \in \mathbb{Z}} w_k u_{t-k}$ and thus

$$c_k = \mathbb{E}[x_t x_{t-k}^*] = \mathbb{E}\left[\left(\sum_{k_1 \in \mathbb{Z}} w_{k_1} u_{t-k_1} \right) \left(\sum_{k_2 \in \mathbb{Z}} w_{k_2} u_{t-k-k_2} \right)^* \right]$$

$$= \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} w_{k_1} w_{k_2}^* \mathbb{E}[u_{t-k_1} u_{t-k-k_2}^*] = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} w_{k_1} w_{k_2}^* \tilde{c}_{k+k_2-k_1}$$



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$$
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c_k = \mathbb{E}[x_t x_{t-k}^*] &= \mathbb{E}\left[\left(\sum_{k_1\in\mathbb{Z}} w_{k_1} u_{t-k_1}\right)\left(\sum_{k_2\in\mathbb{Z}} w_{k_2} u_{t-k-k_2}\right)^*\right] \\
&= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} w_{k_1} w_{k_2}^* \mathbb{E}[u_{t-k_1} u_{t-k-k_2}^*] = \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} w_{k_1} w_{k_2}^* \tilde{c}_{k+k_2-k_1}
\end{aligned}
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- Thus

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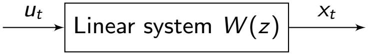
\begin{aligned}
\Phi(e^{i\theta}) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\theta} &= \sum_{k\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} w_{k_1} w_{k_2}^* \tilde{c}_{k+k_2-k_1} e^{-ik\theta} \\
&= \sum_{k_1\in\mathbb{Z}} w_{k_1} e^{-ik_1\theta} \sum_{k_2\in\mathbb{Z}} w_{k_2}^* e^{ik_2\theta} \sum_{k_3\in\mathbb{Z}} \tilde{c}_{k_3} e^{-ik_3\theta} = |W(e^{i\theta})|^2 \Phi_u(e^{i\theta}).
\end{aligned}
$$



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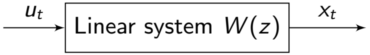
Background

Rational covariance extension problem



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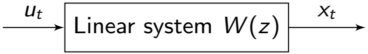
$$\Phi(e^{i\theta}) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\theta} = \sum_{k\in\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} w_{k_1} w_{k_2}^* \tilde{c}_{k+k_2-k_1} e^{-ik\theta}$$

$$= \sum_{k_1\in\mathbb{Z}} w_{k_1} e^{-ik_1\theta} \sum_{k_2\in\mathbb{Z}} w_{k_2}^* e^{ik_2\theta} \sum_{k_3\in\mathbb{Z}} \tilde{c}_{k_3} e^{-ik_3\theta} = |W(e^{i\theta})|^2 \Phi_u(e^{i\theta}).$$

- But u_t is white noise, so $\Phi_u(e^{i\theta}) \equiv 1$. Moreover, $W(z) = b(z)/a(z)$

$$\Phi(e^{i\theta}) = |W(e^{i\theta})|^2 \Phi_u(e^{i\theta}) = |W(e^{i\theta})|^2 = \frac{|b(e^{i\theta})|^2}{|a(e^{i\theta})|^2} = \frac{\sum_{k=-m}^{m} p_k e^{-ik\theta}}{\sum_{k=-n}^{n} q_k e^{-ik\theta}} = \frac{P(e^{i\theta})}{Q(e^{i\theta})}$$

where P and Q are real-valued trigonometric polynomials ($p_{-k} = p_k^*$, $q_{-k} = q_k^*$).



- Summarizing, this gives us the following problem posed by Kalman [1]:

Problem formulation – Rational covariance extension problem

Given a sequence of covariances $c = \{c_k\}_{k=-n}^{n}$ find a positive function $\Phi(e^{i\theta})$ so that

$$\begin{cases} c_k = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} e^{ik\theta}\Phi(e^{i\theta})d\theta, & k = -n, \dots, 0, 1, \dots, n \\ \Phi(e^{i\theta}) = \dfrac{P(e^{i\theta})}{Q(e^{i\theta})}, & P \text{ and } Q \in \bar{\mathfrak{P}}_+^1. \end{cases}$$

- Notation for nonnegative trigonometric polynomials:

$$\bar{\mathfrak{P}}_+^1 = \{p := \{p_k\}_{k=-n}^{n} \in \mathbb{C}^{2n+1} \mid p_{-k} = p_k^*,\ P(e^{i\theta}) := \sum_{k=-n}^{n} p_k e^{-ik\theta},\ P(e^{i\theta}) \geq 0 \text{ for all } \theta \in \mathbb{T}\}$$

[1] R.E. Kalman. Realization of covariance sequences. In the Proceedings of the Toeplitz Memorial Conference, 1981.

Key property in the identification: spectral factorization

$$\frac{P(e^{i\theta})}{Q(e^{i\theta})} \underset{\text{Trivial}}{\overset{\text{Spectral factorization}}{\underset{\leftarrow}{\overset{\rightarrow}{=}}}} \frac{|b(e^{i\theta})|^2}{|a(e^{i\theta})|^2}$$

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Intermission - Connection to analytic interpolation: equivalently characterized as finding a function $f(z)$ that is positive real and of specified degree, i.e.,

- analytic in $\mathbb{D}^C$, i.e., $f(z) = \sum_{k=0}^{\infty} f_k z^{-k}$ for $|z| > 1$,
- $f(z) + f^*(1/z^*) > 0$ on $\mathbb{T}$ $\quad (f(z) + f^*(1/z^*) = \Phi(z)$ on $\mathbb{T})$,
- $f(z)$ takes the form

$$f(z) = \frac{\tilde{b}(z)}{\tilde{a}(z)},$$

for $\tilde{a}$, $\tilde{b}$ monic and of degree n and m, with roots inside the unit disc (Schur polynomials),

fulfilling $f_0 = \frac{1}{2}c_0$ and $f_k = c_k$ for $k = 1, \ldots, n$. This gives interpolation conditions in infinity.

Rational covariance extension problem - Trigonometric moment problem with rationality constraint

Given a sequence of covariances $c = \{c_k\}_{k=-n}^{n}$ find a positive function $\Phi(e^{i\theta})$ so that

$$\begin{cases} c_k = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} e^{ik\theta}\Phi(e^{i\theta})d\theta, & k = -n, \dots, 0, 1, \dots, n \\ \Phi(e^{i\theta}) = \dfrac{P(e^{i\theta})}{Q(e^{i\theta})}, & P \text{ and } Q \in \bar{\mathfrak{P}}_+^1. \end{cases}$$

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- Solvable using convex optimization (next slide)

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- Solvable using convex optimization (next slide)
- Solution exists if and only if

$$T(c) = \begin{bmatrix} c_0 & c_{-1} & \dots & c_{-n} \\ c_1 & c_0 & \dots & c_{-n+1} \\ \vdots & & \ddots & \vdots \\ c_n & c_{n-1} & \dots & c_0 \end{bmatrix} \succ 0.$$

Theorem ([1])

The rational covariance extension problem has a solution if only if $T(c) \succ 0$*. For such* c *and any* $P \in \mathfrak{P}_+^1$*, there is a unique* $\hat{Q}$ *such that* $\Phi = P/\hat{Q}$ *is a solution to the rational covariance extension problem.*

[1] C.I. Byrnes, S.V. Gusev, and A. Lindquist. A convex optimization approach to the rational covariance extension problem. *SIAM Journal on Control and Optimization* 37(1), 211-229, 1998.

Theorem ([1])

The rational covariance extension problem has a solution if only if $T(c) \succ 0$. *For such* c *and any* $P \in \mathfrak{P}_+^1$, *there is a unique* $\hat{Q}$ *such that* $\Phi = P/\hat{Q}$ *is a solution to the rational covariance extension problem. Moreover,* $\Phi = P/\hat{Q}$ *is the optimal solution to the convex problem*

$$(P) \qquad \min_{\Phi \geq 0} \int_{\mathbb{T}} P \log \frac{P}{\Phi} \frac{d\theta}{2\pi}$$

$$\text{subject to} \quad c_k = \int_{\mathbb{T}} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} \quad k = -n, \ldots, 0, 1, \ldots, n.$$

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Theorem ([1])

The rational covariance extension problem has a solution if only if $T(c) \succ 0$. For such c and any $P \in \mathfrak{P}^1_+$, there is a unique $\hat{Q}$ such that $\Phi = P/\hat{Q}$ is a solution to the rational covariance extension problem. Moreover, $\Phi = P/\hat{Q}$ is the optimal solution to the convex problem

$$(P) \qquad \min_{\Phi \geq 0} \int_{\mathbb{T}} P \log \frac{P}{\Phi} \frac{d\theta}{2\pi}$$

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Finally, this $\hat{Q}$ is the unique solution to the dual problem

$$(D) \qquad \min_{q \in \bar{\mathfrak{P}}_+} \langle c, q \rangle - \int_{\mathbb{T}} P \log(Q) \frac{d\theta}{2\pi}.$$

[1] C.I. Byrnes, S.V. Gusev, and A. Lindquist. A convex optimization approach to the rational covariance extension problem. *SIAM Journal on Control and Optimization* 37(1), 211-229, 1998.

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  - G. Picci, and B. Zhu, 2017.

Trigonometric moment problem with rationality constraint

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Multidimensional trigonometric moment problem with rationality constraint

Given a sequence of covariances $c = \{c_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda}$ find a positive function $\Phi(e^{i\boldsymbol{\theta}})$ so that

$$\begin{cases} c_{\mathbf{k}} = \dfrac{1}{(2\pi)^d}\displaystyle\int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})}\Phi(e^{i\boldsymbol{\theta}})d\boldsymbol{\theta}, & k \in \Lambda \\ \Phi(e^{i\boldsymbol{\theta}}) = \dfrac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}, & P \text{ and } Q \in \bar{\mathfrak{P}}_+. \end{cases}$$

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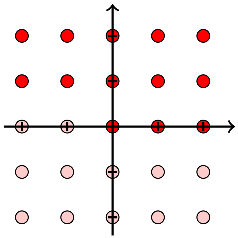
E.g., the rectangular set $\Lambda = \{(k_1, \ldots, k_d) \in \mathbb{Z}^d \ : |k_j| \leq n_j, j = 1, \ldots, d\}$



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- Trigonometric polynomials $\bar{\mathfrak{P}}_+$:

  $$\bar{\mathfrak{P}}_+ = \{p := \{p_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda} \in \mathbb{C}^{|\Lambda|} \mid p_{-\mathbf{k}} = p_{\mathbf{k}}^*,\ P(e^{i\boldsymbol{\theta}}) := \sum_{\mathbf{k}\in\Lambda} p_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})},\ P(e^{i\boldsymbol{\theta}}) \ge 0 \text{ for all } \boldsymbol{\theta} \in \mathbb{T}^d\}$$

  Compare to

  $$\bar{\mathfrak{P}}_+^1 = \{p := \{p_k\}_{k=-n}^n \in \mathbb{C}^{2n+1} \mid p_{-k} = p_k^*,\ P(e^{i\theta}) := \sum_{k=-n}^{n} p_k e^{-ik\theta},\ P(e^{i\theta}) \ge 0 \text{ for all } \theta \in \mathbb{T}\}$$

Important notions for later:

- $\partial\bar{\mathfrak{P}}_+$ denotes the boundary of the set of nonnegative trigonometric polynomials, i.e., $p \in \bar{\mathfrak{P}}_+$ such that $P(e^{i\boldsymbol{\theta}}) = 0$ in at least one point $\boldsymbol{\theta} \in \mathbb{T}^d$.

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- $\bar{\mathfrak{P}}_+$ is a cone: for all $p^1, p^2 \in \bar{\mathfrak{P}}_+$ we have that $\alpha p^1 + \beta p^2 \in \bar{\mathfrak{P}}_+$ for any $\alpha, \beta \geq 0$. To see this

$$\alpha p^1 + \beta p^2 \quad \rightsquigarrow \quad \sum_{\mathbf{k}\in\Lambda} (\alpha p^1_{\mathbf{k}} + \beta p^2_{\mathbf{k}}) e^{-i(\mathbf{k},\boldsymbol{\theta})} = \alpha P^1(e^{i\boldsymbol{\theta}}) + \beta P^2(e^{i\boldsymbol{\theta}}) \geq 0 \quad \forall \boldsymbol{\theta} \in \mathbb{T}^d.$$

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$$\mathfrak{C}_+^1 := \left\{ c \in \mathbb{C}^{2n+1} \mid c_{-k} = c_k^*,\ \langle c, p \rangle := \sum_{k=-n}^{n} c_k p_k^* > 0 \text{ for all } p \in \bar{\mathfrak{P}}_+^1 \setminus \{0\} \right\}$$

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Finally, the theory turns out to not be directly generalizable. Need to allow for solutions that are measures of bounded variation of the form

$$d\mu(\boldsymbol{\theta}) = \Phi(e^{i\boldsymbol{\theta}})dm + d\nu(\boldsymbol{\theta}),$$

where $\Phi = P/Q$ and $d\nu$ is singular with respect to dm (Lebesgue decomposition).

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$$(P) \qquad \min_{d\mu \geq 0} \int_{\mathbb{T}^d} \left(P \log \frac{P}{\Phi} dm + d\mu - P dm \right)$$

$$\text{subject to} \quad c_{\mathbf{k}} = \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \left(\Phi(e^{i\boldsymbol{\theta}}) dm + d\nu \right), \quad \mathbf{k} \in \Lambda.$$

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Finally, for any $c \in \mathfrak{C}_+$ *problem (P) has a solution and it is given by*

$$d\hat{\mu} = \frac{P(e^{i\theta})}{\hat{Q}(e^{i\theta})} dm + d\hat{\nu}$$

where $\hat{Q}$ *is the unique solution to the dual problem*

$$(D) \qquad \min_{q \in \bar{\mathfrak{P}}_+} \langle c, q \rangle - \int_{\mathbb{T}^d} P \log(Q) dm.$$

and $d\hat{\nu}$ *is such that* $\text{supp}(d\hat{\nu}) \subset \{\theta \in \mathbb{T}^d \mid \hat{Q}(e^{i\theta}) = 0\}$.

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$$\mathcal{L}(\Phi, d\nu, q) = -\int_{\mathbb{T}^d} P \log \frac{P}{\Phi} dm + \sum_{\mathbf{k}\in\Lambda} q_{\mathbf{k}}^* \left(c_{\mathbf{k}} - \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \left(\Phi(e^{i\boldsymbol{\theta}})dm + d\nu\right)\right)$$
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  - variation in Φ gives $\Phi = P/Q$ (a.e. dm),
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- Plugging this into the Lagrangian gives the dual problem (D).

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Derivative of the dual objective function $\mathbb{J}(q) = \langle c, q \rangle - \int_{\mathbb{T}^d} P \log(Q) dm$:

$$\frac{\partial \mathbb{J}}{\partial q_{\mathbf{k}}} = c_{\mathbf{k}} - \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \frac{P}{Q} dm.$$

For $d = 1$, and also $d = 2$, this is infinite for $Q \in \partial\bar{\mathfrak{P}}_+$. So the solution is strictly in the interior, i.e., $\hat{Q}(e^{i\boldsymbol{\theta}}) > 0$ for all $\boldsymbol{\theta} \in \mathbb{T}^d$

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Inverse problems view: the primal problem

$$(P) \qquad \begin{aligned} \min_{d\mu \geq 0} \quad & \int_{\mathbb{T}^d} \left(P \log \frac{P}{\Phi} dm + d\mu - P dm \right) \\ \text{subject to} \quad & c_{\mathbf{k}} = \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \left(\Phi(e^{i\boldsymbol{\theta}}) dm + d\nu \right), \quad \mathbf{k} \in \Lambda. \end{aligned}$$

is on the form

$$\begin{aligned} \min_{d\mu \geq 0} \quad & \mathbb{I}(d\mu) \\ \text{subject to} \quad & d\mu \text{ matches } c \text{ exactly,} \end{aligned}$$

where the regularizer $\mathbb{I}$ promotes rational solutions. Change data-matching term.

Theorem

Given a sequence $c = \{c_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda}$, for ε large enough the primal problem

$$(P') \qquad \min_{\Phi>0,\, r} \int_{\mathbb{T}^d} \left(P \log \frac{P}{\Phi} dm + d\mu - P dm \right)$$

$$\text{subject to} \quad r_{\mathbf{k}} = \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \left(\Phi(e^{i\boldsymbol{\theta}}) dm + d\nu \right), \quad \mathbf{k} \in \Lambda,$$

$$\|r - c\|^2 \le \varepsilon^2,$$

has an optimal solution given by

$$d\hat{\mu} = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})} dm + d\hat{\nu}$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D') \qquad \min_{q\in\bar{\mathfrak{P}}_+} \; \langle c, q\rangle - \int_{\mathbb{T}^d} P \log(Q) dm + \varepsilon \|q - e\|,$$

and $d\hat{\nu}$ is such that $\operatorname{supp}(d\hat{\nu}) \subset \{\boldsymbol{\theta} \in \mathbb{T}^d \mid \hat{Q}(e^{i\boldsymbol{\theta}}) = 0\}$. Here, $e \in \mathbb{C}^{|\Lambda|}$, $e_{\mathbf{0}} = 1$ and $e_{\mathbf{k}} = 0$ for $\mathbf{k} \in \Lambda \setminus \{\mathbf{0}\}$.

Outline

- Background
  - Spectral estimation
  - Identification of stochastic linear systems
  - Rational covariance extension
- Multidimensional rational covariance extension
  - A multidimensional trigonometric moment problem
  - Results for exact covariance matching
  - Results for approximate covariance matching
- Example
  - Texture generation by Wiener system identification

- Motivated by the use of thresholded Gaussian random fields to model porous materials [1], we are interested in generating binary textures.

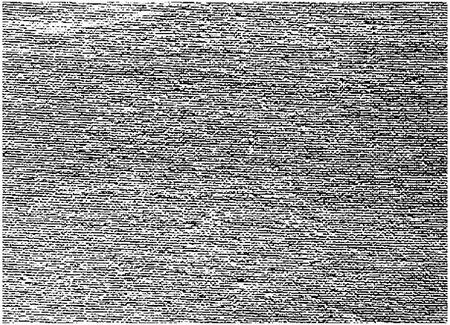

Figure: Example of a texture

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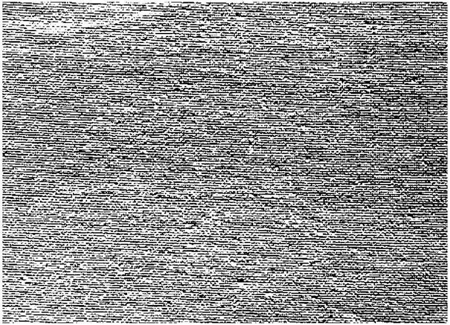
- Motivated by the use of thresholded Gaussian random fields to model porous materials [1], we are interested in generating binary textures.
- We want to estimate a system that can generate similar textures
  $\rightsquigarrow$ multidimensional Wiener systems identification.



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Texture generation by Wiener system identification

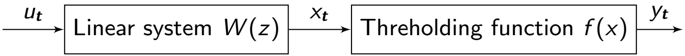

Figure: A Wiener system with thresholding as static nonlinearity.

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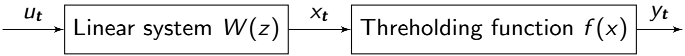

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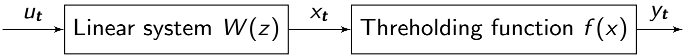

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- $\{u_{\boldsymbol{t}};\ \boldsymbol{t} \in \mathbb{Z}^2\}$ be a zero-mean Gaussian white noise input.
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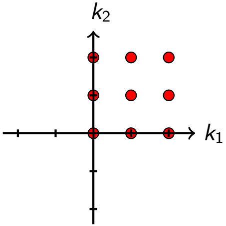
$$x_{\boldsymbol{t}} + \sum_{\mathbf{k} \in \Lambda_+ \setminus \{\mathbf{0}\}} a_{\mathbf{k}}\, x_{\boldsymbol{t}-\mathbf{k}} = \sum_{\mathbf{k} \in \Lambda_+} b_{\mathbf{k}}\, u_{\boldsymbol{t}-\mathbf{k}} \Leftrightarrow W(\mathbf{z}) = \frac{\sum_{\mathbf{k} \in \Lambda_+} b_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}}{\sum_{\mathbf{k} \in \Lambda_+} a_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}} = \frac{b(\mathbf{z})}{a(\mathbf{z})}$$

where $\Lambda_+ \subset \mathbb{Z}^2$ is the support of the filter.



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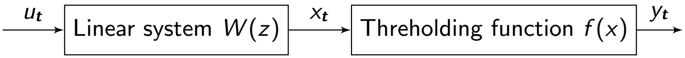

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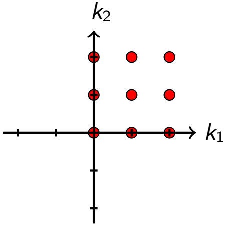
- Thresholding function

$$f(x) = \begin{cases} 1 & x > \tau \\ 0 & \text{otherwise} \end{cases}$$



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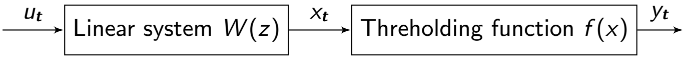


Figure: A Wiener system with thresholding as static nonlinearity.

- $\{u_{\mathbf{t}};\ \mathbf{t} \in \mathbb{Z}^2\}$ be a zero-mean Gaussian white noise input.
- The linear dynamical system is a strictly causal *autoregressive-moving-average* (ARMA) filter

$$x_{\mathbf{t}} + \sum_{\mathbf{k}\in\Lambda_+\setminus\{\mathbf{0}\}} a_{\mathbf{k}}\, x_{\mathbf{t}-\mathbf{k}} = \sum_{\mathbf{k}\in\Lambda_+} b_{\mathbf{k}}\, u_{\mathbf{t}-\mathbf{k}} \Leftrightarrow W(\mathbf{z}) = \frac{\sum_{\mathbf{k}\in\Lambda_+} b_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}}{\sum_{\mathbf{k}\in\Lambda_+} a_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}} = \frac{b(\mathbf{z})}{a(\mathbf{z})}$$

  where $\Lambda_+ \subset \mathbb{Z}^2$ is the support of the filter.

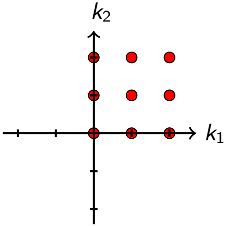


Figure: Example of Λ_+.

- Thresholding function

$$f(x) = \begin{cases} 1 & x > \tau \\ 0 & \text{otherwise} \end{cases}$$

Goal: From samples $(y_{\mathbf{t}})$ we want to identify τ and W.

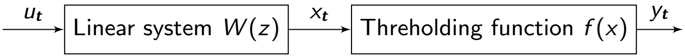


Figure: A Wiener system with thresholding as static nonlinearity.

Example

Texture generation by Wiener system identification

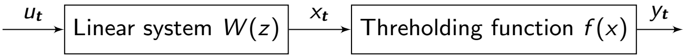
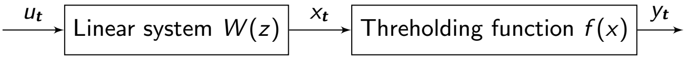


Figure: A Wiener system with thresholding as static nonlinearity.

- Identifying the threshold parameter:
  - Since u_t is zero-mean and Gaussian and $W(\mathbf{z})$ is linear, x_t is zero-mean and Gaussian.

Example

Texture generation by Wiener system identification



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  - In steady-state x_t is second order stationary process, i.e., the covariances are independent of the absolute time $\boldsymbol{t}$: $c_\mathrm{k} := \mathbb{E}[x_t x_{t-\mathrm{k}}]$. Assume that $\mathbb{E}[x_t x_t] = c_0 = 1$ (normalization).

Example

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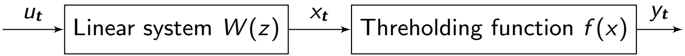

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  - $\mathbb{E}[y_t] = P(y_t = 1) = P(x_t > \tau) = 1 - P(x_t \leq \tau) = 1 - \phi(\tau)$, where ϕ is the Gaussian CDF
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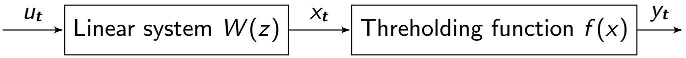

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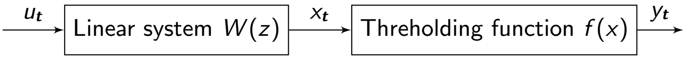
- Estimate covariances $c_\mathbf{k}$:
  - Let $r_\mathbf{k} := \mathbb{E}[y_{t-\mathbf{k}} y_t] - \mathbb{E}[y_{t-\mathbf{k}}]\mathbb{E}[y_t]$ be the covariances of the process y_t.
  - Since x_t is Gaussian a theorem by Price [1] gives the following relationship between the covariances
$$r_\mathbf{k} = \int_0^{c_\mathbf{k}} \frac{1}{2\pi\sqrt{1-s^2}} \exp\left(-\frac{\tau^2}{1+s}\right) ds.$$
    Since the integrand is positive, the mapping can be inverted (numerically).

[1] R. Price. A useful theorem for nonlinear devices having Gaussian inputs. *IRE Transactions on Information Theory*, 4(2), 69-72, 1958.

Example

Texture generation by Wiener system identification



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    Since the integrand is positive, the mapping can be inverted (numerically).
- From the covariances c_k, estimate the linear system $W(\mathbf{z})$.

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Example

Texture generation by Wiener system identification

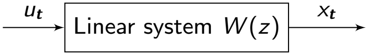
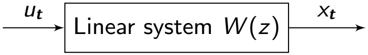


Figure: A linear system.

Estimating the linear system $W(\mathbf{z})$ from from the covariances $c_{\mathbf{k}}$ - looks promising for applying the theory for multidimensional rational covariance extension!

Example
Texture generation by Wiener system identification



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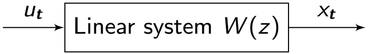
Estimating the linear system $W(\mathbf{z})$ from from the covariances $c_{\mathbf{k}}$ - looks promising for applying the theory for multidimensional rational covariance extension!

- The power spectral density $\Phi(e^{i\boldsymbol{\theta}})$ of a stochastic process $\{x_{\boldsymbol{t}};\ \boldsymbol{t} \in \mathbb{Z}^2\}$ is defined as the nonnegative function such that

$$c_{\mathbf{k}} := \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})}\Phi(e^{i\boldsymbol{\theta}})d\boldsymbol{\theta}, \quad \mathbf{k} \in \mathbb{Z}^2 \qquad \Longleftrightarrow \qquad \Phi(e^{i\boldsymbol{\theta}}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}.$$

Example

Texture generation by Wiener system identification



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- By similar calculations as in the one-dimensional case

$$\Phi(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2 \Phi_u(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2 = \frac{|b(e^{i\boldsymbol{\theta}})|^2}{|a(e^{i\boldsymbol{\theta}})|^2} = \frac{\sum_{\mathbf{k}\in\Lambda} p_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}}{\sum_{\mathbf{k}\in\Lambda} q_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}} = \frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}.$$

where P and Q are trigonometric polynomials, and $\Lambda = \Lambda_+ - \Lambda_+$ (Minkowski set difference).

- Pointed out earlier: in one dimension, spectral factorization as a sum-of-one-square:

$$\frac{P(e^{i\theta})}{Q(e^{i\theta})} \underset{\text{Trivial}}{\overset{\text{Spectral factorization}}{\overset{\rightarrow}{\underset{\leftarrow}{=}}}} \frac{|b(e^{i\theta})|^2}{|a(e^{i\theta})|^2}$$

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- Not true in the higher dimensions - only factorization as sum-of-several-squares can be guaranteed [1, 2]:

$$\frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})} = \frac{\sum_{k=1}^{\ell} |b_k(e^{i\boldsymbol{\theta}})|^2}{\sum_{k=1}^{m} |a_k(e^{i\boldsymbol{\theta}})|^2}.$$

- **Open question**: how to construct a realization from such a spectrum?

[1] M.A. Dritschel. On factorization of trigonometric polynomials. *Integral Equations and Operator Theory*, 49(1), 11-42, 2004.

[2] J.S. Geronimo, and M.J. Lai. Factorization of multivariate positive Laurent polynomials. *Journal of Approximation Theory*, 139(1-2), 327-345, 2006.

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- **Open question**: how to construct a realization from such a spectrum?
- We resort to a heuristic, obtained by “abusing” results in [3].

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[3] J.S. Geronimo, and H.J. Woerdeman. Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables. *Annals of Mathematics*, 839-906, 2004.

Example
Texture generation by Wiener system identification

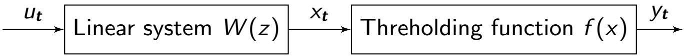


Figure: A Wiener system with thresholding as static nonlinearity.

Algorithm for Wiener system identification with thresholding

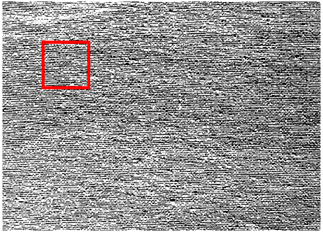
Input: (y_t)

1: Estimate threshold parameter: $\tau_{\text{est}} = \phi^{-1}(1 - E[y_t])$ from the data.
2: Estimate covariances: $r_k := E[y_{t-k}y_t] - E[y_{t-k}]E[y_t]$ from the data.
3: Compute covariances $c_k := E[x_{t-k}x_t]$ by using the relation $r_k = \int_0^{c_k} \frac{1}{2\pi\sqrt{1-s^2}} \exp\left(-\frac{\tau^2}{1+s}\right) ds$
4: Estimate a rational spectrum using the theory developed here.
5: Apply a *heuristic*, approximate factorization procedure.

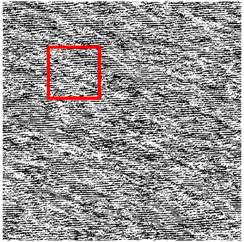
Output: τ_{est}, coefficients for the linear dynamical system

Example

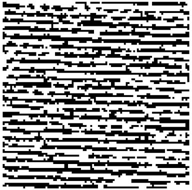
Texture generation by Wiener system identification



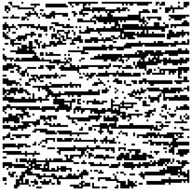
(a) Texture.



(b) Reconstruction.

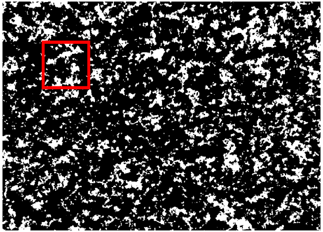


(c) Close-up of the texture.

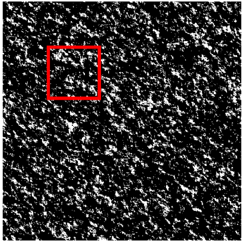


(d) Close-up of the reconstruction.

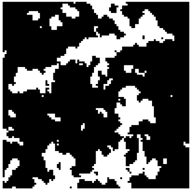
Example
Texture generation by Wiener system identification



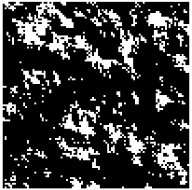
(e) Texture.



(f) Reconstruction.



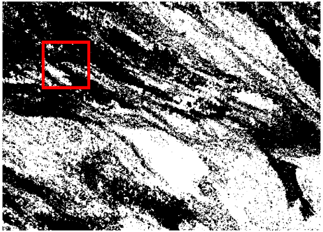
(g) Close-up of the texture.



(h) Close-up of the reconstruction.

Example

Texture generation by Wiener system identification



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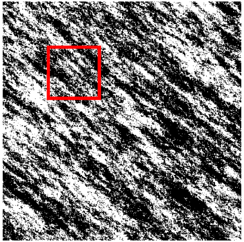
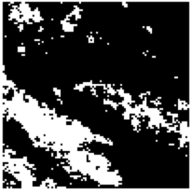
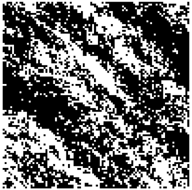


(j) Reconstruction.



(k) Close-up of the texture.



(l) Close-up of the reconstruction.

Conclusion and future work

Conclusions:

- Rational covariance extension - motivated from identification of a linear stochastic system
- Trigonometric moment problem view and convex optimization problem generalized to multidimensional problems
  - We can do estimation of rational multidimensional spectra + impulses using convex optimization
  - Relaxation of exact covariance matching criteria
- Example in Wiener system identification for binary texture generation

Future work/open issues:

- Why do we need a singular measure in dimensions $d \geq 3$?
- What does a spectrum
$$\frac{P(e^{i\theta})}{Q(e^{i\theta})} = \frac{\sum_{k=1}^{\ell} |b_k(e^{i\theta})|^2}{\sum_{k=1}^{m} |a_k(e^{i\theta})|^2}$$
represent in terms of dynamical systems in dimensions $d \geq 2$?
- Good method for approximation as a sum-of-one square?

Thank you for your attention!

Questions?