Texture generation and Wiener system identification by multidimensional rational covariance extension

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Texture generation

In this work we use *Wiener systems* to model and generate textures, see figure 1. In particular we will consider binary textures, i.e., textures where the pixel values are either 0 or 1. The static nonlinearity f is therefore selected to be a *thresholding function*. From a given binary texture the goal is to identify a Wiener system such that when fed with white noise input u_t it generates a texture with similar features. This setup is motivated by [1], where thresholded Gaussian random fields are used to model porous materials for design of surface structures in pharmaceutical film coatings.

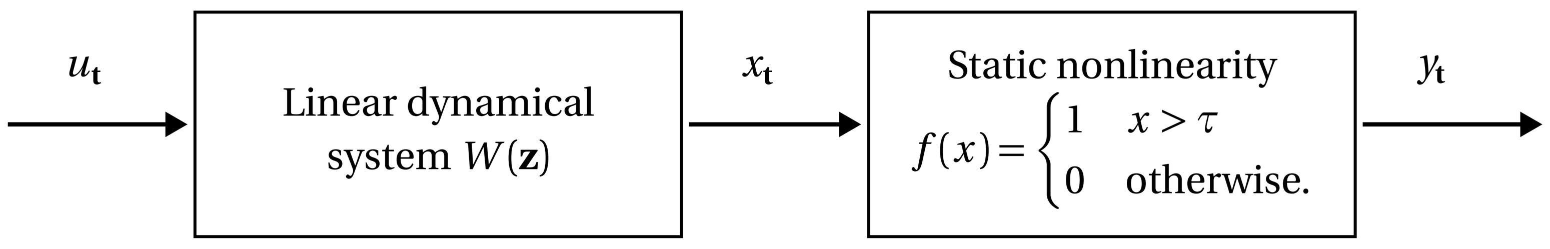

Figure 1: A Wiener system with a thresholding function.

Wiener system identification

To identify the Wiener system we need to identify

1. the thresholding parameter τ,
2. the linear dynamical system $W(\mathbf{z})$.

Let $u_\mathbf{t}$ be a zero-mean white Gaussian noise process and let $x_\mathbf{t}$ be the output of the two-dimensional *autoregressive-moving-average* (ARMA) filter $W(\mathbf{z})$:

$$x_\mathbf{t} + \sum_{\mathbf{k}\in\Lambda_+\setminus\{\mathbf{0}\}} a_\mathbf{k} x_{\mathbf{t}+\mathbf{k}} = \sum_{\mathbf{k}\in\Lambda_+} b_\mathbf{k} u_{\mathbf{t}+\mathbf{k}},$$

where $\Lambda_+ \subset \mathbb{Z}^2$ is the support of the filter. Note that in steady-state $x_\mathbf{t}$ is a stationary process, and we assume that the filter is normalized so that $c_\mathbf{0} = E[x_\mathbf{0}^2] = 1$. Moreover, we get that $W(\mathbf{z})$ has a rational form

$$W(\mathbf{z}) = \frac{\sum_{\mathbf{k}\in\Lambda_+} b_\mathbf{k}\mathbf{z}^\mathbf{k}}{\sum_{\mathbf{k}\in\Lambda_+} a_\mathbf{k}\mathbf{z}^\mathbf{k}} = \frac{b(\mathbf{z})}{a(\mathbf{z})}. \qquad (1)$$

The thresholding parameter can be estimated by noting that

$$E[y_\mathbf{t}] = P(y_\mathbf{t} = 1) = P(x_\mathbf{t} > \tau) = 1 - P(x_\mathbf{t} \le \tau) = 1 - \phi(\tau),$$

where $\phi(\tau)$ is the Gaussian cumulative distribution function $\phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} \exp(-s^2/2)ds$.

To estimate the linear dynamical system $W(\mathbf{z})$, let $r_\mathbf{k} := E[y_{\mathbf{t}+\mathbf{k}} y_\mathbf{t}] - E[y_{\mathbf{t}+\mathbf{k}}]E[y_\mathbf{t}]$ and $c_\mathbf{k} := E[x_{\mathbf{t}+\mathbf{k}} x_\mathbf{t}]$ be the covariances of the process $y_\mathbf{t}$ and $x_\mathbf{t}$ respectively. Since $x_\mathbf{t}$ is a Gaussian process, by [3] we have the following relationship between the covariances

$$r_\mathbf{k} = \int_0^{c_\mathbf{k}} \frac{1}{2\pi\sqrt{1-s^2}} \exp\left(-\frac{\tau^2}{1+s}\right) ds. \qquad (2)$$

Since the integrand is positive, the mapping can be inverted (numerically).

Summarizing, we have following identification procedure [4, 5]:

1. estimate threshold parameter: $\tau_{\text{est}} = \phi^{-1}(1 - E[y_\mathbf{t}])$,
2. estimate covariances: $r_\mathbf{k} := E[y_{\mathbf{t}+\mathbf{k}} y_\mathbf{t}] - E[y_{\mathbf{t}+\mathbf{k}}]E[y_\mathbf{t}]$,
3. compute covariances $c_\mathbf{k} := E[x_{\mathbf{t}+\mathbf{k}} x_\mathbf{t}]$ by using (2),
4. estimate a linear system from the covariances $c_\mathbf{k}$.

Rational covariance extension problem

To identify the linear dynamical system $W(\mathbf{z})$ from the covariances $c_\mathbf{k}$, consider the 2-dimensional stochastic process $x_\mathbf{t} \in \mathbb{R}$, where $\mathbf{t} \in \mathbb{Z}^2$, that is zero-mean and homogeneous. The *power spectral density* of the process is the positive function $\Phi(e^{i\boldsymbol{\theta}})$ defined on the torus $\mathbb{T}^2 = [-\pi, \pi]^2$ such that the covariances $c_\mathbf{k}$ of $x_\mathbf{t}$ are its Fourier coefficients:

$$c_\mathbf{k} := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})} \Phi(e^{i\boldsymbol{\theta}}) d\boldsymbol{\theta}, \quad \mathbf{k} \in \mathbb{Z}^2 \qquad \Longleftrightarrow \qquad \Phi(e^{i\boldsymbol{\theta}}) \sim \sum_{\mathbf{k}\in\mathbb{Z}^2} c_\mathbf{k} e^{-i(\mathbf{k},\boldsymbol{\theta})}.$$

Since $u_\mathbf{t}$ is a white noise process, the spectral density of $x_\mathbf{t}$ is given by $\Phi(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2$. Together with (1) we therefore know that

$$\Phi(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2 = \frac{|b(e^{i\boldsymbol{\theta}})|^2}{|a(e^{i\boldsymbol{\theta}})|^2} = \frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})},$$

where $P(e^{i\boldsymbol{\theta}})$ and $Q(e^{i\boldsymbol{\theta}})$ are *positive trigonometric polynomials* $P(e^{i\boldsymbol{\theta}}) = \sum_{\mathbf{k}\in\Lambda} p_\mathbf{k} e^{-i(\mathbf{k},\boldsymbol{\theta})}$. Using the notation $\mathfrak{P}_+$ = positive trigonometric polynomials, we get the following problem.

Problem formulation – Approximate rational covariance extension

Given a sequence of covariances $c = (c_\mathbf{k})_{\mathbf{k}\in\Lambda}$ find a positive function $\Phi(e^{i\boldsymbol{\theta}})$ so that

$$\begin{cases} c_\mathbf{k} \approx \frac{1}{(2\pi)^2} \int_{\mathbb{T}^d} e^{i(\mathbf{k},\boldsymbol{\theta})} \Phi(e^{i\boldsymbol{\theta}}) d\boldsymbol{\theta}, & \mathbf{k} \in \Lambda \\ \Phi(e^{i\boldsymbol{\theta}}) = \frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}, & P \text{ and } Q \in \mathfrak{P}_+. \end{cases}$$

This problem can be solved using convex optimization, using the following theorem.

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Rational covariance extension problem (cont.)

Theorem 1 – Multidimensional rational covariance extension

Given any sequence c, for ε large enough the primal problem

$$(P) \qquad \min_{\Phi>0,\,\tilde{c}} \int_{\mathbb{T}^2} \left(P \log\frac{P}{\Phi} + \Phi - P\right) d\boldsymbol{\theta}$$

$$\text{subject to} \quad \tilde{c}_\mathbf{k} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})} \Phi(e^{i\boldsymbol{\theta}}) d\boldsymbol{\theta}, \quad \mathbf{k} \in \Lambda,$$

$$\|\tilde{c} - c\|^2 \le \varepsilon^2,$$

has an optimal solution given by

$$\Phi(e^{i\boldsymbol{\theta}}) = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})},$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D) \qquad \min_{q\in\bar{\mathfrak{P}}_+} \langle c, q\rangle - \int_{\mathbb{T}^2} P \log(Q) d\boldsymbol{\theta} + \varepsilon\|q - e\|,$$

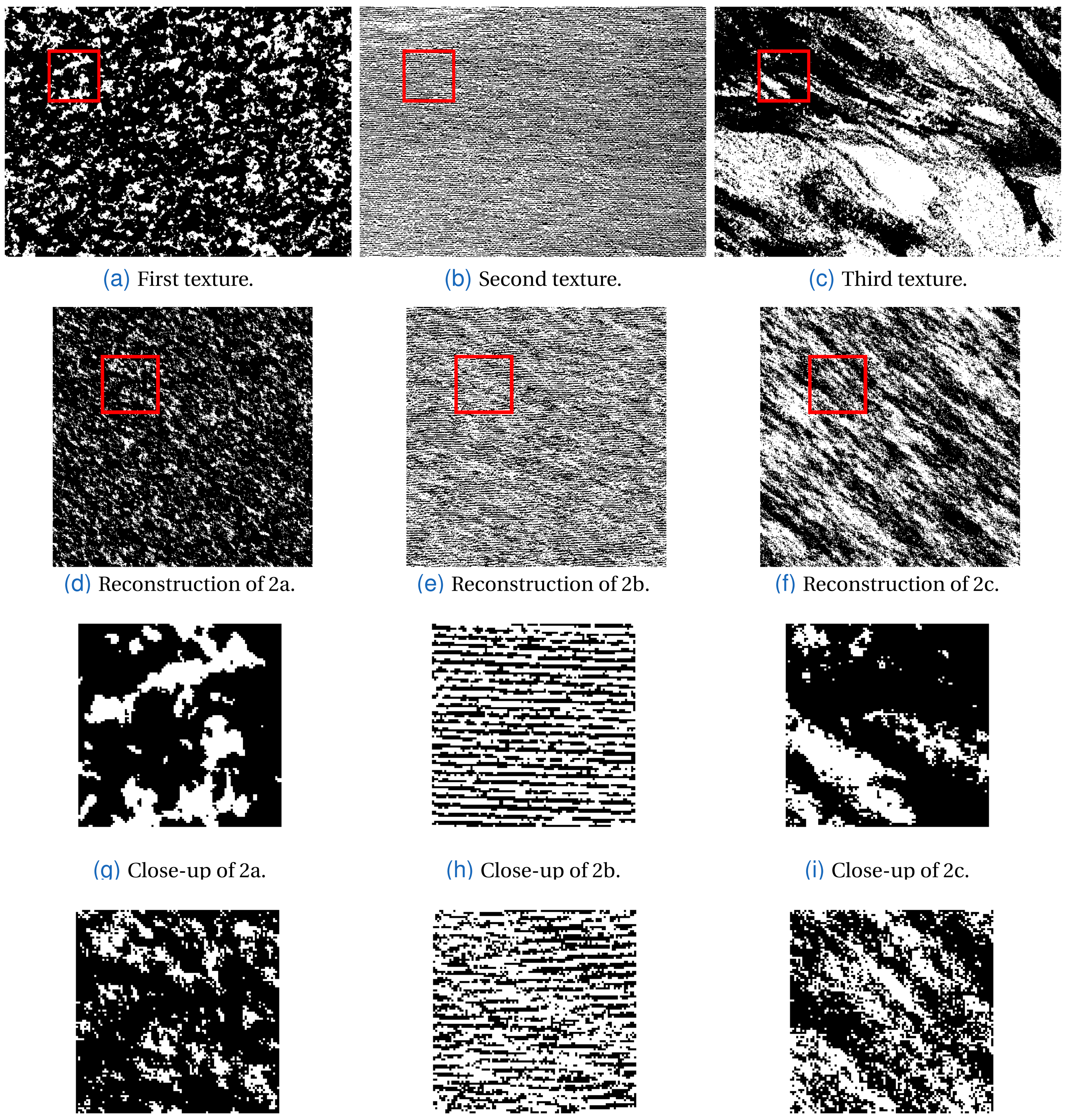
where $e \in \mathbb{C}^{|\Lambda|}$, $e_\mathbf{0} = 1$ and $e_\mathbf{k} = 0$ for $\mathbf{k} \in \Lambda \setminus \{\mathbf{0}\}$.

In one dimension one can use spectral factorization to write P and Q as a sum-of-one-square, $P(e^{i\theta})/Q(e^{i\theta}) = |b(e^{i\theta})|^2/|a(e^{i\theta})|^2$, in order to recover the filter coefficients. However, in the multi-dimensional case only factorization as sum-of-several squares can be guaranteed:

$$\frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})} = \frac{\sum_{k=1}^{\ell} |b_k(e^{i\boldsymbol{\theta}})|^2}{\sum_{k=1}^{m} |a_k(e^{i\boldsymbol{\theta}})|^2},$$

For $m > 1$ the system interpretation of this is unclear and we therefore resort to a heuristic to recover the filter coefficients, as described in [4, 5].

Simulation results



(a) First texture. (b) Second texture. (c) Third texture.

(d) Reconstruction of 2a. (e) Reconstruction of 2b. (f) Reconstruction of 2c.

(g) Close-up of 2a. (h) Close-up of 2b. (i) Close-up of 2c.

(j) Close-up of 2d. (k) Close-up of 2e. (l) Close-up of 2f.

Figure 2: Figures 2a - 2c show three different binary textures, size 1200×900, obtained from textures in the Outex database [2]. In Figures 2d - 2f , reconstructed textures of size 500×500 are shown, and in Figures 2g - 2l close-ups of size 100×100 are shown of the original and reconstructed textures.

Acknowledgments and references

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